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New Limit Distributions for Extremes Under a Nonlinear Normalization

S Ravi, T S Mavitha

Department of Studies in Statistics, University of Mysore, Manasagangotri, Mysuru 570006, India.

Abstract. In this article we look at limit distributions for extremes under a new nonlinear normalization of the form $\exp\{u_n(|\log|x||)^{v_n}\mathrm{sign}(\log|x|)\}\,\mathrm{sign}(x)$ which we have called as exponential norming. The corresponding limit laws are called as e-max stable laws. We study e-max stable laws, their stability property, their max domains of attraction, comparison between p-max and e-max domains and give examples of distribution functions in e-max domains, some of which do not belong to l-max and p-max domains.

1. Introduction and motivation

Extreme value distributions are well known and have been studied extensively in literature on extreme value theory. These are used as approximations to distributions of normalized partial maxima $M_n = \max\{X_1, X_2, \ldots, X_n\}$ of independent, identically distributed (iid) random variables (rvs) $X_1, X_2, \ldots, X_n$ with common distribution function (df) F. The df F is said to belong to the l-max domain of attraction of a nondegenerate df G under linear normalization, denoted by $F \in \mathcal{D}_l(G)$, if there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x),\ x \in \mathcal{C}(G), \tag{1}$$

$C(G)$ being the set of all continuity points of G. The df G in (1) is called an extreme value df. It is well known that there are only three types of possible nondegenerate limiting dfs G in (1), the l-max stable laws, satisfying the stability relation $G^n(A_n x + B_n) = G(x)$, $x \in \mathbb{R}$, $n \geq 1$, for some constants $A_n > 0$, $B_n \in \mathbb{R}$, and these are given in Appendix A.1. Here, two dfs F and G are of the same type if $F(x) = G(Ax + B)$ for all x, for some constants $A > 0$ and $B \in \mathbb{R}$. For necessary and sufficient conditions for F to satisfy (1) for a given G, we refer to Galambos (1978), Resnick (1987) and Embrechts et al. (1997).

Pancheva (1984) introduced a nonlinear normalization called power normalization. The df F is said to belong to the p-max domain of attraction of a nondegenerate df H under power normalization, denoted by $F \in \mathcal{D}_p(H)$, if for some norming constants $\alpha_n > 0$ and $\beta_n > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{1/\beta_n} \mathrm{sign}(M_n) \leq x\right) = \lim_{n\to\infty} F^n(\alpha_n |x|^{\beta_n} \mathrm{sign}(x)) = H(x),\ x \in \mathcal{C}(H), \tag{2}$$

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Email addresses: `ravi@statistics.uni-mysore.ac.in` (S Ravi), `mavithats@gmail.com` (T S Mavitha)

where sign(x) = -1, 0 or 1 according as $x < 0$, $= 0$ or > 0. The possible p-types of limiting dfs H in (2) are the p-max stable laws satisfying the stability relation $H^n(A_n|x|^{B_n}\text{sign}(x)) = H(x), x \in \mathbb{R}, n \geq 1,$ for some constants $A_n > 0$, $B_n > 0$, given in Appendix A.2. Here, two dfs F and H are of the same p-type if $F(x) = H(\alpha|x|^\beta\text{sign}(x))$ for all x, for some constants $\alpha > 0, \beta > 0$. Necessary and sufficient conditions for F to satisfy (2) given in Appendix A.3 were derived in Mohan and Ravi (1993), see also Christoph and Falk (1996) and Ravi and Mavitha (2015) for some recent work. All dfs mentioned in this article are given only for x values for which they are in the interval $(0, 1)$.

The following proposition gives a chain of equivalences which can be used to obtain p-max domain of attraction from l-max domain of attraction and vice-versa. The proof of the proposition is given in the next section. Here and elsewhere, F_X denotes the df of a rv X.

Proposition 1.1. *$F_X \in \mathcal{D}_l(F_\xi) \Leftrightarrow F_{\exp(X)} \in \mathcal{D}_p(F_{\exp(\xi)}) \Leftrightarrow F_{-\exp(-X)} \in \mathcal{D}_p(F_{-\exp(-\xi)})$, where F_ξ denotes an l-max stable df, $F_{\exp(\xi)}$ and $F_{-\exp(-\xi)}$ denote p-max stable dfs.*

It is of interest to extend this chain of equivalences from power normalization to the next possible normalization and to see what kind of norming and limit laws arise. Though not obvious at the first instance, this norming is found to be of the form

$$\exp\{u_n(|\log|x||)^{v_n}\text{sign}(\log|x|)\}\,\text{sign}(x)$$

and we have called this as exponential norming and the corresponding limit laws as e-max stable laws.

This article looks at e-max stable laws, their stability property, max domains of attraction, comparison between p-max and e-max domains and examples of dfs in e-max domains, some of which do not belong to p-max domains and hence l-max domains. Section 2 gives definition of an e-max stable law, e-max domain of attraction and obtains the e-max stable laws from p-max stable laws. Stability property satisfied by an e-max stable law is mentioned and proved in this section. In Section 3 we obtain necessary and sufficient conditions for a df to belong to the e-max domains of attraction of the e-max stable laws. A comparison result between p-max and e-max domains is stated and proved in Section 4 which shows that every df belonging to the p-max domain of attraction of a p-max stable law necessarily belongs to the e-max domain of attraction of some e-max stable law and that the converse is not true. This shows that e-max stable laws attract more dfs to their max domains than the p-max stable laws. In Section 5 we give some examples of dfs in e-max domains, some of which do not belong to p-max domains and hence l-max domains. We have given the l-max stable laws, p-max stable laws, criteria for a df to belong to p-max domains of the p-max stable laws and comparison of l-max and p-max domains of attractions in the Appendix in Section 6 for ease of reference.

2. Exponential norming, the e-max stable laws and stability properties

Before defining the e-max stable laws, we give the proof of Proposition 1.1 below.

Proof of Proposition 1.1. Suppose that $F_X \in \mathcal{D}_l(F_\xi)$ for some l-max stable df F_ξ. That is, there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\lim_{n\to\infty} F_X^n(a_n x + b_n) = F_\xi(x)$, $x \in \mathbb{R}$. Note that the l-max stable laws are all continuous. For $x \leq 0$, $F^n_{\exp(X)}(-\alpha_n(-x)^{\beta_n}) = P^n(e^X \leq -\alpha_n(-x)^{\beta_n}) = 0$ with $\alpha_n = \exp(b_n)$, $\beta_n = a_n$, and for $x > 0$,

$$\begin{aligned}\lim_{n\to\infty} F^n_{\exp(X)}(\alpha_n x^{\beta_n}) &= \lim_{n\to\infty} P^n(e^X \leq \exp(b_n)x^{a_n}) = \lim_{n\to\infty} P^n(X \leq a_n(\log x) + b_n),\\ &= \lim_{n\to\infty} F^n_X(a_n(\log x) + b_n) = F_\xi(\log x) = P(\xi \leq \log x) = F_{\exp(\xi)}(x),\end{aligned}$$

so that $F_{\exp(X)} \in \mathcal{D}_p(F_{\exp(\xi)})$.

Conversely, if $F_{\exp(X)} \in \mathcal{D}_p(F_{\exp(\xi)})$ for some p-max stable df $F_{\exp(\xi)}$, then for some norming constants $\alpha_n > 0$, $\beta_n > 0$, we have

$$\begin{aligned}\lim_{n\to\infty} F^n_{\exp(X)}(\alpha_n \mid x \mid^{\beta_n} \text{sign}(x)) &= F_{\exp(\xi)}(x),\ x \in \mathbb{R},\\ \Longleftrightarrow \lim_{n\to\infty} P^n(e^X \leq \alpha_n \mid x \mid^{\beta_n} \text{sign}(x)) &= P(e^\xi \leq x),\ x \in \mathbb{R}. \qquad (3)\end{aligned}$$

If $x \leq 0$, then both sides in (3) are equal to 0. If $x > 0$, then using (3), we have

$$\begin{aligned}\lim_{n\to\infty} P^n(e^X \leq \alpha_n x^{\beta_n}) = P(e^\xi \leq x) &\iff \lim_{n\to\infty} P^n(X \leq \beta_n(\log x) + \log\alpha_n) = P(\xi \leq \log x),\\ &\iff \lim_{n\to\infty} F_X^n(\beta_n(\log x) + \log\alpha_n) = F_\xi(\log x),\\ &\implies \lim_{n\to\infty} F_X^n(a_n y + b_n) = F_\xi(y), y \in \mathbb{R},\end{aligned}$$

so that $F_X \in \mathcal{D}_l(F_\xi)$ with $a_n = \beta_n,\, b_n = \log\alpha_n$.

Similarly, if $F_X \in \mathcal{D}_l(F_\xi)$ for some l-max stable df F_ξ, then for some norming constants $a_n > 0, b_n \in \mathbb{R}$, $F^n_{-\exp(-X)}(\exp(-b_n)x^{a_n}) = P^n(-e^{-X} \leq \exp(-b_n)x^{a_n}) = 1,\ 0 \leq x,$ and for $x < 0$,

$$\begin{aligned}\lim_{n\to\infty} F^n_{-\exp(-X)}(-\exp(-b_n)(-x)^{a_n}) &= \lim_{n\to\infty} P^n(-e^{-X} \leq -\exp(-b_n)(-x)^{a_n}),\\ &= \lim_{n\to\infty} P^n(X \leq a_n(-\log(-x)) + b_n),\\ &= \lim_{n\to\infty} F_X^n(a_n(-\log(-x)) + b_n),\\ &= F_\xi(-\log(-x)) = P(\xi \leq -\log(-x)) = F_{-\exp(-\xi)}(x),\end{aligned}$$

so that $F_{-\exp(-X)} \in \mathcal{D}_p(F_{-\exp(-\xi)})$.

Conversely, if $F_{-\exp(-X)} \in \mathcal{D}_p(F_{-\exp(-\xi)})$ for some p-max stable df $F_{-\exp(-\xi)}$, then for some norming constants $\alpha_n > 0,\ \beta_n > 0$, we have

$$\begin{aligned}\lim_{n\to\infty} F^n_{-\exp(-X)}(\alpha_n \mid x \mid^{\beta_n} \text{sign}(x)) &= F_{-\exp(-\xi)}(x),\ x \in \mathbb{R},\\ \iff \lim_{n\to\infty} P^n(-e^{-X} \leq \alpha_n \mid x \mid^{\beta_n} \text{sign}(x)) &= P(-e^{-\xi} \leq x),\ x \in \mathbb{R}.\end{aligned} \tag{4}$$

If $x \geq 0$, then both sides in (4) are equal to 1. If $x < 0$, then using (4), we have

$$\begin{aligned}&\lim_{n\to\infty} P^n(-e^{-X} \leq -\alpha_n(-x)^{\beta_n}) = P(-e^{-\xi} \leq x)\\ &\iff \lim_{n\to\infty} P^n(X \leq \beta_n(-\log(-x)) - \log\alpha_n) = P(\xi \leq -\log(-x)),\\ &\iff \lim_{n\to\infty} F_X^n(\beta_n(-\log(-x)) - \log\alpha_n) = F_\xi(-\log(-x)),\\ &\implies \lim_{n\to\infty} F_X^n(a_n y + b_n) = F_\xi(y), y \in \mathbb{R},\end{aligned}$$

so that $F_X \in \mathcal{D}_l(F_\xi)$ with $a_n = \beta_n,\, b_n = -\log\alpha_n$, proving the proposition. □

Remark 2.1. *In view of the proposition, if rv ξ has an l-max stable df, then the dfs of e^ξ and $-e^{-\xi}$ are the p-max stable laws listed in Appendix A.2. A similar result is used below to derive the e-max stable laws from the p-max stable laws.*

We define e-max stable laws, e-max domains and e-types of dfs below. After this the e-max stable laws are derived. In the following definition F is the common df of iid rvs $X_1, \ldots, X_n$ and $M_n = \max\{X_1, \ldots, X_n\}$.

Definition 2.2. *(i) A nondegenerate df U is said to be an e-max stable law if there exists a df F and norming constants $u_n > 0, v_n > 0$ such that*

$$\begin{aligned}&\lim_{n\to\infty} P\left(\exp\left\{\left(\frac{|\log|M_n||}{u_n}\right)^{1/v_n} sign(\log|M_n|)\right\} sign(M_n) \leq x\right)\\ &= \lim_{n\to\infty} P\left(M_n \leq \exp\left\{(u_n(|\log|x||)^{v_n} sign(\log|x|))\right\} sign(x)\right)\\ &= \lim_{n\to\infty} F^n(\exp\left\{(u_n(|\log|x||)^{v_n} sign(\log|x|))\right\} sign(x))\\ &= U(x), \quad x \in \mathcal{C}(U).\end{aligned} \tag{5}$$

(ii) A df F is said to belong to the e-max domain of attraction of a nondegenerate df U under e-normalization, denoted by $F \in \mathcal{D}_e(U)$, if for some norming constants $u_n > 0$ and $v_n > 0$, (5) holds.

(iii) Two dfs F and U are of the same e-type if $F(x) = U(\exp\{(u|\log|x||^v) sign(\log|x|)\} sign(x))$, $x \in \mathbb{R}$, for some constants $u > 0, v > 0$.

The following theorem derives the e-max stable laws from the p-max stable laws by proving a chain of equivalences to obtain e-max domains of attraction from p-max domains of attraction.

Theorem 2.3. *$F_X \in \mathcal{D}_p(F_\xi) \Leftrightarrow F_{\exp(X)} \in \mathcal{D}_e(F_{\exp(\xi)}) \Leftrightarrow F_{-\exp(-X)} \in \mathcal{D}_e(F_{-\exp(-\xi)})$, where F_ξ is a p-max stable law and $F_{\exp(\xi)}$ and $F_{-\exp(-\xi)}$ denote e-max stable laws.*

Proof. If F_X belongs to $\mathcal{D}_p(F_\xi)$ for some p-max stable law F_ξ then there exist norming constants $\alpha_n > 0$, $\beta_n > 0$ such that $\lim_{n\to\infty} F_X^n(\alpha_n \mid x \mid^{\beta_n} \text{sign}(x)) = F_\xi(x), x \in \mathbb{R}$. Note that the p-max stable laws are all continuous. For $x \leq 0$, $F^n_{\exp(X)}(\exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log \mid x \mid))\text{sign}(x)) = P^n(e^X \leq -\exp(\alpha_n \mid \log \mid x \mid\mid^{\beta_n} \text{sign}(\log \mid x \mid))) = 0$ with $u_n = \alpha_n, v_n = \beta_n, n \geq 1$. And for $x > 0$,

$$\begin{aligned}
\lim_{n\to\infty} F^n_{\exp(X)}(\exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log x))) &= \lim_{n\to\infty} P^n(e^X \leq \exp(\alpha_n \mid \log x \mid^{\beta_n} \text{sign}(\log x))), \\
&= \lim_{n\to\infty} P^n(X \leq \alpha_n \mid \log x \mid^{\beta_n} \text{sign}(\log x)), \\
&= \lim_{n\to\infty} F^n_X(\alpha_n \mid \log x \mid^{\beta_n} \text{sign}(\log x)), \\
&= F_\xi(\log x) = P(\xi \leq \log x) = F_{\exp(\xi)}(x),
\end{aligned}$$

so that $F_{\exp(X)} \in \mathcal{D}_e(F_{\exp(\xi)})$.

Conversely, if $F_{\exp(X)} \in \mathcal{D}_e(F_{\exp(\xi)})$ for some e-max stable df $F_{\exp(\xi)}$, then for some norming constants $u_n > 0, v_n > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} F^n_{\exp(X)}(\exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log \mid x \mid))\text{sign}(x)) &= F_{\exp(\xi)}(x), x \in \mathbb{R}, \\
\Longleftrightarrow \lim_{n\to\infty} P^n(e^X \leq \exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log \mid x \mid))\text{sign}(x)) &= P(e^\xi \leq x), x \in \mathbb{R}. \qquad (6)
\end{aligned}$$

If $x \leq 0$, then both sides in (6) are equal to 0. If $x > 0$, then using (6), we have

$$\begin{aligned}
& \lim_{n\to\infty} P^n(e^X \leq \exp(u_n \mid \log x \mid^{v_n} \text{sign}(\log x))) = P(e^\xi \leq x), \\
& \Longleftrightarrow \lim_{n\to\infty} P^n(X \leq u_n \mid \log x \mid^{v_n} \text{sign}(\log x)) = P(\xi \leq \log x), \\
& \Longleftrightarrow \lim_{n\to\infty} F^n_X(\alpha_n \mid \log x \mid^{\beta_n} \text{sign}(\log x)) = F_\xi(\log x), \\
& \Longrightarrow \lim_{n\to\infty} F^n_X(\alpha_n \mid y \mid^{\beta_n} \text{sign}(y)) = F_\xi(y), y \in \mathbb{R},
\end{aligned}$$

so that $F_X \in \mathcal{D}_p(F_\xi)$ with $\alpha_n = u_n, \beta_n = v_n$.

Similarly, if $F_X \in \mathcal{D}_p(F_\xi)$ for some p-max stable df F_ξ, then for some norming constants $\alpha_n > 0, \beta_n > 0$,

$$F^n_{-\exp(-X)}(\exp(\alpha_n \mid \log \mid x \mid\mid^{\beta_n} \text{sign}(\log \mid x \mid))\text{sign}(x)) = 1, \quad 0 \leq x,$$

and for $x < 0$,

$$\begin{aligned}
& \lim_{n\to\infty} F^n_{-\exp(-X)}(-\exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log \mid x \mid))) \\
&= \lim_{n\to\infty} P^n(-e^{-X} \leq -\exp(u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\log \mid x \mid))), \\
&= \lim_{n\to\infty} P^n(X \leq u_n \mid \log \mid x \mid\mid^{v_n} \text{sign}(\mid \log \mid x \mid\mid)), \\
&= \lim_{n\to\infty} F^n_X(\alpha_n \mid \log \mid x \mid\mid^{\beta_n} \text{sign}(\mid \log \mid x \mid\mid)), \\
&= F_\xi(-\log(-x)) = P(\xi \leq -\log(-x)) = F_{-\exp(-\xi)}(x),
\end{aligned}$$

so that $F_{-\exp(-X)} \in \mathcal{D}_e(F_{-\exp(-\xi)})$ with $u_n = \alpha_n, v_n = \beta_n$.

Conversely, if $F_{-\exp(-X)} \in \mathcal{D}_e(F_{-\exp(-\xi)})$ for some e-max stable df $F_{-\exp(-\xi)}$, then for some norming constants $u_n > 0, v_n > 0$, we have, for $x \in \mathbb{R}$,

$$\begin{aligned} \lim_{n\to\infty} F^n_{-\exp(-X)}(\exp(u_n \mid \log \mid x \mid\mid^{v_n} \operatorname{sign}(\log \mid x \mid))\operatorname{sign}(x)) &= F_{-\exp(-\xi)}(x), \\ \Leftrightarrow \quad \lim_{n\to\infty} P^n(-e^{-X} \le \exp(u_n \mid \log \mid x \mid\mid^{v_n} \operatorname{sign}(\log \mid x \mid))\operatorname{sign}(x)) &= P(-e^{-\xi} \le x). \end{aligned} \tag{7}$$

If $x \ge 0$, then both sides in (7) are equal to 1. If $x < 0$, then from (7), we have

$$\begin{aligned} &\lim_{n\to\infty} P^n(-e^{-X} \le -\exp(u_n \mid \log \mid x \mid\mid^{v_n} \operatorname{sign}(\log \mid x \mid))) = P(-e^{-\xi} \le x), \\ &\iff \lim_{n\to\infty} P^n(X \le u_n \mid \log \mid x \mid\mid^{v_n} \operatorname{sign}(\mid \log \mid x \mid\mid)) = P(\xi \le -\log(-x)), \\ &\iff \lim_{n\to\infty} F^n_X(\alpha_n \mid \log \mid x \mid\mid^{\beta_n} \operatorname{sign}(\mid \log \mid x \mid\mid)) = F_\xi(-\log(-x)), \\ &\implies \lim_{n\to\infty} F^n_X(\alpha_n \mid y \mid^{\beta_n} \operatorname{sign}(y)) = F_\xi(y),\ y \in \mathbb{R}, \end{aligned}$$

so that $F_X \in \mathcal{D}_p(F_\xi)$ with $\alpha_n = u_n, \beta_n = v_n$. Finally, using (5), it is clear that dfs $F_{\exp(\xi)}$ and $F_{-\exp(-\xi)}$ are e-max stable laws whenever F_ξ is p-max stable, proving the theorem. □

Using the above theorem, we get the e-max stable laws listed below wherein the first six e-max stable dfs have right end point $r(F) = \sup\{x : F(x) < 1\} > 0$ and the subsequent six e-max stable dfs have $r(F) \le 0$.

2.1. The e-max stable laws

$$\begin{aligned}
\text{(i) Loglog-Fréchet}: \quad U_{1,\alpha}(x) &= \exp(-(\log\log x)^{-\alpha}),\ x \ge e. \\
\text{(ii) Loglog-Weibull}: \quad U_{2,\alpha}(x) &= \exp(-(-\log\log x)^{\alpha}),\ 1 \le x < e. \\
\text{(iii) Standard log-Fréchet}: \quad H_{1,1}(x) &= \exp(-(\log x)^{-1}),\ x \ge 1. \\
\text{(iv) Inverse loglog-Fréchet}: \quad U_{3,\alpha}(x) &= \exp(-(-\log(-\log x))^{-\alpha}),\ 1/e \le x < 1. \\
\text{(v) Inverse loglog-Weibull}: \quad U_{4,\alpha}(x) &= \exp(-(\log(-\log x))^{\alpha}),\ 0 \le x < 1/e. \\
\text{(vi) Standard Uniform}: \quad H_{2,1}(x) &= x,\ 0 \le x < 1. \\
\text{(vii) Negative loglog-Fréchet}: \quad U_{5,\alpha}(x) &= \exp(-(\log(-\log(-x)))^{-\alpha}),\ -1/e \le x < 0. \\
\text{(viii) Negative loglog-Weibull}: \quad U_{6,\alpha}(x) &= \exp(-(-\log(-\log(-x)))^{\alpha}),\ -1 \le x < -1/e. \\
\text{(ix) Standard inverse log-Fréchet}: \quad H_{3,1}(x) &= \exp(-(-\log(-x))^{-1}),\ -1 \le x < 0. \\
\text{(x) Negative inverse loglog-Fréchet}: \quad U_{7,\alpha}(x) &= \exp(-(-\log\log(-x))^{-\alpha}), -e \le x < -1. \\
\text{(xi) Negative inverse loglog-Weibull}: \quad U_{8,\alpha}(x) &= \exp(-(\log\log(-x))^{\alpha}),\ x \le -e. \\
\text{(xii) Standard inverse log-Weibull}: \quad H_{4,1}(x) &= -1/x,\ x \le -1.
\end{aligned}$$

Remark 2.4. *Note that the l-max stable laws Fréchet, Weibull and Gumbel and the p-max stable laws standard Fréchet and standard Weibull laws are not e-max stable laws but the p-max stable laws standard log-Fréchet, standard uniform, standard inverse log-Fréchet and standard inverse log-Weibull laws are e-max stable laws.*

2.2. Stability property

The following result shows that the e-max stable laws satisfy a stability property which implies that if $X_1, \ldots, X_n$ are iid with common df as an e-max stable law, then the df of e-normalized partial maxima $M_n = \max\{X_1, \ldots, X_n\}$ is exactly equal to the same e-max stable law for every $n \ge 1$, which, in particular, implies that an e-max stable law belongs to the e-max domain of attraction of itself.

Theorem 2.5. *If U is an e-max stable law then U satisfies the stability property:*

$$U^n(\exp\{(u_n|\log|x||^{v_n})sign(\log|x|)\}sign(x)) = U(x),\ x\in\mathbb{R},\ n\geq 1,$$

for some constants $u_n>0,\ v_n>0$.

Proof. The stability properties follow from the following observations wherein norming constants u_n and v_n are specified for each of the e-max stable laws.

(i) We have, for $n\geq 1,\ e\leq x,$ with $u_n=1,\ v_n=n^{1/\alpha}$,

$$\begin{aligned} U^n_{1,\alpha}(e^{u_n(\log x)^{v_n}}) &= \left(e^{-(\log\log e^{u_n(\log x)^{v_n}})^{-\alpha}}\right)^n, \quad e^{u_n(\log x)^{v_n}}\geq e,\\ &= e^{-n(\log(u_n(\log x)^{v_n}))^{-\alpha}} = e^{-n(n^{1/\alpha}\log\log x)^{-\alpha}}\\ &= e^{-(\log\log x)^{-\alpha}} = U_{1,\alpha}(x),\ x\geq e. \end{aligned}$$

(ii) For $n\geq 1,\ 1\leq x<e,$ with $u_n=1,\ v_n=n^{-1/\alpha}$, we have

$$\begin{aligned} U^n_{2,\alpha}(e^{u_n(\log x)^{v_n}}) &= \left(e^{-(-\log\log e^{u_n(\log x)^{v_n}})^{\alpha}}\right)^n, \quad 1\leq e^{u_n(\log x)^{v_n}}<e,\\ &= e^{-n(-\log(u_n(\log x)^{v_n}))^{\alpha}} = e^{-n(-n^{-1/\alpha}\log\log x)^{\alpha}}\\ &= e^{-(-\log\log x)^{\alpha}} = U_{2,\alpha}(x),\ 1\leq x<e. \end{aligned}$$

(iii) We have, for $n\geq 1,\ 1\leq x,$ with $u_n=n,\ v_n=1$,

$$\begin{aligned} H^n_{1,1}(e^{u_n(\log x)^{v_n}}) &= \left(e^{-(\log e^{u_n(\log x)^{v_n}})^{-1}}\right)^n, \quad e^{u_n(\log x)^{v_n}}>1,\\ &= e^{-n(u_n(\log x)^{v_n})^{-1}} = e^{-(\log x)^{-1}} = H_{1,1}(x),\ x\geq 1. \end{aligned}$$

(iv) We have, for $n\geq 1,\ e^{-1}\leq x<1,$ with $u_n=1,\ v_n=n^{1/\alpha}$,

$$\begin{aligned} U^n_{3,\alpha}(e^{-u_n(-\log x)^{v_n}}) &= \left(e^{-(-\log(-\log e^{-u_n(-\log x)^{v_n}}))^{-\alpha}}\right)^n, \quad e^{-1}\leq e^{-u_n(-\log x)^{v_n}}<1,\\ &= e^{-n(-\log(u_n(-\log x)^{v_n}))^{-\alpha}}\\ &= e^{-(-\log(-\log x))^{-\alpha}} = U_{3,\alpha}(x),\ e^{-1}\leq x<1. \end{aligned}$$

(v) We have, for $n\geq 1,\ 0\leq x<e^{-1},$ with $u_n=1,\ v_n=n^{-1/\alpha}$,

$$\begin{aligned} U^n_{4,\alpha}(e^{-u_n(-\log x)^{v_n}}) &= \left(e^{-(\log(-\log e^{-u_n(-\log x)^{v_n}}))^{\alpha}}\right)^n, \quad 0\leq e^{-u_n(-\log x)^{v_n}}<e^{-1},\\ &= e^{-n(\log(u_n(-\log x)^{v_n}))^{\alpha}}\\ &= e^{-(\log(-\log x))^{\alpha}} = U_{4,\alpha}(x),\ 0\leq x<e^{-1}. \end{aligned}$$

(vi) We have, for $n\geq 1,\ 0\leq x<1,$ with $u_n=n^{-1},\ v_n=1$,

$$\begin{aligned} H^n_{2,1}(e^{-u_n(-\log x)^{v_n}}) &= \left(e^{-u_n(-\log x)^{v_n}}\right)^n,\ 0<e^{-u_n(-\log x)^{v_n}}<1,\\ &= e^{-(-\log x)} = x = H_{2,1}(x),\ 0\leq x<1. \end{aligned}$$

(vii) We have, for $n\geq 1,\ -e^{-1}\leq x<0,$ with $u_n=1,\ v_n=n^{1/\alpha}$,

$$\begin{aligned} U^n_{5,\alpha}(-e^{-u_n(-\log(-x))^{v_n}}) &= \left(e^{-(\log(-\log e^{-u_n(-\log(-x))^{v_n}}))^{-\alpha}}\right)^n, \quad -e^{-1}\leq -e^{-u_n(-\log(-x))^{v_n}}<0\\ &= e^{-n(\log(u_n(-\log(-x))^{v_n}))^{-\alpha}}\\ &= e^{-(\log(-\log(-x)))^{-\alpha}} = U_{5,\alpha}(x),\ -e^{-1}\leq x<0. \end{aligned}$$

(viii) We have, for $n\geq 1,\ -1\leq x<-e^{-1},$ with $u_n=1,\ v_n=n^{-1/\alpha}$,

$$\begin{aligned} U^n_{6,\alpha}(-e^{-u_n(-\log(-x))^{v_n}}) &= \left(e^{-(-\log(-\log e^{-u_n(-\log(-x))^{v_n}}))^{\alpha}}\right)^n, \quad -1\leq -e^{-u_n(-\log(-x))^{v_n}}<-e^{-1},\\ &= e^{-n(-\log(u_n(-\log(-x))^{v_n}))^{\alpha}}\\ &= e^{-(-\log(-\log(-x)))^{\alpha}} = U_{6,\alpha}(x),\ -1\leq x<-e^{-1}. \end{aligned}$$

(ix) We have, for $n \geq 1$, $-1 \leq x < 0$, with $u_n = n$, $v_n = 1$,

$$\begin{aligned} H^n_{3,1}(-e^{-u_n(-\log(-x))^{v_n}}) &= \left(e^{-(-\log e^{-u_n(-\log(-x))^{v_n}})^{-1}}\right)^n, \quad -1 \leq -e^{-u_n(-\log(-x))^{v_n}} < 0, \\ &= e^{-n(u_n(-\log(-x))^{v_n})^{-1}} \\ &= e^{-(-\log(-x))^{-1}} = H_{3,1}(x), \ -1 \leq x < 0. \end{aligned}$$

(x) We have, for $n \geq 1$, $-e \leq x < -1$, with $u_n = 1$, $v_n = n^{1/\alpha}$,

$$\begin{aligned} U^n_{7,\alpha}(-e^{u_n(\log(-x))^{v_n}}) &= \left(e^{-(-\log\log e^{u_n(\log(-x)^{v_n})})^{-\alpha}}\right)^n, \quad -e \leq -e^{u_n(\log(-x))^{v_n}} < -1, \\ &= e^{-n(-\log(u_n(\log(-x))^{v_n}))^{-\alpha}} \\ &= e^{-(-\log\log(-x))^{-\alpha}} = U_{7,\alpha}(x), \ -e \leq x < -1. \end{aligned}$$

(xi) We have, for $n \geq 1$, $x \leq -e$, with $u_n = 1$, $v_n = n^{1/\alpha}$,

$$\begin{aligned} U^n_{8,\alpha}(-e^{u_n(\log(-x))^{v_n}}) &= \left(e^{-(\log\log e^{u_n(\log(-x)^{v_n})})^{-\alpha}}\right)^n, \quad -e^{u_n(\log(-x))^{v_n}} < -e, \\ &= e^{-n(\log(u_n(\log(-x))^{v_n}))^{-\alpha}} = e^{-(\log\log(-x))^{-\alpha}} = U_{8,\alpha}(x), \ x \leq -e. \end{aligned}$$

(xii) We have, for $n \geq 1$, $x \leq -1$, with $u_n = n^{-1}$, $v_n = 1$,

$$\begin{aligned} H^n_{4,1}(-e^{u_n(\log(-x))^{v_n}}) &= \left(e^{-u_n(\log(-x)^{v_n}}\right)^n, \quad -e^{u_n(\log(-x))^{v_n}} < -1, \\ &= e^{-nu_n(\log(-x))^{v_n}} = e^{-(\log(-x))} = H_{4,1}(x), \ x \leq -1. \end{aligned}$$

□

3. Necessary and sufficient conditions for a df to belong to $\mathcal{D}_e(.)$

In this section, criteria are given for a df to belong to the e-max domain of attraction of e-max stable laws. For a df F, F^- is defined as $F^-(y) = \inf\{x \in \mathbb{R} : F(x) > y\}$, $y \in \mathbb{R}$, $\to$ is used to denote 'tends to' and $\max\{a, b\}$ is denoted by $a \vee b$ for $a \in \mathbb{R}, b \in \mathbb{R}$.

Theorem 3.1. *A df $F \in \mathcal{D}_e(U_{1,\alpha})$ for some $\alpha > 0$ iff (i) $r(F) = \infty$ and (ii) $\lim_{t\to\infty} \frac{1 - F(e^{e^{tx}})}{1 - F(e^{e^t})} = x^{-\alpha}$, $x > 0$. Here we can take $u_n = 1$, $v_n = \log\log F^-(1 - 1/n)$.*

Proof. If $F \in \mathcal{D}_e(U_{1,\alpha})$ then from (5), $\lim_{n\to\infty} F^n(e^{u_n(\log x)^{v_n}}) = e^{-(\log\log x)^{-\alpha}}$, $x \geq e$, for some norming constants $u_n > 0, v_n > 0$. Putting $x = e$, we get $\lim_{n\to\infty} F^n(e^{u_n}) = 0$. Defining $Y = \log(a \vee X)$ for some a, $0 < a < 1$, we have

$$G(y) = P(Y \leq y) = P(\log(a \vee X) \leq y) = F(e^y), \quad y \geq \log a, \text{ and } r(G) = \log r(F). \tag{8}$$

With $\alpha_n = u_n$, $\beta_n = v_n$, trivially, for $x < 1$, $0 \leq G^n(\alpha_n|x|^{\beta_n}\,\mathrm{sign}(x)) \leq G^n(u_n) = F^n(e^{u_n}) \to 0$ and for $x \geq 1$,

$$\begin{aligned} \lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n}\,\mathrm{sign}(x)) &= \lim_{n\to\infty} F^n(e^{\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{u_n\,(\log e^x)^{v_n}}), \\ &= e^{-(\log\log e^x)^{-\alpha}} = e^{-(\log x)^{-\alpha}} = H_{1,\alpha}(x), \end{aligned}$$

so that $G \in \mathcal{D}_p(H_{1,\alpha})$. Therefore, from Theorem 6.1(i) in Appendix A.3., $r(G) = \infty$ and $\lim_{t\to\infty} \frac{1 - G(e^{tx})}{1 - G(e^t)} = x^{-\alpha}$ and hence $r(F) = \infty$ and $\lim_{t\to\infty} \frac{1 - F(e^{e^{tx}})}{1 - F(e^{e^t})} = \lim_{t\to\infty} \frac{1 - G(e^{tx})}{1 - G(e^t)} = x^{-\alpha}, x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (8), we have $r(G) = \log r(F) = \infty$ and $\lim_{t\to\infty} \dfrac{1 - G(e^{tx})}{1 - G(e^t)} = \lim_{t\to\infty} \dfrac{1 - F(e^{e^{tx}})}{1 - F(e^{e^t})} = x^{-\alpha}$, $x > 0$. Hence from Theorem 6.1(i) in Appendix A.3., $G \in \mathcal{D}_p(H_{1,\alpha})$ with $\alpha_n = 1$, $\beta_n = \log G^{-}(1-1/n)$. Therefore, with $u_n = 1$, $v_n = \beta_n = \log\log F^{-}(1-1/n)$, $F^n(e^{(\log x)^{v_n}}) = G^n((\log x)^{v_n}) \to H_{1,\alpha}(\log x) = e^{-(\log\log x)^{-\alpha}} = U_{1,\alpha}(x)$, $x \geq e$, so that $F \in \mathcal{D}_e(U_{1,\alpha})$, proving the theorem. $\square$

Theorem 3.2. *A df $F \in \mathcal{D}_e(U_{2,\alpha})$, $\alpha > 0$ iff (i) $1 < r(F) < \infty$ and (ii) $\lim_{t\to\infty} \dfrac{1 - F(e^{\log r(F) e^{-x/t}})}{1 - F(e^{\log r(F) e^{-1/t}})}$ $= x^{\alpha}$, $x > 0$. Norming constants can be chosen as $u_n = \log r(F)$, $v_n = \log\log r(F) - \log\log F^{-}(1 - 1/n)$.*

Proof. If $F \in \mathcal{D}_e(U_{2,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(e^{u_n(\log x)^{v_n}}) = e^{-(-\log\log x)^{\alpha}}$, $1 \leq x < e$. Putting $x = 1$ we then get $F^n(1) \to 0$ and putting $x = e$ we get $F^n(e^{u_n}) \to 1$. Defining $Y = \log(a \vee X)$, for some a, $0 < a < 1$, we get G as in (8) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x < 0$, $0 \leq G^n(\alpha_n \mid x \mid^{\beta_n} \operatorname{sign}(x)) \leq G^n(0) = F^n(1) \to 0$, and for $x \geq 1$, $G^n(\alpha_n x^{\beta_n}) \geq G^n(\alpha_n) = F^n(e^{u_n}) \to 1$. For $0 \leq x < 1$, we have

$$\begin{aligned}\lim_{n\to\infty} G^n(\alpha_n x^{\beta_n}) &= \lim_{n\to\infty} F^n(e^{\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{u_n (\log e^x)^{v_n}}),\\ &= \exp(-(-\log\log e^x)^{\alpha}) = \exp(-(-\log x)^{\alpha}),\end{aligned}$$

so that $G \in \mathcal{D}_p(H_{2,\alpha})$. Therefore from Theorem 6.1(ii) in Appendix A.3., $0 < r(G) < \infty$ and $\lim_{t\to\infty} \dfrac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} = x^{\alpha}$. So, $1 < r(F) < \infty$ and $\lim_{t\to\infty} \dfrac{1 - F(e^{\log r(F)e^{-x/t}})}{1 - F(e^{\log r(F)e^{-1/t}})} = \lim_{t\to\infty} \dfrac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} =$ x^{α}, $x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (8), we have $1 < r(F) < \infty$ so that $0 < r(G) < \infty$ and $\lim_{t\to\infty} \dfrac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} = \lim_{t\to\infty} \dfrac{1 - F(e^{\log r(F)e^{-x/t}})}{1 - F(e^{\log r(F)e^{-1/t}})} = x^{\alpha}$, $x > 0$. Then from Theorem 6.1(ii) in Appendix A.3., $G \in \mathcal{D}_p(H_{2,\alpha})$ with $\alpha_n = r(G)$, $\beta_n = \log r(G) - \log G^{-}(1-1/n)$. Therefore, with $u_n = \log r(F)$, $v_n = \log\log r(F) - \log\log F^{-}(1 - 1/n)$, $F^n(e^{u_n(\log x)^{v_n}}) = G^n(u_n(\log x)^{v_n}) \to H_{2,\alpha}(\log x) = \exp(-(-\log\log x)^{\alpha}) = U_{2,\alpha}(x)$, $1 \leq x < e$, so that $F \in \mathcal{D}_e(U_{2,\alpha})$, proving the theorem. $\square$

Theorem 3.3. *A df $F \in \mathcal{D}_e(H_{1,1})$ iff (i) $r(F) > 1$ and (ii) $\lim_{t\uparrow\log r(F)} \dfrac{1 - F(e^{te^{xf(t)}})}{1 - F(e^t)} = e^{-x}$, for some positive valued function f. If (ii) holds for some f then $\int_a^{\log r(F)} \frac{1-F(e^x)}{x}dx < \infty$ for $0 < a < \log r(F)$ and (ii) holds with $f(t) = \dfrac{1}{1 - F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x}dx$. The norming constants here may be chosen as $u_n = \log F^{-}(1 - 1/n)$, $v_n = f(u_n)$.*

Proof. If $F \in \mathcal{D}_e(H_{1,1})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(e^{u_n(\log x)^{v_n}}) = e^{-(\log x)^{-1}}$, $x > 1$. Putting $x = 1$, we get $F^n(1) \to 0$. Defining $Y = \log(a \vee X)$ for some a, $0 < a < 1$, we have G as in (8) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \leq 0$, $0 \leq G^n(\alpha_n \mid x \mid^{\beta_n} \operatorname{sign}(x)) \leq G^n(0) = F^n(1) \to 0$. For $x > 0$ we have

$$\lim_{n\to\infty} G^n(\alpha_n x^{\beta_n}) = \lim_{n\to\infty} F^n(e^{\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{u_n (\log e^x)^{v_n}}) = e^{-(\log e^x)^{-1}} = e^{-1/x} = \Phi(x),$$

so that $G \in \mathcal{D}_p(\Phi)$. Therefore, from Theorem 6.1(v) in Appendix A.3., $r(G) > 0$ and $\lim_{t\uparrow r(G)} \dfrac{1 - G(te^{xg(t)})}{1 - G(t)} =$ e^{-x}, for some positive valued function g, and if it holds for some g, then $\dfrac{1}{1 - G(t)} \int_t^{r(G)} \frac{1-G(x)}{x}dx < \infty$ and it holds with $g(t) = \dfrac{1}{1 - G(t)} \int_t^{r(G)} \frac{1-G(x)}{x}dx$. Re-writing these in terms of F, we get $r(F) > 1$

and $\lim\limits_{t\uparrow \log r(F)} \dfrac{1-F(e^{te^{xf(t)}})}{1-F(e^t)} = \lim\limits_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = e^{-x}$, for some positive valued function f and if this holds for some function f, then $\dfrac{1}{1-F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x} dx < \infty$, and we can take $f(t) = \dfrac{1}{1-F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x} dx$, proving (i) and (ii).

Conversely, if (i) and (ii) hold, then defining G as in (8), $r(F) > 1$ which implies that $r(G) > 0$ and $\lim\limits_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = \lim\limits_{t\uparrow \log r(F)} \dfrac{1-F(e^{te^{xf(t)}})}{1-F(e^t)} = e^{-x}$, where g and f are as above. Hence from Theorem 6.1(v) in Appendix A.3., $G \in \mathcal{D}_p(\Phi)$. Therefore, with $u_n = \log F^{-}(1-1/n)$ and $v_n = f(u_n)$, $F^n(e^{u_n(\log x)^{v_n}}) = G^n(u_n(\log x)^{v_n}) \to \Phi(\log x) = e^{-(\log x)^{-1}} = H_{1,1}(x)$, $x > 0$, so that $F \in \mathcal{D}_e(H_{1,1})$, proving the theorem. □

Theorem 3.4. *A df* $F \in \mathcal{D}_e(U_{3,\alpha})$ *iff (i)* $r(F) = 1$ *and (ii)* $\lim\limits_{t\to\infty} \dfrac{1-F(e^{-e^{-ty}})}{1-F(e^{-e^{-t}})} = y^{-\alpha}, y > 0$. *Norming constants can be chosen as* $u_n = 1, v_n = -\log(-\log F^{-}(1-1/n))$.

Proof. If $F \in \mathcal{D}_e(U_{3,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim\limits_{n\to\infty} F^n(e^{-u_n(-\log x)^{v_n}}) = \exp(-(-\log(-\log x))^{-\alpha})$, $1/e \le x < 1$. Putting $x = 1/e$, we get $\lim\limits_{n\to\infty} F^n(e^{-u_n}) \to 0$ and putting $x = 1$, we get $\lim\limits_{n\to\infty} F^n(1) = 1$. Defining $Y = \log(a \vee X)$ for some a, $0 < a < 1$, we have G as in (8) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \le -1$, $0 \le G^n(-\alpha_n(-x)^{\beta_n}) \le G^n(-\alpha_n) = F^n(e^{-u_n}) \to 0$ and for $x \ge 0$, $G^n(\alpha_n x^{\beta_n}) \ge G^n(0) = F^n(1) \to 1$. For $-1 \le x < 0$, we have

$$\begin{aligned}\lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n}\ \mathrm{sign}(x)) &= \lim_{n\to\infty} F^n(e^{-\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{-u_n(-\log e^x)^{v_n}}),\\ &= e^{-(-\log(-\log e^x))^{-\alpha}} = e^{-(-\log(-x))^{-\alpha}} = H_{3,\alpha}(x),\end{aligned}$$

so that $G \in \mathcal{D}_p(H_{3,\alpha})$. Therefore, from Theorem 6.1(iii) in Appendix A.3., $r(G) = 0$ and $\lim\limits_{t\to\infty} \dfrac{1-G(-e^{-tx})}{1-G(-e^{-t})} = x^{-\alpha}$. So, $r(F) = e^{r(G)} = 1$ and $\lim\limits_{t\to\infty} \dfrac{1-F(e^{-e^{-tx}})}{1-F(e^{-e^{-t}})} = \lim\limits_{t\to\infty} \dfrac{1-G(-e^{-tx})}{1-G(-e^{-t})} = x^{-\alpha}, x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (8), we have $r(G) = \log r(F) = 0$ and $\lim\limits_{t\to\infty} \dfrac{1-G(-e^{-tx})}{1-G(-e^{-t})} = \lim\limits_{t\to\infty} \dfrac{1-F(e^{-e^{-tx}})}{1-F(e^{-e^{-t}})} = x^{-\alpha}$, $x > 0$. Then from Theorem 6.1(iii) in Appendix A.3., $G \in \mathcal{D}_p(H_{3,\alpha})$ with $\alpha_n = 1, \beta_n = -\log(-G^{-}(1-1/n))$. So, with $u_n = 1, v_n = \beta_n = -\log(-\log F^{-}(1-1/n))$, $F^n(e^{-(-\log x)^{v_n}}) = G^n((-(-\log x))^{v_n}) \to H_{3,\alpha}(\log x) = e^{-(-\log(-\log x))^{-\alpha}} = U_{3,\alpha}(x)$, $1/e \le x < 1$, proving the theorem. □

Theorem 3.5. *A df* $F \in \mathcal{D}_e(U_{4,\alpha})$ *iff (i)* $0 < r(F) < 1$ *and (ii)* $\lim\limits_{t\to\infty} \dfrac{1-F(e^{\log(r(F))e^{y/t}})}{1-F(e^{\log(r(F))e^{1/t}})}$ $= y^{\alpha}$, $y > 0$. *Norming constants can be chosen as* $u_n = -\log r(F)$, $v_n = \log\log F^{-}(1-1/n) - \log\log r(F)$.

Proof. If $F \in \mathcal{D}_e(U_{4,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim\limits_{n\to\infty} F^n(e^{-u_n(-\log x)^{v_n}}) = e^{-(\log(-\log x))^{\alpha}}$, $0 \le x < 1/e$. Putting $x = 0$, we get $F^n(0) \to 0$, and putting $x = 1/e$ we get $F^n(e^{-u_n}) \to 1$. Defining $Y = \log(a \vee X)$, for some a, $0 < a < r(F)$, we have G as in (8) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \ge -1$, $1 \ge G^n(\alpha_n \mid x \mid^{\beta_n}\ \mathrm{sign}(x)) \ge G^n(-\alpha_n) = F^n(e^{-u_n}) \to 1$ and for $x < -1$,

$$\begin{aligned}\lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n}\ \mathrm{sign}(x)) &= \lim_{n\to\infty} F^n(e^{-\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{-u_n(-\log e^x)^{v_n}}),\\ &= e^{-(-\log(-\log e^x))^{\alpha}} = e^{-(-\log(-x))^{\alpha}} = H_{4,\alpha}(x),\end{aligned}$$

so that $G \in \mathcal{D}_p(H_{4,\alpha})$. Therefore from Theorem 6.1(iv) in Appendix A.3., $r(G) < 0$ and $\lim_{t\to\infty} \dfrac{1-G(r(G)e^{x/t})}{1-G(r(G)e^{1/t})} = x^\alpha$. So, $0 < r(F) < 1$ and $\lim_{t\to\infty} \dfrac{1-F(e^{\log r(F) e^{x/t}})}{1-F(e^{\log r(F) e^{1/t}})} = \lim_{t\to\infty} \dfrac{1-G(r(G)e^{x/t})}{1-G(r(G)e^{1/t})} = x^\alpha$, $x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (8), $0 < r(F) < 1$ which implies that $r(G) < 0$ and $\lim_{t\to\infty} \dfrac{1-G(r(G)e^{x/t})}{1-G(r(G)e^{1/t})} = \lim_{t\to\infty} \dfrac{1-F(e^{\log r(F) e^{x/t}})}{1-F(e^{\log r(F) e^{1/t}})} = x^\alpha$, $x > 0$. Hence from Theorem 6.1(iv) in Appendix A.3., $G \in \mathcal{D}_p(H_{4,\alpha})$ with $\alpha_n = -r(G)$, $\beta_n = \log G^-(1-1/n) - \log r(G)$. So, with $u_n = -\log r(F)$, $v_n = \log\log F^-(1-1/n) - \log\log r(F)$, $F^n(e^{-u_n(-\log x)^{v_n}}) = G^n(-u_n(-\log x)^{v_n}) \to H_{4,\alpha}(\log x) = \exp(-(\log(-\log x))^\alpha) = U_{4,\alpha}(x), 0 \le x < 1/e$, proving the theorem. □

Theorem 3.6. *A df* $F \in \mathcal{D}_e(H_{2,1})$ *iff (i)* $0 < r(F) \le 1$ *and (ii)* $\lim_{t\uparrow \log r(F)} \dfrac{1-F(e^{te^{xf(t)}})}{1-F(e^t)} = e^x$, *for some positive valued function* f. *If (ii) holds for some* f *then* $\int_t^{\log r(F)} \frac{1-F(e^x)}{x}dx < \infty$ *for* $t < \log r(F)$ *and (ii) holds with the choice* $f(t) = -\dfrac{1}{1-F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x}dx$. *The norming constants here may be chosen as* $u_n = -\log F^-(1-1/n)$, $v_n = f(-u_n)$.

Proof. If $F \in \mathcal{D}_e(H_{2,1})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(e^{-u_n(-\log x)^{v_n}}) = x$, $0 < x < 1$. Putting $x = 1$, $F^n(1) \to 1$. Defining $Y = \log(a \vee X)$ for some a, $0 < a < r(F)$, we have G as in (8) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \ge 0$, $1 \ge G^n(\alpha_n x^{\beta_n}) \ge G^n(0) = F^n(1) \to 1$. For $x < 0$,

$$\lim_{n\to\infty} G^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) = \lim_{n\to\infty} F^n(e^{-\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(e^{-u_n(-\log e^x)^{v_n}}) = e^x = \Psi(x),$$

so that $G \in \mathcal{D}_p(\Psi)$. Therefore, from Theorem 6.1(vi) in Appendix A.3., $r(G) \le 0$ and $\lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = e^x$, for some positive valued function g, and if this holds for some positive valued function g then $-\dfrac{1}{1-G(t)} \int_t^{r(G)} \frac{1-G(x)}{x}dx < \infty$ and it holds with $g(t) = -\dfrac{1}{1-G(t)} \int_t^{r(G)} \frac{1-G(x)}{x}dx$. Thus $0 < r(F) \le 1$ and $\lim_{t\uparrow \log r(F)} \dfrac{1-F(e^{te^{xf(t)}})}{1-F(e^t)} = \lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = e^x$, for some positive valued function f and if this holds for some f then $-\dfrac{1}{1-F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x}dx < \infty$, and it holds with $f(t) = -\dfrac{1}{1-F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^x)}{x}dx$, proving (i) and (ii).

Conversely, if (i) and (ii) hold, then defining G as in (8), $0 < r(F) \le 1$ which implies that $r(G) \le 0$ and $\lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = \lim_{t\uparrow \log r(F)} \dfrac{1-F(e^{te^{xf(t)}})}{1-F(e^t)} = e^x$, where g and f are as above. Then from Theorem 6.1(vi) in Appendix A.3., $G \in \mathcal{D}_p(\Psi)$. So, with $u_n = -\log F^-(1-1/n)$ and $v_n = f(-u_n)$, $F^n(e^{-u_n(-\log x)^{v_n}}) = G^n(u_n(\log x)^{v_n}) \to \Phi(\log x) = x = H_{2,1}(x), 0 \le x < 1$, proving the theorem. □

Theorem 3.7. *A df* $F \in \mathcal{D}_e(U_{5,\alpha})$ *iff (i)* $r(F) = 0$ *and (ii)* $\lim_{t\to\infty} \dfrac{1-F(-e^{-e^{tx}})}{1-F(-e^{-e^t})} = x^{-\alpha}, x > 0$. *Norming constants can be chosen as* $u_n = 1$, $v_n = \log(-\log(-F^-(1-1/n)))$.

Proof. If $F \in \mathcal{D}_e(U_{5,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that

$$\lim_{n\to\infty} F^n(-e^{-u_n(-\log(-x))^{v_n}}) = \exp(-(\log(-\log(-x)))^{-\alpha}),\ -1/e \le x < 0.$$

Putting $x = -1/e$ we get $F^n(-e^{-u_n}) \to 0$ and putting $x = 0$ we get $F^n(0) \to 1$ so that $r(F) \le 0$. Defining $Y = -\log(-X)$, we have

$$G(y) = P(Y \le y) = P(-\log(-X) \le y) = F(-e^{-y}),\ y \in \mathbb{R},\ \text{ and } r(F) = -e^{-r(G)}. \tag{9}$$

With $\alpha_n = u_n,\ \beta_n = v_n,$ for $x \le 1,\quad 0 \le G^n(\alpha_n \mid x \mid^{\beta_n} \operatorname{sign}(x)) \le G^n(\alpha_n) = F^n(-e^{-u_n}) \to 0,$ and for $x > 1,$ we have

$$\begin{aligned} \lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n} \operatorname{sign}(x)) &= \lim_{n\to\infty} F^n(-e^{-\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{-u_n(-\log e^{-x})^{v_n}}), \\ &= e^{-(\log(-\log e^{-x}))^{-\alpha}} = e^{-(\log x)^{-\alpha}} = H_{1,\alpha}(x), \end{aligned}$$

so that $G \in \mathcal{D}_p(H_{1,\alpha})$. Therefore, from Theorem 6.1(i) in Appendix A.3., $r(G) = \infty$ and $\lim_{t\to\infty} \frac{1 - G(e^{tx})}{1 - G(e^t)} = x^{-\alpha}$. So, $r(F) = -e^{-r(G)} = 0$ and $\lim_{t\to\infty} \frac{1 - F(-e^{-e^{tx}})}{1 - F(-e^{-e^t})} = \lim_{t\to\infty} \frac{1 - G(e^{tx})}{1 - G(e^t)} = x^{-\alpha}, x > 0,$ proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (9), we have $r(G) = -\log(-r(F)) = \infty$ and $\lim_{t\to\infty} \frac{1 - G(e^{tx})}{1 - G(e^t)} = \lim_{t\to\infty} \frac{1 - F(-e^{-e^{tx}})}{1 - F(-e^{-e^t})} = x^{-\alpha},\ x > 0.$ Then from Theorem 6.1(i) in Appendix A.3., $G \in \mathcal{D}_p(H_{1,\alpha})$ with $\alpha_n = 1,\ \beta_n = \log G^{-}(1 - 1/n)$. Therefore, with $u_n = 1,\ v_n = \log(-\log(-F^{-}(1 - 1/n))),\ F^n(-e^{-(-\log(-x))^{v_n}}) = G^n((-\log(-x))^{v_n}) \to H_{1,\alpha}(-\log(-x)) = e^{-(\log(-\log(-x)))^{-\alpha}} = U_{5,\alpha}(x),\ -1/e \le x < 0,$ so that $F \in \mathcal{D}_e(U_{5,\alpha})$, proving the theorem. □

Theorem 3.8. *A df* $F \in \mathcal{D}_e(U_{6,\alpha})$ *iff (i)* $-1 < r(F) < 0$ *and (ii)* $\lim_{t\to\infty} \frac{1 - F(-e^{\log(-r(F))e^{-x/t}})}{1 - F(-e^{\log(-r(F))e^{-1/t}})} = x^{\alpha},\ x > 0.$ *Norming constants can be chosen as* $u_n = -\log(-r(F)),\ v_n = \log\log(-r(F)) - \log\log(-F^{-}(1 - 1/n))$.

Proof. If $F \in \mathcal{D}_e(U_{6,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0,$ so that $\lim_{n\to\infty} F^n(-e^{-u_n(-\log(-x))^{v_n}}) = \exp(-(-\log(-\log(-x)))^{\alpha}),\ -1 \le x < -1/e.$ Putting $x = -1$ we get $F^n(-1) \to 0$ so that $r(F) > -1$ and putting $x = -1/e$ we get $F^n(-e^{-u_n}) \to 1$ and hence $r(F) \le 0$. Defining $Y = -\log(-X)$ we get G as in (9) and with $\alpha_n = u_n,\ \beta_n = v_n,$ for $x \le 0,\ 0 \le G^n(\alpha_n \mid x \mid^{\beta_n} \operatorname{sign}(x)) \le G^n(0) = F^n(-1) \to 0$ and for $x > 1,\ G^n(\alpha_n x^{\beta_n}) > G^n(\alpha_n) = F^n(-e^{-u_n}) \to 1$. For $0 < x \le 1,$ we have

$$\begin{aligned} \lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n} \operatorname{sign}(x)) &= \lim_{n\to\infty} F^n(-e^{-\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{-u_n(-\log e^{-x})^{v_n}}), \\ &= e^{-(-\log(-\log e^{-x}))^{\alpha}} = e^{-(-\log x)^{\alpha}} = H_{2,\alpha}(x), \end{aligned}$$

so that $G \in \mathcal{D}_p(H_{2,\alpha})$. Therefore, from Theorem 6.1(ii) in Appendix A.3., $0 < r(G) < \infty$ and $\lim_{t\to\infty} \frac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} = x^{\alpha}$. So, $-1 < r(F) < 0$ and $\lim_{t\to\infty} \frac{1 - F(-e^{\log(-r(F))e^{-x/t}})}{1 - F(-e^{\log(-r(F))e^{-1/t}})} = \lim_{t\to\infty} \frac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} = x^{\alpha},\ x > 0,$ proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, defining G as in (9), $-1 < r(F) < 0$ which implies that $0 < r(G) < \infty$ and $\lim_{t\to\infty} \frac{1 - G(r(G)e^{-x/t})}{1 - G(r(G)e^{-1/t})} = \lim_{t\to\infty} \frac{1 - F(-e^{\log(-r(F))e^{-x/t}})}{1 - F(-e^{\log(-r(F))e^{-1/t}})} = x^{\alpha},\ x > 0.$ Then from Theorem 6.1(ii) in Appendix A.3., $G \in \mathcal{D}_p(H_{2,\alpha})$ with $\alpha_n = r(G)$ and $\beta_n = \log r(G) - \log G^{-}(1 - 1/n)$. So, with $u_n = -\log(-r(F)),\ v_n = \log\log(-r(F)) - \log\log(-F^{-}(1 - 1/n)),\ F^n(-e^{-u_n(-\log(-x))^{v_n}}) = G^n(u_n(-\log(-x))^{v_n}) \to H_{2,\alpha}(-\log(-x)) = \exp(-(-\log(-\log(-x)))^{\alpha}) = U_{6,\alpha}(x),\ -1 \le x < -1/e$ so that $F \in \mathcal{D}_e(U_{6,\alpha})$, proving the theorem. □

Theorem 3.9. *A df* $F \in \mathcal{D}_e(H_{3,1})$ *iff (i)* $-1 < r(F) \leq 0$ *and (ii)* $\lim_{t \uparrow -\log(-r(F))} \dfrac{1-F(-e^{-te^{yf(t)}})}{1-F(-e^{-t})}$ $= e^{-y}$ *for some positive valued function* f. *If (ii) holds for some* f *then* $\int_t^{-\log(-r(F))} \frac{1-F(-e^{-x})}{x} dx < \infty$ *for* $0 < t < -\log(-r(F))$ *and (ii) holds with the choice* $f(t) = \dfrac{1}{1-F(-e^{-t})} \int_t^{-\log(-r(F))} \frac{1-F(-e^{-y})}{y} dy.$ *The norming constants here may be chosen as* $u_n = -\log(-F^{-}(1-1/n))$, $v_n = f(u_n)$.

Proof. If $F \in \mathcal{D}_e(H_{3,1})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(-e^{-u_n(-\log(-x))^{v_n}}) = \exp(-(-\log(-x))^{-1})$, $-1 < x \leq 0$. Putting $x = -1$, we get $F^n(-1) \to 0$ and putting $x = 0$, we get $F^n(0) \to 1$ so that $r(F) \leq 0$. Defining $Y = -\log(-X)$ we have G as in (9) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \leq 0$, $0 \leq G^n(\alpha_n \mid x \mid^{\beta_n} \operatorname{sign}(x)) \leq G^n(0) = F^n(-1) \to 0$, and for $x > 0$, we have

$$\lim_{n\to\infty} G^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) \;=\; \lim_{n\to\infty} F^n(-e^{-\alpha_n x^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{-u_n(-\log e^{-x})^{v_n}}) = e^{-1/x} = \Phi(x),$$

so that $G \in \mathcal{D}_p(\Phi)$. Therefore, from Theorem 6.1(v), $r(G) > 0$ and $\lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = e^{-x}$ for some positive valued function g, and if this holds for some g then $\dfrac{1}{1-G(t)} \int_t^{r(G)} \dfrac{1-G(x)}{x} dx < \infty$ and it holds with $g(t) = \dfrac{1}{1-G(t)} \int_t^{r(G)} \dfrac{1-G(x)}{x} dx$. Re-writing in terms of F we get $-1 < r(F) \leq 0$ and $\lim_{t\uparrow -\log(-r(F))} \dfrac{1-F(-e^{-te^{xf(t)}})}{1-F(-e^{-t})} = \lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = e^{-x}$, which holds for some positive valued function and if this holds for some positive valued function then $\dfrac{1}{1-F(-e^{-t})} \int_t^{-\log(-r(F))} \dfrac{1-F(-e^{-x})}{x} dx < \infty$, and it holds with $f(t) = \dfrac{1}{1-F(-e^{-t})} \int_t^{-\log(-r(F))} \dfrac{1-F(-e^{-x})}{x} dx$, proving (i) and (ii).

Conversely, if (i) and (ii) hold, then defining G as in (9), $-1 < r(F) \leq 0$ which implies that $r(G) > 0$ and $\lim_{t\uparrow r(G)} \dfrac{1-G(te^{xg(t)})}{1-G(t)} = \lim_{t\uparrow -\log(-r(F))} \dfrac{1-F(-e^{-te^{xf(t)}})}{1-F(-e^{-t})} = e^{-x}$, where g and f are as above. Hence from Theorem 6.1(v) in Appendix A.3., $G \in \mathcal{D}_p(\Phi)$. So, with $u_n = -\log(-F^{-}(1-1/n))$, $v_n = f(u_n)$, $F^n(-e^{-u_n(-\log(-x))^{v_n}}) = G^n(u_n(-\log(-x))^{v_n}) \to \Phi(-\log(-x)) = \exp(-(-\log(-x))^{-1}) = H_{3,1}(x)$, $-1 \leq x < 0$, so that $F \in \mathcal{D}_e(H_{3,1})$, proving the theorem. □

Theorem 3.10. *A df* $F \in \mathcal{D}_e(U_{7,\alpha})$ *iff (i)* $r(F) = -1$ *and (ii)* $\lim_{t\to\infty} \dfrac{1-F(-e^{e^{-tx}})}{1-F(-e^{e^{-t}})} = x^{-\alpha}$, $x > 0$. *Norming constants can be chosen as* $u_n = 1$, $v_n = -\log\log(-F^{-}(1-1/n))$.

Proof. If $F \in \mathcal{D}_e(U_{7,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(-e^{u_n(\log(-x))^{v_n}}) = \exp(-(-\log\log(-x))^{-\alpha})$, $-e \leq x < -1$. Putting $x = -e$ we get $F^n(-e^{u_n}) \to 0$ and putting $x = -1$ we get $F^n(-1) \to 1$ so that $r(F) \leq -1$. Defining $Y = -\log(-X)$ we have G as in (9) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \leq 1$, $0 \leq G^n(-\alpha_n(-x)^{\beta_n}) \leq G^n(-\alpha_n) = F^n(-e^{u_n}) \to 0$ and for $x > 0$, $G^n(\alpha_n x^{\beta_n}) > G^n(0) = F^n(-1) \to 1$. For $-1 \leq x < 0$, we have

$$\begin{aligned}\lim_{n\to\infty} G^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) &= \lim_{n\to\infty} F^n(-e^{\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{u_n(\log e^{-x})^{v_n}}),\\ &= e^{-(-\log\log e^{-x}))^{-\alpha}} = e^{-(-\log(-x))^{-\alpha}} = H_{3,\alpha}(x),\end{aligned}$$

so that $G \in \mathcal{D}_p(H_{3,\alpha})$. Therefore, from Theorem 6.1(iii) in Appendix A.3., $r(G) = 0$ and $\lim_{t\to\infty} \frac{1 - G(-e^{-tx})}{1 - G(-e^{-t})} = x^{-\alpha}$. So, $r(F) = -e^{-r(G)} = -1$ and $\lim_{t\to\infty} \frac{1 - F(-e^{e^{-tx}})}{1 - F(-e^{e^{-t}})} = \lim_{t\to\infty} \frac{1 - G(-e^{-tx})}{1 - G(-e^{-t})} = x^{-\alpha}, x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (9) we have $r(G) = -\log(-r(F)) = 0$ and $\lim_{t\to\infty} \frac{1 - G(-e^{-tx})}{1 - G(-e^{-t})} = \lim_{t\to\infty} \frac{1 - F(-e^{e^{-tx}})}{1 - F(-e^{e^{-t}})} = x^{-\alpha}$, $x > 0$. Then from Theorem 6.1(iii) in Appendix A.3., $G \in \mathcal{D}_p(H_{3,\alpha})$ with $\alpha_n = 1$ and $\beta_n = -\log(-G^-(1 - 1/n))$. So, with $u_n = 1$ and $v_n = \beta_n = -\log\log(-F^-(1 - 1/n))$, $F^n(-e^{(\log(-x))^{v_n}}) = G^n((-\log(-x))^{v_n}) \to H_{3,\alpha}(-\log(-x)) = \exp(-(-\log\log(-x))^{-\alpha}) = U_{7,\alpha}(x)$, $-e \le x < -1$, proving the theorem. □

Theorem 3.11. *A df* $F \in \mathcal{D}_e(U_{8,\alpha})$ *iff (i)* $r(F) < -1$ *and (ii)* $\lim_{t\to\infty} \frac{1 - F(e^{-\log(-r(F))e^{x/t}})}{1 - F(-e^{\log(-r(F))e^{1/t}})} = x^{\alpha}$, $x > 0$. *Norming constants can be chosen as* $u_n = \log(-r(F))$, $v_n = \log\log(-F^-(1 - 1/n)) - \log\log(-r(F))$.

Proof. If $F \in \mathcal{D}_e(U_{8,\alpha})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(-e^{u_n(\log(-x))^{v_n}}) = \exp(-(\log\log(-x))^{\alpha})$, $x \le -e$. Putting $x = -e$, we get $F^n(-e^{u_n}) \to 1$ and for $x \ge 0$, by (5), $F^n(e^{u_n(\log x)^{v_n}}) \to 1$, so that $r(F) \le 0$. Defining $Y = -\log(-X)$ we have G as in (9) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $-1 < x < 0$, $1 \ge G^n(-\alpha_n(-x)^{\beta_n}) \ge G^n(-\alpha_n) = F^n(-e^{u_n}) \to 1$, and for $x \le -1$, we have

$$\begin{aligned}
\lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n}\ \mathrm{sign}(x)) &= \lim_{n\to\infty} F^n(-e^{\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{u_n\,(\log e^{-x})^{v_n}}), \\
&= \exp(-(\log\log e^{-x}))^{\alpha} = H_{4,\alpha}(x),
\end{aligned}$$

so that $G \in \mathcal{D}_p(H_{4,\alpha})$. Therefore, from Theorem 6.1(iv) in Appendix A.3., $r(G) < 0$ and $\lim_{t\to\infty} \frac{1 - G(r(G)e^{x/t})}{1 - G(r(G)e^{1/t})} = x^{\alpha}$. Thus $r(F) < -1$ and $\lim_{t\to\infty} \frac{1 - F(e^{-\log(-r(F))e^{x/t}})}{1 - F(-e^{\log(-r(F))e^{1/t}})} = \lim_{t\to\infty} \frac{1 - G(r(G)e^{x/t})}{1 - G(r(G)e^{1/t})} = x^{\alpha}$, $x > 0$, proving (i) and (ii).

Conversely, if (i) and (ii) hold for some $\alpha > 0$, then defining G as in (9), $r(F) < -1$ which implies that $r(G) < 0$ and $\lim_{t\to\infty} \frac{1 - G(r(G)e^{x/t})}{1 - G(r(G)e^{1/t})} = \lim_{t\to\infty} \frac{1 - F(-e^{\log(-r(F))e^{x/t}})}{1 - F(-e^{\log(-r(F))e^{1/t}})} = x^{\alpha}$, $x > 0$. Then from Theorem 6.1(iv) in Appendix A.3., $G \in \mathcal{D}_p(H_{4,\alpha})$ with $\alpha_n = -r(G)$ and $\beta_n = \log G^-(1 - 1/n) - \log r(G)$. So, with $u_n = \log(-r(F))$, $v_n = \log\log(-F^-(1 - 1/n)) - \log\log(-r(F))$, $F^n(-e^{u_n(\log(-x))^{v_n}}) = G^n(u_n(-\log(-x))^{v_n}) \to H_{4,\alpha}(-\log(-x)) = \exp(-(\log\log(-x)))^{\alpha}) = U_{8,\alpha}(x)$, $x \le -e$, proving the theorem. □

Theorem 3.12. *A df* $F \in \mathcal{D}_e(H_{4,1})$ *iff (i)* $r(F) \le -1$ *and (ii)* $\lim_{t\uparrow -\log(-r(F))} \frac{1 - F(-e^{-te^{yf(t)}})}{1 - F(-e^{-t})} = e^{y}$, *for some positive valued function* f. *If (ii) holds for some* f *then* $\int_t^{-\log(-r(F))} \frac{1-F(-e^{-x})}{x} dx < \infty$ *for* $0 < t < -\log(-r(F))$ *and (ii) holds with* $f(t) = -\frac{1}{1 - F(-e^{-t})} \int_t^{-\log(-r(F))} \frac{1-F(-e^{-y})}{y} dy$. *Norming constants may be chosen as* $u_n = -\log(-F^-(1 - 1/n))$, $v_n = f(-u_n)$.

Proof. If $F \in \mathcal{D}_e(H_{4,1})$ then (5) holds for some norming constants $u_n > 0, v_n > 0$, so that $\lim_{n\to\infty} F^n(-e^{u_n(\log(-x))^{v_n}}) = -1/x$, $x \le -1$. Putting $x = -1$, we get $F^n(-1) \to 1$ so that $r(F) \le -1$. Defining $Y = -\log(-X)$ we have G as in (9) and with $\alpha_n = u_n$, $\beta_n = v_n$, for $x \ge 0$, $1 \ge G^n(\alpha_n x^{\beta_n}) \ge G^n(0) = F^n(-1) \to 1$. For $x < 0$, we have $\lim_{n\to\infty} G^n(\alpha_n|x|^{\beta_n}\ \mathrm{sign}(x)) = \lim_{n\to\infty} F^n(-e^{\alpha_n(-x)^{\beta_n}}) = \lim_{n\to\infty} F^n(-e^{u_n\,(\log e^{-x})^{v_n}}) = e^{x} = \Psi(x)$, so that $G \in \mathcal{D}_p(\Psi)$. Therefore, from Theorem 6.1(vi) in Appendix A.3., $r(G) \leq 0$ and $\lim_{t \uparrow r(G)} \dfrac{1 - G(te^{xg(t)})}{1 - G(t)} = e^x$ for some positive valued function g, and if this holds for some function g then $-\dfrac{1}{1 - G(t)} \int_t^{r(G)} \frac{1-G(x)}{x} dx < \infty$, and it holds with the choice $g(t) = -\dfrac{1}{1 - G(t)} \int_t^{r(G)} \frac{1-G(x)}{x} dx$. Re-writing in terms of F we get $r(F) \leq -1$ and $\lim_{t \uparrow (-\log(-r(F)))} \dfrac{1 - F(-e^{-te^{xf(t)}})}{1 - F(-e^{-t})} = \lim_{t \uparrow r(G)} \dfrac{1 - G(te^{xg(t)})}{1 - G(t)} = e^x$, for some positive valued function f and if this holds for some f then $-\dfrac{1}{1 - F(-e^{-t})} \int_t^{-\log(-r(F))} \frac{1-F(-e^{-x})}{x} dx < \infty$, and it holds with the choice $f(t) = -\dfrac{1}{1 - F(-e^{-t})} \int_t^{-\log(-r(F))} \frac{1-F(-e^{-x})}{x} dx$, proving (i) and (ii).

Conversely, if (i) and (ii) hold, then defining G as in (9), $r(F) \leq -1$ which implies that $r(G) \leq 0$ and $\lim_{t \uparrow r(G)} \dfrac{1 - G(te^{xg(t)})}{1 - G(t)} = \lim_{t \uparrow (-\log(-r(F)))} \dfrac{1 - F(-e^{-te^{xf(t)}})}{1 - F(-e^{-t})} = e^x$, with g and f as above. Then from Theorem 6.1(vi) in Appendix A.3. $G \in \mathcal{D}_p(\Psi)$. So, with $u_n = -\log(-F^{-}(1 - 1/n))$, $v_n = f(-u_n)$, $F^n(-e^{u_n(\log(-x))^{v_n}}) = G^n(u_n(-\log(-x))^{v_n}) \to \Phi(-\log(-x)) = -1/x = H_{4,1}(x)$, $x \leq -1$, proving the theorem. $\square$

4. Comparison between p-max and e-max domains

The result below compares $\mathcal{D}_p(.)$ and $\mathcal{D}_e(.)$ and shows that every df in a $\mathcal{D}_p(.)$ necessarily belongs to some $\mathcal{D}_e(.)$ and that the converse is not always true. So, e-max stable laws attract more dfs than the p-max stable laws. Proofs of results in the previous section and Theorem 6.2 in Appendix A.4. are extensively used in the proof below and the proof is different from the one given in Mohan and Ravi (1993) for Theorem 6.2 in Appendix A.4. Two dfs F and G are said to be tail equivalent if $r(F) = r(G)$ and $\lim_{t \to r(F)} \dfrac{1 - F(t)}{1 - G(t)} = A$, for some constant $A > 0$, and in such a case, if one belongs to a max domain of some df, then the other df also belongs to the same max domain with the same kind of normalization, see for example, Resnick (1987). We use this result in the proofs below.

Theorem 4.1. *Let F be a df.*

$$
\begin{array}{cll}
(a) & \left.\begin{array}{l} (i)\ F \in \mathcal{D}_p(H_{1,\alpha}) \\ (ii)\ F \in \mathcal{D}_p(\Phi),\ r(F) = \infty \end{array}\right\} & \Longrightarrow F \in \mathcal{D}_e(H_{1,1}),\ r(F) = \infty. \\
(b) & F \in \mathcal{D}_p(\Phi),\ 1 < r(F) < \infty & \Longleftrightarrow F \in \mathcal{D}_e(H_{1,1}),\ 1 < r(F) < \infty. \\
(c) & \left.\begin{array}{l} (i)\ F \in \mathcal{D}_p(H_{2,\alpha}),\ r(F) = 1 \\ (ii)\ F \in \mathcal{D}_p(\Phi),\ r(F) = 1 \end{array}\right\} & \Longrightarrow F \in \mathcal{D}_e(H_{2,1}),\ r(F) = 1. \\
(d) & F \in \mathcal{D}_p(\Phi),\ 0 < r(F) < 1 & \Longleftrightarrow F \in \mathcal{D}_e(H_{2,1}),\ 0 < r(F) < 1. \\
(e) & \left.\begin{array}{l} (i)\ F \in \mathcal{D}_p(H_{3,\alpha}) \\ (ii)\ F \in \mathcal{D}_p(\Psi),\ r(F) = 0 \end{array}\right\} & \Longrightarrow F \in \mathcal{D}_e(H_{3,1}),\ r(F) = 0. \\
(f) & F \in \mathcal{D}_p(\Psi),\ -1 < r(F) < 0 & \Longleftrightarrow F \in \mathcal{D}_e(H_{3,1}),\ -1 < r(F) < 0. \\
(g) & \left.\begin{array}{l} (i)\ F \in \mathcal{D}_p(H_{4,\alpha}),\ r(F) = -1 \\ (ii)\ F \in \mathcal{D}_p(\Psi),\ r(F) = -1 \end{array}\right\} & \Longrightarrow F \in \mathcal{D}_e(H_{4,1}),\ r(F) = -1. \\
(h) & F \in \mathcal{D}_p(\Psi),\ r(F) < -1 & \Longleftrightarrow F \in \mathcal{D}_e(H_{4,1}),\ r(F) < -1. \\
(i) & F \in \mathcal{D}_p(H_{2,\alpha}),\ 1 < r(F) < \infty & \Longleftrightarrow F \in \mathcal{D}_e(U_{2,\alpha}),\ 1 < r(F) < \infty. \\
(j) & F \in \mathcal{D}_p(H_{2,\alpha}),\ r(F) < 1 & \Longleftrightarrow F \in \mathcal{D}_e(U_{4,\alpha}),\ 0 < r(F) < 1. \\
(k) & F \in \mathcal{D}_p(H_{4,\alpha}),\ -1 < r(F) < 0 & \Longleftrightarrow F \in \mathcal{D}_e(U_{6,\alpha}),\ -1 < r(F) < 0. \\
(l) & F \in \mathcal{D}_p(H_{4,\alpha}),\ r(F) < -1 & \Longleftrightarrow F \in \mathcal{D}_e(U_{8,\alpha}),\ r(F) < -1.
\end{array}
$$

Remark 4.2. *Note that $\mathcal{D}_e(U_{1,\alpha}) \cup \mathcal{D}_e(U_{3,\alpha}) \cup \mathcal{D}_e(U_{5,\alpha}) \cup \mathcal{D}_e(U_{7,\alpha})$ contain dfs which do not belong to any of the $\mathcal{D}_p(\cdot)'s$.*

Proof of (a)(i): If $X \sim F \in \mathcal{D}_p(H_{1,\alpha})$ then by Theorem 6.1(i) in Appendix A.3., $r(F) = \infty$ and $F^n(x^{\beta_n}) \to e^{-(\log x)^{-\alpha}}$, $x > 1$, for some $\beta_n > 0$. Defining $Y = \log(a \vee X)$ for some a, $0 < a < 1$, we have G as in (8) with $r(G) = \log r(F) = \infty$ and $G^n(a_n y + b_n) = F^n(e^{a_n y + b_n}) = F^n(e^{b_n}(e^y)^{a_n}) \to H_{1,\alpha}(e^y) = \Phi_\alpha(y)$, $y > 0$, so that $G \in \mathcal{D}_l(\Phi_\alpha)$. Hence, from Theorem 6.2(i) in Appendix A.4., $G \in \mathcal{D}_p(\Phi)$. Therefore, as in the proof of converse part of Theorem 3.3, the df of $e^Y = a \vee X$ belongs to $\mathcal{D}_e(H_{1,1})$, and since $\lim_{t\to\infty} \dfrac{P(a \vee X > t)}{P(X > t)} = 1$, trivially $a \vee X$ and X are tail equivalent and we get $F \in \mathcal{D}_e(H_{1,1})$ with norming constants $u_n = \log F^{-}(1 - 1/n)$, $v_n = h(u_n)$, where $h(t) = \dfrac{1}{1 - F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^s)}{s} ds$, completing the proof.

Proof of (a)(ii): If $X \sim F \in \mathcal{D}_p(\Phi)$ with $r(F) = \infty$ then $F^n(\alpha_n x^{\beta_n}) \to e^{-1/x}$, $x > 0$ for some constants $\alpha_n > 0, \beta_n > 0$. Defining G as in (8), $\lim_{n\to\infty} G^n(\beta_n y + \log \alpha_n) = \lim_{n\to\infty} F^n(\alpha_n (e^y)^{\beta_n}) = \Phi(e^y) = \Lambda(y), y \in \mathbb{R}$, and hence $G \in \mathcal{D}_l(\Lambda)$ with $r(G) = \log(r(F)) = \infty$. So, by Theorem 6.2(ii) in Appendix A.4., $G \in \mathcal{D}_p(\Phi)$ and as in the proof of converse part of Theorem 3.3, the df of $e^Y = (a \vee X)$ belongs to $\mathcal{D}_e(H_{1,1})$ which implies that $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \log F^{-}(1 - 1/n)$, $v_n = h(u_n)$, where $h(t) = \dfrac{1}{1 - F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^s)}{s} ds$, proving (a)(ii).

Proof of (b): The proof of the necessity part of (b) is similar to that of (a)(ii) above and is omitted.

If $X \sim F \in \mathcal{D}_e(H_{1,1})$ with $1 < r(F) < \infty$ then $F^n(e^{u_n(\log x)^{v_n}}) \to e^{-(\log x)^{-1}}$, $x > 1$, for some constants $u_n > 0, v_n > 0$. Hence for some a, $0 < a < 1$, $Y = \log(a \vee X) \sim G$ with G as in (8) and $G^n(u_n y^{v_n}) = F^n(e^{u_n y^{v_n}}) = F^n(e^{u_n(\log e^y)^{v_n}}) \to e^{-1/y} = \Phi(y), y > 0$. So $G \in \mathcal{D}_p(\Phi), 0 < r(G) < \infty$. Therefore, from Theorem 6.2(iii) in Appendix A.4., $G \in \mathcal{D}_l(\Lambda), 0 < r(G) < \infty$. Hence from Proposition 1.1, the df of $e^Y = a \vee X$ belongs to $\mathcal{D}_p(\Phi), 1 < r(F) < \infty$, and since $a \vee X$ and X are tail equivalent, the df of X belongs to $\mathcal{D}_p(\Phi)$, proving (b).

Proof of (c)(i): If $X \sim F \in \mathcal{D}_p(H_{2,\alpha})$ with $r(F) = 1$, then $F^n(x^{\beta_n}) \to e^{-(-\log x)^{\alpha}}$, $0 < x < 1$, for some constants $\beta_n > 0$. Defining G as in (8), we have $G^n(\beta_n y) = F^n((e^y)^{\beta_n}) \to H_{2,\alpha}(e^y) = \Psi_\alpha(y), y < 0$, so that $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) = \log(r(F)) = 0$. Hence from Theorem 6.2(vi) in Appendix A.4., $G \in \mathcal{D}_p(\Psi)$. So, as in the converse part of the proof of Theorem 3.6, the df of $e^Y = (a \vee X)$ belongs to $\mathcal{D}_e(H_{2,1})$ and since $a \vee X$ and X are tail equivalent, we have $F \in \mathcal{D}_e(H_{2,1})$ with $u_n = -\log F^{-}(1 - 1/n)$, $v_n = h(-u_n)$, where $h(t) = -\dfrac{1}{1 - F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^s)}{s} ds$, proving (c)(i).

Proof of (c)(ii): As in the proof of (a) (ii) above, if $X \sim F \in \mathcal{D}_p(\Phi)$ with $r(F) = 1$, then $F^n(\alpha_n x^{\beta_n}) \to e^{-1/x}$, $x > 0$, for some constants $\alpha_n > 0, \beta_n > 0$. Defining G as in (8), we have $G^n(\beta_n y + \log \alpha_n) = F^n(\alpha_n (e^y)^{\beta_n}) \to \Phi(e^y) = \Lambda(y), y \in \mathbb{R}$, and hence $G \in \mathcal{D}_l(\Lambda)$ with $r(G) = \log(r(F)) = 0$. So, from Theorem 6.2(v) in Appendix A.4., $G \in \mathcal{D}_p(\Psi)$ so that the df of $e^Y = (a \vee X)$ belongs to $\mathcal{D}_e(H_{2,1})$ as in the proof of converse part of Theorem 3.6, which implies that $F \in \mathcal{D}_e(H_{2,1})$ with $u_n = -\log F^{-}(1 - 1/n)$, $v_n = h(-u_n)$, where $h(t) = -\dfrac{1}{1 - F(e^t)} \int_t^{\log r(F)} \frac{1-F(e^s)}{s} ds$, proving (c)(ii).

Proof of (d): The proof of the necessity part of (d) is similar to that of (c)(ii) above and is omitted.

Now, if $X \sim F \in \mathcal{D}_e(H_{2,1})$, $0 < r(F) < 1$, then $F^n(e^{-u_n(-\log x)^{v_n}}) \to x$, $0 \le x < 1$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (8) for some $a, 0 < a < r(F)$, we have $G^n(-u_n(-y)^{v_n}) = F^n(e^{-u_n(-y)^{v_n}}) = F^n(e^{-u_n(-\log e^y)^{v_n}}) \to e^{-(-\log e^y)} = e^y = \Psi(y)$, $y < 0$, so that $G \in \mathcal{D}_p(\Psi)$, with $r(G) = \log(r(F)) < 0$. Hence from Theorem 6.2(iv) in Appendix A.4., $G \in \mathcal{D}_l(\Lambda)$. From Proposition 1.1, this implies that the df of $e^Y = a \vee X$ belongs to $\mathcal{D}_p(\Phi)$, $0 < r(F) < 1$, proving (d).

Proof of (e)(i): If $X \sim F \in \mathcal{D}_p(H_{3,\alpha})$ then $r(F) = 0$ and $F^n(-\alpha_n(-x)^{\beta_n}) \to e^{-(-\log(-x))^{-\alpha}}$, $-1 < x < 0$, for some norming constants $\alpha_n > 0, \beta_n > 0$. Defining G as in (9), we have $G^n(\beta_n y - \log \alpha_n) = F^n(-e^{-\beta_n y} e^{\log \alpha_n}) = F^n(-\alpha_n(e^{-y})^{\beta_n}) \to e^{-(-\log(-(-e^{-y})))^{\alpha}}, -1 < -e^{-y} < 0, = \Phi_\alpha(y)$, $y > 0$. So $G \in \mathcal{D}_l(\Phi_\alpha)$ with $r(G) = -\log(-r(F)) = \infty$. Hence from Theorem 6.2(i) in Appendix A.4., $G \in \mathcal{D}_p(\Phi)$

and as in the proof of converse part of Theorem 3.9, the df of $X = -e^{-Y}$ belongs to $\mathcal{D}_e(H_{3,1})$, proving (e)(i).

Proof of (e)(ii): If $X \sim F \in \mathcal{D}_p(\Psi)$ with $r(F) = 0$ then $F^n(-\alpha_n(-x)^{\beta_n}) \to e^x$, $x < 0$, for some norming constants $\alpha_n > 0, \beta_n > 0$. Defining G as in (9), $G^n(\beta_n y - \log\alpha_n) = F^n(-e^{-\beta_n y}e^{\log\alpha_n})$ $= F^n(-\alpha_n(e^{-y})^{\beta_n}) \to e^{-e^{-y}} = \Lambda(y)$, $y \in \mathbb{R}$. So, $G \in \mathcal{D}_l(\Lambda)$ with $r(G) = -\log(-r(F)) = \infty$. Hence from Theorem 6.2(ii) in Appendix A.4., $F \in \mathcal{D}_p(\Phi)$ so that, as in the proof of converse part of Theorem 3.9, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(H_{3,1})$, proving (e)(ii).

Proof of (f): If $X \sim F \in \mathcal{D}_p(\Psi)$ with $-1 < r(F) < 0$ then proceeding as in the proof of e(ii), we have $G \in \mathcal{D}_l(\Lambda)$ with $0 < r(G) < \infty$ and hence from Theorem 6.2(iii) in Appendix A.4., $G \in \mathcal{D}_p(\Phi)$ so that the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(H_{3,1})$.

If $X \sim F \in \mathcal{D}_e(H_{3,1})$ with $-1 < r(F) < 0$ then $F^n(-e^{-u_n(-\log(-x))^{v_n}}) \to e^{-(-\log(-x))} = H_{3,1}(x)$, $-1 \le x \le 0$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (9), $G^n(u_n y^{v_n}) = F^n(-e^{-u_n(y)^{v_n}}) = F^n(-e^{-u_n(-\log|-e^{-y}|)^{v_n}}) \to e^{-(-\log e^{-y})^{-1}} = e^{-1/y} = \Phi(y)$, $y > 0$, so that $G \in \mathcal{D}_p(\Phi)$ with $r(G) = -\log(-r(F))$, $0 < r(G) < \infty$. Hence from Theorem 6.2(iii) in Appendix A.4., $G \in \mathcal{D}_l(\Lambda)$, $0 < r(G) < \infty$, and so from Proposition 1.1, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_p(\Psi)$, $-1 < r(F) < 0$, proving (f).

Proof of (g)(i): If $X \sim F \in \mathcal{D}_p(H_{4,\alpha})$ with $r(F) = -1$ then $F^n(-\alpha_n(-x)^{\beta_n}) \to e^{-(\log(-x))^\alpha}$, $x < -1$, for some norming constants $\alpha_n > 0, \beta_n > 0$. Defining G as in (9), we have $G^n(\beta_n y - \log\alpha_n) = F^n(-e^{-\beta_n y}e^{\log\alpha_n}) = F^n(-\alpha_n(e^{-y})^{\beta_\alpha}) \to e^{-(\log(e^{-y}))^\alpha}$, $-e^{-y} < -1, = \Psi_\alpha(y)$, $y < 0$. So, $G \in \mathcal{D}_l(\Psi_\alpha)$, $r(G) = -\log(-r(F)) = 0$. Hence from Theorem 6.2(vi) in Appendix A.4., $G \in \mathcal{D}_p(\Psi)$ so that as in the proof of converse part of Theorem 3.12, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(H_{4,1})$, proving (g)(i).

Proof of (g)(ii) and direct part of (h): If $X \sim F \in \mathcal{D}_p(\Psi)$, $r(F) \le -1$, then proceeding as in the proof of (e)(ii), $G \in \mathcal{D}_l(\Lambda)$, $r(G) \le 0$ and hence from Theorem 6.2(iv) and (v) in Appendix A.4., $G \in \mathcal{D}_p(\Psi)$. So, from the proof of converse part of Theorem 3.12, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(H_{4,1})$, proving (g)(ii) and the direct part of (h).

Proof of converse part of (h): If $X \sim F \in \mathcal{D}_e(H_{4,1})$ with $r(F) < -1$ then $F^n(-e^{u_n(\log|x|)^{v_n}}) \to e^{-\log(-x)} = H_{4,1}(x)$, $x < -1$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (9), $G^n(-u_n(-y)^{v_n}) = F^n(-e^{u_n(-y)^{v_n}}) = F^n(-e^{u_n(\log|-e^{-y}|)^{v_n}}) \to e^{-\log e^{-y}} = e^y = \Psi(y)$, $y < 0$, so that $G \in \mathcal{D}_p(\Psi)$, $r(G) = -\log(-r(F)) < 0$. Hence, from Theorem 6.2(iv) in Appendix A.4., $G \in \mathcal{D}_l(\Lambda)$, $r(G) < 0$ and so by Proposition 1.1, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_p(\Psi)$, $r(F) < -1$, proving the converse part of (h).

Proof of (i): If $X \sim F \in \mathcal{D}_p(H_{2,\alpha})$, $r(F) > 1$, then proceeding as in the proof of (c)(i), $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) > 0$ and hence from Theorem 6.2(vii) in Appendix A.4., $G \in \mathcal{D}_p(H_{2,\alpha})$. So, as in the proof of converse part of Theorem 3.2, the df of $e^Y = (a \vee X)$ and hence that of X, belongs to $\mathcal{D}_e(U_{2,\alpha})$.

Conversely, if $X \sim F \in \mathcal{D}_e(U_{2,\alpha})$ with $1 < r(F) < \infty$ then $F^n(e^{u_n(\log x)^{v_n}}) \to e^{-(-\log\log x)^\alpha}$, $1 \le x < e$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (8) for some $a, 0 < a < 1$, $G^n(u_n x^{v_n}) = F^n(e^{u_n x^{v_n}}) = F^n(e^{u_n(\log e^x)^{v_n}}) \to H_{2,\alpha}(x)$, $0 \le x < 1$. Hence $G \in \mathcal{D}_p(H_{2,\alpha})$. From Theorem 6.2(vii) in Appendix A.4., $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) = \log r(F) > 0$ which implies that the df of $e^Y = a \vee X$ belongs to $\mathcal{D}_p(H_{2,\alpha})$ by Proposition 1.1. Since $a \vee X$ and X are tail equivalent, we have $F \in \mathcal{D}_p(H_{2,\alpha})$, $r(F) > 1$, proving (i).

Proof of (j): If $X \sim F \in \mathcal{D}_p(H_{2,\alpha})$ with $r(F) < 1$, then proceeding as in the proof of (c)(i), $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) < 0$ and hence from Theorem 6.2(viii) in Appendix A.4., $G \in \mathcal{D}_p(H_{4,\alpha})$. So, as in the proof of converse part of Theorem 3.5, the df of $e^Y = (a \vee X)$ and hence that of X, belongs to $\mathcal{D}_e(U_{4,\alpha})$.

Conversely, if $X \sim F \in \mathcal{D}_e(U_{4,\alpha})$ with $0 < r(F) < 1$ then $F^n(e^{u_n|\log x|^{v_n} sign(\log x)}) \to e^{-(\log|\log x|)^\alpha}$, $0 < x < 1/e$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (8) for some $a, 0 < a < r(F)$, $G^n(-u_n(-x)^{v_n}) = F^n(e^{-u_n(-x)^{v_n}}) = F^n(e^{-u_n(-\log e^x)^{v_n}}) \to H_{4,\alpha}(x)$, $x < -1$, so that $G \in \mathcal{D}_p(H_{4,\alpha})$. Hence, from Theorem 6.2(viii) in Appendix A.4., $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) = \log r(F) < 0$ so that the df of $e^Y = a \vee X$ and hence that of X belongs to $\mathcal{D}_p(H_{2,\alpha})$ with $r(F) < 1$, proving (j).

Proof of (k): If $X \sim F \in \mathcal{D}_p(H_{4,\alpha})$ with $-1 < r(F) < 0$, then proceeding as in the proof of g(i), we have $G \in \mathcal{D}_l(\Psi_\alpha)$ with $0 < r(G) < \infty$. Hence, from Theorem 6.2 (vii) in Appendix A.4., $G \in \mathcal{D}_p(H_{2,\alpha})$ so that, as in the proof of converse part of Theorem 3.8, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(U_{6,\alpha})$.

If $X \sim F \in \mathcal{D}_e(U_{6,\alpha})$ with $-1 < r(F) < 0$, then $F^n(-e^{u_n|\log|x||^{v_n} sign(\log|x|)}) \to e^{-(-\log(-\log(-x)))^\alpha}$, $-1 \le x < -1/e$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (9), $G^n(u_n x^{v_n}) = F^n(-e^{-u_n x^{v_n}}) = F^n(-e^{-u_n(-\log(-(-e^{-x})))^{v_n}}) \to e^{-(-\log x)^\alpha} = H_{2,\alpha}(x)$, $0 < x < 1$, so that $G \in \mathcal{D}_p(H_{2,\alpha})$, $r(G) = -\log(-r(F))$, $0 < r(G) < \infty$. Hence, from Theorem 6.2(vii) in Appendix A.4., $G \in \mathcal{D}_l(\Psi_\alpha)$, $r(G) > 0$, and from Proposition 1.1, the df of $X = -e^{-Y}$ belongs to $\mathcal{D}_p(H_{4,\alpha})$, $-1 < r(F) < 0$, proving (k).

Proof of (l): If $X \sim F \in \mathcal{D}_p(H_{4,\alpha})$ with $r(F) < -1$ then proceeding as in the proof of g(i), we have $G \in \mathcal{D}_l(\Psi_\alpha)$ with $r(G) < 0$ and hence from Theorem 6.2(viii) in Appendix A.4., $G \in \mathcal{D}_p(H_{4,\alpha})$. So, as in the proof of Theorem 3.11, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_e(U_{8,\alpha})$.

If $X \sim F \in \mathcal{D}_e(U_{8,\alpha})$ with $r(F) < -1$ then $F^n(-e^{u_n(\log(-x))^{v_n}}) \to e^{-(\log(\log(-x)))^\alpha}$, $x < -e$, for some norming constants $u_n > 0, v_n > 0$. Defining G as in (9), $G^n(-u_n(-y)^{v_n}) = F^n(-e^{u_n(-y)^{v_n}}) = F^n(-e^{u_n(\log(-(-e^{-y})))^{v_n}}) \to e^{-(\log(-y))^{v_n}} = H_{4,\alpha}(y), y < -1$, and $G \in \mathcal{D}_p(H_{4,\alpha})$ with $r(G) = -\log(-r(F)) < 0$. From the Theorem 6.2(viii) in Appendix A.4., $G \in \mathcal{D}_l(\Psi_\alpha)$, $r(G) < 0$, and hence from Proposition 1.1, the df of $-e^{-Y} = X$ belongs to $\mathcal{D}_p(H_{4,\alpha})$, $r(F) < -1$, proving (l) and the theorem. □

5. Examples

The following examples give dfs belonging to the max domains of attraction of the e-max stable laws and these can be verified directly by using (5). Note that $\mathcal{D}_l(.) \subset \mathcal{D}_p(.) \subset \mathcal{D}_e(.)$. In the discussion below, $\alpha = 1$.

(i) $F_1(x) = 1 - (\log\log x)^{-1}$, $x \ge e^e$, belongs to $\mathcal{D}_e(U_{1,\alpha})$ with $u_n = 1$ and $v_n = n$. But $F \notin \mathcal{D}_p(\cdot)$, see, Mohan and Ravi (1993).

(ii) $F_2(x) = 1 - (-\log(-\log x))^{-1}$, $e^{-1/e} < x < 1$, belongs to $\mathcal{D}_e(U_{3,\alpha})$ with $u_n = 1$ and $v_n = n$. But $F_2 \notin \mathcal{D}_p(\cdot)$, as if at all F_2 belongs to any of these domains then $F_2 \in \mathcal{D}_l(\Psi) \cup \mathcal{D}_l(\Lambda) \cup \mathcal{D}_p(H_{2,\alpha}) \cup \mathcal{D}_p(\Phi)$, as $r(F_2) = 1$. But, from Theorem 2.1.2 of Galambos (1978), $F_2 \notin \mathcal{D}_l(\Psi)$ and hence from Theorem 6.2(vii) in Appendix A.4., $F_2 \notin \mathcal{D}_p(H_{2,\alpha})$. Further from Theorem 2.1.3 of Galambos (1978), $F_2 \notin \mathcal{D}_l(\Lambda)$, and hence from Theorem 6.2(iii) in Appendix A.4., $F_2 \notin \mathcal{D}_p(\Phi)$.

(iii) $F_3(x) = 1 - (\log(-\log(-x))^{-1}$, $-e^{-e} < x < 0$, belongs to $\mathcal{D}_e(U_{5,\alpha})$ with $u_n = 1$ and $v_n = n$. But $F_3 \notin \mathcal{D}_p(\cdot)$, as if at all F_3 belongs to any of these domains then $F_3 \in \mathcal{D}_l(\Psi) \cup \mathcal{D}_l(\Lambda) \cup \mathcal{D}_p(H_{3,\alpha}) \cup \mathcal{D}_p(\Psi)$, as $r(F_3) = 0$. But, from Theorem 2.1.2 of Galambos (1978), $F_3 \notin \mathcal{D}_l(\Psi)$ and from Theorem 6.1(iii) in Appendix A.3., $F_3 \notin \mathcal{D}_p(H_{3,\alpha})$ and also from Theorem 2.1.3 of Galambos (1978), $F_3 \notin \mathcal{D}_l(\Lambda)$, and hence from Theorem 6.2(iv) in Appendix A.4., $F_3 \notin \mathcal{D}_p(\Psi)$.

(iv) $F_4(x) = 1 - (-\log\log(-x))^{-1}$, $-e^{1/e} < x < -1$, belongs to $\mathcal{D}_e(U_{7,\alpha})$ with $u_n = 1$ and $v_n = n$. But $F_4 \notin \mathcal{D}_p(\cdot)$, as if at all F_4 belongs to any of these domains then $F_4 \in \mathcal{D}_l(\Psi) \cup \mathcal{D}_l(\Lambda) \cup \mathcal{D}_p(H_{4,\alpha}) \cup \mathcal{D}_p(\Psi)$, as $r(F_4) = -1$. But, from Theorem 2.1.2 of Galambos (1978), $F_2 \notin \mathcal{D}_l(\Psi)$ and hence from Theorem 6.2(viii) in Appendix A.4., $F_2 \notin \mathcal{D}_p(H_{4,\alpha})$ and also from Theorem 2.1.3 of Galambos (1978), $F_2 \notin \mathcal{D}_l(\Lambda)$, and hence from Theorem 6.2(iv) in Appendix A.4., $F_2 \notin \mathcal{D}_p(\Psi)$.

(v) $F_5(x) = 1 - e^{-(\log\log x)^2}$, $x \ge e$, belongs to $\mathcal{D}_e(H_{1,1})$, with $u_n = e^{\sqrt{\log n}}$ and $v_n = (2\sqrt{\log n})^{-1}$. Note that $F_5 \notin \mathcal{D}_p(H_{1,\alpha}) \cup \mathcal{D}_p(\Phi)$. For, if $F_5(x) \in \mathcal{D}_p(H_{1,\alpha})$ then from the proof of a(i) of Theorem 4.1, $G_5(y) = F_5(e^y) = 1 - e^{-(\log x)^2}$, $x \ge 1 \in \mathcal{D}_l(\Phi_\alpha)$, a contradiction established in Mohan and Ravi (1993). Also, if $F_5 \in \mathcal{D}_p(\Phi)$ then from the proof of c(ii) of Theorem 4.1, $G_5 \in \mathcal{D}_l(\Lambda)$, again a contradiction established in Mohan and Ravi (1993).

(vi) $F_6(x) = 1 - e^{-(-\log(-\log x))^2}$, $1/e < x < 1$, belongs to $\mathcal{D}_e(H_{2,1})$, with $u_n = e^{\sqrt{\log(1/n)}}$ and $v_n = (2\sqrt{\log(1/n)})^{-1}$. Note that $F_6 \notin \mathcal{D}_p(H_{2,\alpha}) \cup \mathcal{D}_p(\Phi)$. For, if $F_6 \in \mathcal{D}_p(H_{2,\alpha})$ then from the proof of c(i) of Theorem 4.1, $G_6(y) = F_6(e^y) = 1 - e^{-(-\log(-x))^2}$, $-1 < x < 0 \in \mathcal{D}_l(\Psi_\alpha)$, a contradiction which is easy to establish. Also, if $F_6 \in \mathcal{D}_p(\Phi)$, then from the proof of c(ii) of Theorem 4.1, $G_6 \in \mathcal{D}_l(\Lambda)$, again a contradiction which is easy to establish.

(vii) $F_7(x) = 1 - e^{-(\log(-\log(-x))^2}$, $-1/e < x < 0$, belongs to $\mathcal{D}_e(H_{3,1})$, with $u_n = e^{\sqrt{\log n}}$ and $v_n = (2\sqrt{\log n})^{-1}$. Note that $F_7 \notin \mathcal{D}_p(H_{3,\alpha}) \cup \mathcal{D}_p(\Psi)$. For, if $F_7(x) \in \mathcal{D}_p(H_{3,\alpha})$, then from the proof of e(i) of Theorem 4.1, $G_7(y) = F_7(-e^{-y}) = 1 - e^{-(\log x)^2}$, $x \geq 1, \in \mathcal{D}_l(\Psi_\alpha)$, a contradiction established in Mohan and Ravi (1993). Also, if $F_7 \in \mathcal{D}_p(\Psi)$, then from the proof of e(ii) of Theorem 4.1, $G_7 \in \mathcal{D}_l(\Lambda)$, a contradiction established in Mohan and Ravi (1993).

(viii) $F_8(x) = 1 - e^{-(-\log\log(-x))^2}$, $-e < x < -1$, belongs to $\mathcal{D}_e(H_{4,1})$, with $u_n = e^{\sqrt{\log(1/n)}}$ and $v_n = (2\sqrt{\log(1/n)})^{-1}$. Note that $F_8 \notin \mathcal{D}_p(H_{4,\alpha}) \cup \mathcal{D}_p(\Psi)$. For, if $F_8 \in \mathcal{D}_p(H_{4,\alpha})$, then from the proof of g(i) of Theorem 4.1, $G_8(y) = F_8(-e^{-y}) = 1 - e^{-(-\log(-x))^2}$, $-1 < x < 0, \in \mathcal{D}_l(\Psi_\alpha)$, a contradiction which is easy to establish. Also, if $F_8 \in \mathcal{D}_p(\Psi)$, then from the proof of g(ii) of Theorem 4.1, $G_8 \in \mathcal{D}_l(\Lambda)$, a contradiction which is easy to establish.

(ix) The following examples pertain to some standard dfs and can be deduced using Theorem 4.1 and Theorem 6.2 in Appendix A.4.

(a) The standard normal df $F \in \mathcal{D}_l(\Lambda)$ with $a_n = \sqrt{2\log n} - \dfrac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}$, $b_n = (\sqrt{2\log n})^{-1}$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = b_n$, $\beta_n = \dfrac{a_n}{b_n}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(b) The uniform df over (0,1) $F \in \mathcal{D}_l(\Psi_\alpha)$ with $a_n = 1/n$, $b_n = 1$ so that (1) holds; $F \in \mathcal{D}_p(H_{2,\alpha})$ with $\alpha_n = b_n$, $\beta_n = \dfrac{a_n}{b_n}$ so that (2) holds and $F \in \mathcal{D}_e(H_{2,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(c) The standard exponential df $F \in \mathcal{D}_l(\Lambda)$ with $a_n = 1$, $b_n = \log n$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = b_n$, $\beta_n = \dfrac{a_n}{b_n}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(d) The Cauchy df $F \in \mathcal{D}_l(\Phi_\alpha)$ with $a_n = \dfrac{n}{\pi}$, $b_n = 0$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = a_n$, $\beta_n = \dfrac{1}{\alpha}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(e) The lognormal df $F \in \mathcal{D}_l(\Lambda)$ with $a_n = \dfrac{b_n}{\sqrt{2\log n}}$, $b_n = \exp(1 + \sqrt{2\log n} - \dfrac{\log 4\pi + \log\log n}{2\sqrt{2\log n}})$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = b_n$, $\beta_n = \dfrac{a_n}{b_n}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(f) The Gamma (α, β) df $F \in \mathcal{D}_l(\Lambda)$ with $a_n = 1/\beta$, $b_n = \dfrac{1}{\beta}(\log n + (\alpha - 1)\log\log n - \log\Gamma(\alpha))$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = a_n$, $\beta_n = \dfrac{a_n}{b_n}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(g) The loggamma(α, β) df $F \in \mathcal{D}_l(\Phi_\alpha)$ with $a_n = \left(\dfrac{n(\log n)^{\beta-1}}{\Gamma(\beta)}\right)^{1/\alpha}$, $b_n = 0$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = a_n$, $\beta_n = \dfrac{1}{\alpha}$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

(h) The Pareto df $F \in \mathcal{D}_l(\Phi_\alpha)$ with $a_n = n^{1/\alpha}$, $b_n = 0$ so that (1) holds; $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = a_n$, $\beta_n = 1/\alpha$ so that (2) holds and $F \in \mathcal{D}_e(H_{1,1})$ with $u_n = \beta_n$, $v_n = \alpha_n$ so that (5) holds.

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6. Appendix

A.1. **The l-max stable laws** (Galambos, 1978) For parameter $\alpha > 0$, the following are the different types of l-max stable laws:

$$\begin{aligned} \text{the Fréchet law,} \quad \Phi_\alpha(x) &= \exp\{-x^{-\alpha}\}, \quad 0 < x; \\ \text{the Weibull law,} \quad \Psi_\alpha(x) &= \exp\{-(-x)^{\alpha}\}, \quad x < 0; \\ \text{and the Gumbel law,} \quad \Lambda(x) &= \exp\{-\exp(-x)\}, \ x \in \mathbb{R}. \end{aligned}$$

A.2. **The p-max stable laws** (Mohan and Ravi, 1993) For parameter $\alpha > 0$, the following are the different p-types of p-max stable laws:

$$\begin{aligned} \text{the log-Fréchet law,} \quad H_{1,\,\alpha}(x) &= \exp\{-(\log x)^{-\alpha}\}, \quad 1 < x; \\ \text{the log-Weibull law,} \quad H_{2,\,\alpha}(x) &= \exp\{-(-\log x)^{\alpha}\}, \quad 0 < x < 1; \\ \text{the standard Fréchet law,} \quad \Phi(x) &= \Phi_1(x), \ x \in \mathbb{R}; \\ \text{the inverse log-Fréchet law,} \quad H_{3,\,\alpha}(x) &= \exp\{-(-\log(-x))^{-\alpha}\}, \quad -1 < x < 0; \\ \text{the inverse log-Weibull law,} \quad H_{4,\,\alpha}(x) &= \exp\{-(\log(-x))^{\alpha}\}, \quad x < -1; \\ \text{and the standard Weibull law,} \quad \Psi(x) &= \Psi_1(x), \ x \in \mathbb{R}. \end{aligned}$$

A.3. **Criteria for a df to belong to $\mathcal{D}_p(.)$** (Mohan and Ravi, 1993)

Theorem 6.1.

(i) A df $F \in \mathcal{D}_p(H_{1,\alpha})$ iff $r(F) = \infty$ and $\lim_{t\to\infty} \dfrac{1 - F(e^{tx})}{1 - F(e^t)} = x^{-\alpha}, \ x > 0$. And the norming constants may be chosen as $\alpha_n = 1$ and $\beta_n = \log F^{-}(1 - 1/n)$.

(ii) A df $F \in \mathcal{D}_p(H_{2,\alpha})$ iff $0 < r(F) < \infty$ and $\lim_{t\to\infty} \dfrac{1 - F(r(F)e^{-x/t})}{1 - F(r(F)e^{-1/t})} = x^{\alpha}, \ x > 0$. And the norming constants may be chosen as $\alpha_n = r(F)$ and $\beta_n = \log r(F) - \log F^{-}(1 - 1/n)$.

(iii) A df $F \in \mathcal{D}_p(H_{3,\alpha})$ iff $r(F) = 0$ and $\lim_{t\to\infty} \dfrac{1 - F(-e^{-tx})}{1 - F(-e^{-t})} = x^{-\alpha}, \ x > 0$. And the norming constants may be chosen as $\alpha_n = 1$ and $\beta_n = -\log(-F^{-}(1 - 1/n))$.

(iv) A df $F \in \mathcal{D}_p(H_{4,\alpha})$ iff $r(F) < 0$ and $\lim_{t\to\infty} \dfrac{1 - F(r(F)e^{x/t})}{1 - F(r(F)e^{1/t})} = x^{\alpha}, \ x > 0$. And the norming constants may be chosen as $\alpha_n = -r(F)$ and $\beta_n = \log F^{-}(1 - 1/n) - \log r(F)$.

(v) A df $F \in \mathcal{D}_p(\Phi)$ iff $r(F) > 0$ and $\lim_{t\uparrow r(F)} \dfrac{1 - F(te^{xf(t)})}{1 - F(t)} = e^{-x}, \ x > 0$, for some positive function f and if this holds for some positive function f then $\dfrac{1}{1 - F(t)} \int_t^{r(F)} \dfrac{1 - F(x)}{x} dx < \infty$, and it holds with the choice $f(t) = \dfrac{1}{1 - F(t)} \int_t^{r(F)} \dfrac{1 - F(x)}{x} dx$. And the norming constants may be chosen as $\alpha_n = F^{-}(1 - 1/n)$ and $\beta_n = f(\alpha_n)$.

(vi) A df $F \in \mathcal{D}_p(\Psi)$ iff $r(F) \leq 0$ and $\lim_{t\uparrow r(F)} \dfrac{1 - F(te^{xf(t)})}{1 - F(t)} = e^{x}, \ x > 0$, for some positive function f and if this holds for some positive function f then $-\dfrac{1}{1 - F(t)} \int_t^{r(F)} \frac{1-F(x)}{x} dx < \infty$, and it holds with the choice $f(t) = -\dfrac{1}{1 - F(t)} \int_t^{r(F)} \frac{1-F(x)}{x} dx$. And the norming constants may be chosen as $\alpha_n = -F^{-}(1 - 1/n)$ and $\beta_n = f(-\alpha_n)$.

A.4. **Comparison of $\mathcal{D}_l(.)$ and $\mathcal{D}_p(.)$ domains** (Mohan and Ravi, 1993)

Theorem 6.2. *Let F be a df.*

(i) If $F \in \mathcal{D}_l(\Phi_\alpha)$ with norming constants $a_n > 0, b_n \in \mathbb{R}$, then $F \in \mathcal{D}_p(\Phi)$ with $\alpha_n = a_n$ and $\beta_n = 1/\alpha$.

(ii) *If* $r(F) = \infty$ *and* $F \in \mathcal{D}_l(\Lambda)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(\Phi)$ *with* $\alpha_n = b_n$ *and* $\beta_n = \frac{a_n}{b_n}$.

(iii) *If* $0 < r(F) < \infty$ *and* $F \in \mathcal{D}_l(\Lambda)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(\Phi)$ *with* $\alpha_n = b_n$ *and* $\beta_n = \frac{a_n}{b_n}$. *Conversely, if* $r(F) < \infty$ *and* $F \in \mathcal{D}_p(\Phi)$ *with norming constants* $\alpha_n > 0, \beta_n > 0$, *then* $F \in \mathcal{D}_l(\Lambda)$ *with* $a_n = \alpha_n\beta_n,\ b_n = \alpha_n$.

(iv) *If* $r(F) < 0$ *and* $F \in \mathcal{D}_l(\Lambda)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(\Psi)$ *with* $\alpha_n = -b_n$ *and* $\beta_n = -\frac{a_n}{b_n}$. *Conversely, if* $r(F) < 0$ *and* $F \in \mathcal{D}_p(\Psi)$ *with norming constants* $\alpha_n > 0, \beta_n > 0$, *then* $F \in \mathcal{D}_l(\Lambda)$ *with* $a_n = \alpha_n\beta_n,\ b_n = -\alpha_n$.

(v) *If* $r(F) = 0$ *and* $F \in \mathcal{D}_l(\Lambda)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(\Psi)$ *with* $\alpha_n = -b_n$ *and* $\beta_n = -\frac{a_n}{b_n}$.

(vi) *If* $r(F) = 0$ *and* $F \in \mathcal{D}_l(\Psi_\alpha)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(\Psi)$ *with* $\alpha_n = a_n$ *and* $\beta_n = -1/\alpha$.

(vii) *If* $r(F) > 0$ *and* $F \in \mathcal{D}_l(\Psi_\alpha)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(H_{2,\alpha})$ *with* $\alpha_n = b_n$ *and* $\beta_n = \frac{a_n}{b_n}$. *Conversely, if* $F \in \mathcal{D}_p(H_{2,\alpha})$ *with norming constants* $\alpha_n > 0, \beta_n > 0$, *then* $F \in \mathcal{D}_l(\Psi_\alpha)$ *with* $a_n = \alpha_n\beta_n,\ b_n = \alpha_n$.

(viii) *If* $r(F) < 0$ *and* $F \in \mathcal{D}_l(\Psi_\alpha)$ *with norming constants* $a_n > 0, b_n \in \mathbb{R}$, *then* $F \in \mathcal{D}_p(H_{4,\alpha})$ *with* $\alpha_n = -b_n$ *and* $\beta_n = -\frac{a_n}{b_n}$. *Conversely, if* $F \in \mathcal{D}_p(H_{4,\alpha})$ *with norming constants* $\alpha_n > 0, \beta_n > 0$, *then* $F \in \mathcal{D}_l(\Psi_\alpha)$ *with* $a_n = \alpha_n\beta_n,\ b_n = -\alpha_n$.

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