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Efficient Estimation Capacity Index for 3.2^{n-m} Fractional Factorial Designs

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Abstract. Among the irregular fractions of 2^n factorials, the $3/2^m$ fractions have greater practical value because their alias patterns are known. Based on alias pattern, the variances and covariances of all estimable factorial effects can be known and hence also A-efficiency of design. This paper introduces an efficient estimation capacity (EEC) index to assess designs of $3/2^m$ fraction of 2^n factorial. It is simply a count generated from the alias pattern. It serves as a surrogate for finding the most A-efficient design among the designs of same runs. Higher the index, higher the A-efficiency, that is lower the variance and covariance of 3.2^{n-m} design for interaction model. This index has opened up comparison among 3.2^{n-m} fractions of variable resolutions between resolutions II to IV. The role of EEC index as compared to other assessment criteria for irregular factorial designs namely A-, df-efficiency, generalized resolution and minimum moment aberration has been discussed.

1. Introduction

There are many situations in which an experimenter must estimate the important effects and interactions of a symmetrical factorial experiment with as few trials as possible. There are instances where say, all two-factor interactions are important and a fractional replicate design, which permits orthogonal estimates of all main effects (MEs) and two-factor interactions (TFIs) requires more trials than one can afford to make. If the experimenter is restricted to designs of the type represented by a $1/2^m$ fraction of the 2^n experiment, he must either abandon the investigation or choose a more highly fractionated design, and assume that several of the TFIs are negligible.

Consider, for example, a situation where it is desirable to estimate the MEs and TFIs in an experiment involving six factors, each having two levels in less than 28 trials. It is known that a 1/2 replicate of the 2^6 experiment defined by at least five factors interaction, as generator allows uncorrelated estimates of all MEs and TFIs, when three-factor and higher order interactions are negligible. However, this design requires 32 trials, exceeding the maximum number which can be made. A higher order fraction 1/4 replicate of the 2^6 experiment having 16 runs allows uncorrelated estimates of all MEs and half of the TFIs assuming half of the TFIs as negligible. The number of TFIs to be considered negligible would increase with the increase in the number of factors. For more details, refer Chen and Cheng (2004), who developed estimation index indicating estimation capacity of a regular fractional factorial designs.

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It seems reasonable to inquire whether a design can be constructed with 24 runs which will yield information on all MEs and TFIs although partially correlated. A $3/2^3$ replicate of the 2^6 factorial is such a design. Such irregular fractions were considered long back by Addelman (1961), John(1961, 1962) and Patel (1963), but it still possesses potential to explore some novelties. Addelman (1961) showed that the 3.2^{n-m} replicate designs introduce correlations between some of the estimates of up to 1/2, although these correlations do not affect individual tests on the parameters. Patel (1963) showed that 3.2^{n-m} factorial are partially duplicated designs and can be studied and assessed through 2^{n-m} factorial alias sets. Authors/ Researchers have restricted discussion to designs of resolution III and higher, fearing loss of orthogonality in estimation of MEs. However, 3.2^{n-m} design based on defining contrasts containing a ME or TFI do yield estimation of all the MEs and TFIs as equally as a resolution III design, in fact with minimum variances and covariances.

One can base selection of a good 3.2^{n-m} design on answers to one or more of the following practical questions.

1. Does there exist a design, may be of resolution less than three that can estimate all the MEs and TFIs?
2. Does there exist an economical design suitable for MEs and TFIs model?
3. Does there exist an irregular design for MEs and TFIs model that allows orthogonal estimation of few MEs?

Since factorial estimates from irregular designs are correlated, one must focus on reducing the variances and covariances of the estimates. Other forms of two-level irregular designs studied for estimation of MEs and TFIs model by Rechtschaffner (1967), Tobias (1996), Mee (2004) and Tang and Zhou (2009, 2013) are useful but for them alias patterns are hardly known.

Many criteria have been developed to characterize irregular fractions of two level factorials, but $3/2^m$ fraction of 2^n has not been focused. A criterion, df-efficiency proposed by Daniel (1956) measures how many more factorial effects can be estimated, thus higher the df-efficiency, higher the rank of design information matrix. Another criterion, A-efficiency measures how far average variance and covariance are minimum thus, higher the A-efficiency lower the value of diagonal and off-diagonal elements of variance-covariance matrix (Kuhfeld, 1997). This A-efficiency also indicates the amount of orthogonality in a design. A formula of A-efficiency specially for 3.2^{n-m} designs was given, Mee (2004). One more indicator is the generalized resolution proposed by Deng and Tang (1999) for irregular designs. It is resolution of a design plus, a less than unity value indicating the minimum percentage of runs utilized in estimation of a factorial effect. Its value means higher the use of design runs in estimation of all factorial effects. However, there are two problems with this criteria, first, there exist several designs of equal generalized resolution, and second, it is incompetent to asses the amount of non-orthogonality (confounding) among the contrasts of factorial effects of interest, because it is defined on the basis of design matrix and not the information matrix. Consequently, designs with equal generalized resolution value can have different A-efficiencies. Recently introduced, minimum moment aberration (MMA) criterion by Xu (2003) can rank irregular fractions of the same generalized resolution in terms of improved estimability of MEs than TFIs but, it cannot assess the amount of partial confounding among these factorial effects. Alike generalized resolution, MMA protects MEs from getting confounded with another ME, but does not protect MEs from getting confounded with two TFIs, leaving behind only 1/3 orthogonal runs used exclusively for its estimation. All the above criteria are suitable for assessing Plackett Burman and saturated type of irregular designs, which are respectively orthogonal and non-orthogonal designs, but not for 3.2^{n-m} fractions as they do not make use of the alias pattern.

This article introduces an index for selection of the most A-efficient 3.2^{n-m} design suitable for interaction model, simply based on alias pattern. This new criterion called efficient estimation capacity (EEC) index assesses 3.2^{n-m} designs and demonstrates that high EEC indexed design would estimate MEs and TFIs with minimum variance and covariance that is, high A-efficiency. A design with lower EEC index would be df-efficient, and a design with moderate EEC index would be MMA design. Further it is showed that, some 3.2^{n-m} designs having resolution III or less, possess special properties and provide two new 3.2^{n-m} designs for (n, m)=(5, 2), (6, 3) of practical importance. Comparison of designs with relevant saturated designs is given.

The article is organized in four sections. Section 2 narrates some basics of estimation of factorial effects which serves as an essential basis for the definition and explanation of the new index in Section 3. EEC index and its role in selection of good 3.2^{n-m} design is exhaustively discussed for 5 and 6 factors and verified for 7, 8 and 9 factors. Characterization of 24 run designs in 5 and 6 factors is given in Section 4. Treatment combinations and variances-covariances of estimable factorial effects of all 24 runs designs in 5 and 6 factors and alias sets of 5 to 9 factors suitable for interaction model are given in Appendix.

2. Estimation of factorial effects of 3.2^{n-m} irregular fractions

2.1. 3.2^{n-m} design

As commonly done, firstly, a 2^{n-m} design generator or defining contrast is chosen. Then combining three distinct 2^{n-m} designs by the generator gives a 3.2^{n-m} design. The choice of 2^{n-m} design generator is instrumental in determining the resolution and the corresponding alias set is instrumental in determining the efficiency with which the MEs and TFIs would be estimable.

2.2. The model and estimation of factorial effects

Let the observations resulting from a 3.2^{n-m} fractional factorial experiment are expressed as,

$$Y = X\beta + \varepsilon \tag{1}$$

Where Y is a vector of observation, X is the coefficient matrix for β-vector consisting of an intercept term, MEs and all estimable TFIs. Accordingly, we consider X matrix as juxtaposed matrix of a unity column representing general mean effect, 3.2^{n-m} design in ± 1 levels representing ME contrasts for n factors and two way products of n columns resulting in $n(n-1)/2$ columns of TFIs. The design matrix X is of order $3.2^{n-m} \times n_e$, n_e denotes the number of estimable factorial effects and ε is vector of independent normally distributed errors with mean zero and common variance σ^2.

Similar to Addelman(1961), the design matrix X is expressed such that the number of MEs and TFIs aliased in an alias relation form a block of r columns, $r = 1, 2$ or 3. Note that, r must not be more than 3 for $X'X$ to be invertible, because additional number of effects are unavoidably fully confounded. Then the information matrix $X'X$ consists of 2^{n-m} blocks of the form $2^{n-m+2}I_r - 2^{n-m}J_r$ and $(X'X)^{-1}$ consists of 2^{n-m} blocks of the form,

$$\frac{1}{2^{n-m+2}}[I_r + \frac{1}{4-r}J_r], \tag{2}$$

of order r, where I_r is an identity matrix of order r and J_r is a unity matrix of 1's of order r. Then estimate of β in (1) is,

$$\hat{\beta} = (X'X)^{-1}X'Y$$

with variance covariance matrix

$$v(\hat{\beta}) = (X'X)^{-1}\sigma^2. \tag{3}$$

In particular, using (2) and (3), the variance of i-th ($i = 1, ..., n_e$) model term and covariance between i-th and j-th ($i \neq j = 1, ..., n_e$) model terms for factorial effects appearing in blocks of order r, r=1, 2, 3 are given by,

$$Var(\hat{\beta}_i) = \begin{cases} 3\sigma^2/2^{n-m+3}, & \text{if r=1, 2;} \\ 4\sigma^2/2^{n-m+3}, & \text{if r=3.} \end{cases} \tag{4}$$

$$Cov(\hat{\beta}_i, \hat{\beta}_j) = \begin{cases} 0, & \text{if r=1;} \\ \sigma^2/2^{n-m+3}, & \text{if r=2;} \\ 2\sigma^2/2^{n-m+3}, & \text{if r=3.} \end{cases} \tag{5}$$

A 1/m replicate of a 2^n experiment will allow uncorrelated estimates of all or some MEs and TFIs when the three and higher factor interactions are negligible with variances $\sigma^2/2^{n-m-2}$. In comparison, a 3.2^{n-m} replicate design results in slight correlated estimates of all MEs and TFIs having smaller variance, variance is at most $\sigma^2/2^{n-m-1}$. Specific 3.2^{n-m} designs permit uncorrelated estimate of a few factorial effects, one such design exist for two level experiments in 5- and 6-factors.

3. EEC index for 3.2^{n-m} irregular fractions

Let, n_r^d denote the number of MEs and TFIs in blocks of order r, $r = 1, 2, 3$, n_{12}^d stands for $n_1^d + n_2^d$ and $\sigma_1^2 = (2^{n-m+3})^{-1}\sigma^2$. Then from variances in (4), the total variance of a 3.2^{n-m} design d for model (1) is given by,

$$V(d) = (3n_{12}^d + 4n_3^d)\sigma_1^2 \tag{6}$$

Similarly using (5), the total covariance of a 3.2^{n-m} design d is given by,

$$Cov(d) = (n_2^d + 2n_3^d)\sigma_1^2 \tag{7}$$

Theorem 1: Let d_1 and d_2 be two 3.2^{n-m} designs with number of estimable factorial effects $n_e^{d_1}$ and $n_e^{d_2}$ respectively, then $V(d_1) = V(d_2) + [4m_d - m_{12}]\sigma_1^2$.
Proof: Let m_d, m_{12} and m_3 defined as $m_d = n_e^{d_1} - n_e^{d_2}, m_2 = n_2^{d_1} - n_2^{d_2}, m_{12} = n_{12}^{d_1} - n_{12}^{d_2}, m_3 = n_3^{d_1} - n_3^{d_2}$, denote the differences of number of estimable factorial effects between two comparable designs d_1 and d_2 respectively, in whole design, in blocks of order one or two and in blocks of order three. From (6),

$$\begin{aligned} V(d_1) &= (3n_{12}^{d_1} + 4n_3^{d_1})\sigma_1^2 \\ &= [3(m_{12} + n_{12}^{d_2}) + 4(m_3 + n_3^{d_2})]\sigma_1^2 \\ &= [3n_{12}^{d_1} + 4n_3^{d_2} + 3m_{12} + 4(m_d - m_{12})]\sigma_1^2 \\ V(d_1) &= V(d_2) + [4m_d - m_{12}]\sigma_1^2 \end{aligned} \tag{8}$$

Corollary 1: Let d_1 and d_2 be two 3.2^{n-m} designs. If $n_e^{d_1} = n_e^{d_2}$ and $n_{12}^{d_1} > n_{12}^{d_2}$ then $V(d_1) < V(d_2)$. Result holds from (8) because $m_d = 0$ and $m_{12} > 0$.
Corollary 2: Let d_1 and d_2 be two 3.2^{n-m} designs. If $n_e^{d_1} < n_e^{d_2}$ and $n_{12}^{d_1} \geq n_{12}^{d_2}$ then $V(d_1) < V(d_2)$. Result holds from (8) because $m_d < 0$ and $m_{12} \geq 0$.
Corollary 3: Let d_1 and d_2 be two 3.2^{n-m} designs. If $n_e^{d_1} > n_e^{d_2}$ and $n_{12}^{d_1} > 4m_d + n_{12}^{d_2}$ then $V(d_1) < V(d_2)$. Result holds from (8) because $m_{12} > 4m_d$. However, the condition in Corollary 3 is nonexisting in practice. Following the Corollary 2, switching d_1 with d_2 we get that design with greater n_{12}^d value will have smaller variance. Similar results also hold among total covariances of 3.2^{n-m} designs.
Theorem 2: Let d_1 and d_2 be two 3.2^{n-m} designs with number of estimable factorial effects $n_e^{d_1}$ and $n_e^{d_2}$ respectively, then $Cov(d_1) = Cov(d_2) + [2m_d - m_2]\sigma_1^2$.
Proof: As above using (7) and $n_{12}^d = n_2^d$.

Above theorems and corollaries imply that variance and covariance of 3.2^{n-m} designs are decreasing function of n_{12}^d, higher the n_{12}^d value, lower the total variance and covariance. Equivalent trend is followed by the design average variance and covariance because n_e^d values for a class of 3.2^{n-m} designs do not differ greatly. Hence, we define,

$$\text{EEC Index} = n_{12}^d \tag{9}$$

EEC rank =1 for design d^* if $n_{12}^{d^*} = \max_d(n_{12}^d)$. EEC ranks all designs and are given in descending order of their EEC index values.

Use of EEC index is illustrated below for 3.2^{n-m} designs listed in Table 1. Table 1 shows generators for construction of 3.2^{5-2}, 3.2^{6-3} and 3.2^{7-4} irregular fractional factorial designs suitable for MEs and TFIs model, along with counts and list of the estimable MEs and TFIs, and counts of paired aliases and triple aliases.

Example 1: Consider designs d_1^5 and d_3^5 shown in Table 1. For these designs $n_e^{d_1} = n_e^{d_3} = 16$, EEC index of $d_1 = n_{12}^{d_1}$=16 > EEC index of $d_3 = n_{12}^{d_3}$ =13 and $n_3^{d_3}$ =3. This illustrates Corollary 1 and that design d_1 is A-efficient than d_3 because, using (6), $V(d_1) = 48\sigma_1^2 = 0.75\sigma^2 < V(d_3) = 51\sigma_1^2 = 0.80\sigma^2$, also average $V(d_1) = 0.0469\sigma^2 <$ average $V(d_3) = 0.0498\sigma^2$, using (7) $Cov(d_1) = 16\sigma_1^2 = 0.25\sigma^2 < Cov(d_3) = 19\sigma_1^2 = 0.28\sigma^2$, average $Cov(d_1) = 0.0156\sigma^2 <$ average $Cov(d_3) = 0.0176\sigma^2$ and A-efficiency of $d_1 = 88.89\% >$ A-efficiency of $d_3 = 81.93\%$.
Example 2: Consider designs d_1^6 and d_2^6 shown in Table 1. For these designs $n_e^{d_1} = 18 < n_e^{d_2} = 19$, EEC index of $d_1 = n_{12}^{d_1} = 12 >$ EEC index of $d_2 = n_{12}^{d_2}$ =10 and $n_3^{d_1} = 6$, $n_3^{d_2}$ =9. This illustrates Corollary 2 and that design d_1 is A-efficient than d_2 because, using (6), $V(d_1) = 0.9375\sigma^2 < V(d_2) = 1.0313\sigma^2$, also average $V(d_1) = 0.0521\sigma^2 <$ average $V(d_2) = 0.0543\sigma^2$, using (7) $Cov(d_1) = 0.3750\sigma^2 < Cov(d_2) = 0.4375\sigma^2$, average $Cov(d_1) = 0.0208\sigma^2 <$ average $Cov(d_2) = 0.0230\sigma^2$ and A-efficiency of $d_1 = 78.79\% >$ A-efficiency of $d_2 = 74.07\%$.
Example 3: Consider designs d_4^6 and d_6^6 shown in Table 1. For these designs, $n_e^{d_6} = 21 > n_e^{d_4}$=19, EEC index of $d_4 = n_{12}^{d_4} = 7 >$ EEC index of $d_6 = n_{12}^{d_6} = 6$. This illustrates that, Corollary 3 is not satisfied because $n_{12}^{d_6} \ngtr n_{12}^{d_4}$ by $4m_d$, however, Corollary 2 is satisfied, and accordingly, average $V(d_4) = 0.0567 <$ average $V(d_6) = 0.0580$ and average $Cov(d_4) = 0.0247 <$ average $Cov(d_6) = 0.0268$ and A-efficiency of $d_4 = 71.11\% >$ A-efficiency of $d_6 = 69.84\%$.

Table 1: Generators and Estimable factorial effects of 3.2^{5-2}, 3.2^{6-3}, 3.2^{7-3}, 3.2^{8-4} and 3.2^{9-4} designs

Design	Generator	n_e	n_k	Estimable factorial effects
d_1^5	I=ABCDE=ABC	16	n_2=16	I A B C D E AB AC AD AE BC BD BE CD CE DE
			n_3=00	-
d_2^5	I=ABCDE=ACDE	16	n_2=16	I A B C D E AB AC AD AE BC BD BE CD CE DE
			n_3=00	-
d_3^5	I=ABC=CDE	16	n_1=01	I
			n_2=12	A B D E AC AD AE BC BD BE CD CE
			n_3=03	C AB DE
d_4^5	I=ABCD=BCDE	16	n_1=03	B C D
			n_2=04	I A E AE
			n_3=09	AB AC AD BC BD BE CD CE DE
d_5^5	I=ABD=BE	16	n_1=03	C AC CD
			n_2=04	I BC BE CE
			n_3=09	A B D E AB AD AE BD DE
d_1^6	I=ABCD=ABCDE=ACDEF	18	n_2=12	A C D AD AE AF BD CD CE CF DE DF
			n_3=06	I B E F BE BF
d_2^6	I=ABCD=BCDEF=ACDE	19	n_2=10	I C D AC BC BE CE CF DE DF
			n_3=09	A B E F AB AE AF BF EF
d_3^6	I=ABCD=BDEF=ABDEF	19	n_2=10	I A B D AB BC BE BF DE DF
			n_3=09	C E F AC AE AF CE CF EF
d_4^6	I=ABCD=BCEF=CDE	19	n_1=01	I
			n_2=06	E F BC BD CD EF
			n_3=12	A B C D AE AF BE BF CE CF DE DF
d_5^6	I=ABCD=BCDE=CDEF	18	n_1=02	C D
			n_2=04	A B E F
			n_3=12	I AE AF BC BD BE BF CE CF DE DF EF
d_6^6	I=ABCDE=ABF=AE	21	n_2=06	I C D AE BC BD
			n_3=15	A B E F AB AC AD BE BF CD CE CF DE DF EF
$d_{e_1}^6$	I=ABCD=ACDEF	1	–	AF
d_1^7	I=ABCD=BDFH=DEG	29	$n_1 = 5$	I A C AE CE
			$n_2 = 18$	B D F G AB AD AF AG BC BE BG CD CF CG DE DF EF EG
			$n_3 = 6$	E AC BD BF DG FG
d_2^7	I=ABCDE=CDEFG=ADFG	29	$n_1 = 6$	F G CF CG EF EG
			$n_2 = 14$	I A B C D E AC AE BC BD BE CD CE DE
			$n_3 = 9$	AB AD AF AG BF BG DF DG FG

Table 1 conti..

Design	Generator	n_e	n_k	Estimable factorial effects
d_1^8	ABCDE=ABCFG=CDGH=BEFH	34	$n_1 = 3$	H AH DF
			$n_2 = 16$	I A B C D E F G AB AC AD AE AF AG BC DE
			$n_3 = 15$	BD BE BF BG BH CD CE CF CG CH DG DH EH FH GH
d_2^8	I=ABCDE=ABFGH=ACF=BEF	37	$n_1 = 1$	I
			$n_2 = 18$	A B D H AB AE AG BC BF CD CE CF DE DH EG FG FH GH
			$n_3 = 18$	C E F G AC AD AF AH BD BE BG BH CG CH DF DG EF EH
d_1^9	I=ABCDE=ABFGH=ACF=BEGJ	46	$n_1 = 16$	I A B E J AE AG AJ BE BG BJ EG GJ
			$n_2 = 24$	C D F H AC AD AF AH BC BD BF BH CE CG CJ DE DG DJ EF EH FG FJ GH HJ
			$n_3 = 6$	G AB CD CF CH DF DH EJ FH
d_2^9	I=ABCDE=DEFGH=GHJ=BFGJ	46	$n_1 = 13$	I D E AB AD AE AF BC CD CE CF DH
			$n_2 = 24$	A B C F J AG AH AJ BD BE BH BJ CG CH CJ DF DG DJ EF EG EH EJ FG FH GH
			$n_3 = 9$	G H AC BF BG DE FJ GJ HJ

3.1. Role of EEC index and comparison

Resolution of a regular fractional factorial serves as an indicator that, all or some of the factorial effects of given order are independently estimable. A resolution V design is required for independent estimation of MEs and TFIs model terms, when three and higher factor interactions are negligible. Only, 2^{5-1} resolution V design is both variance efficient and df-efficient for independent estimation of MEs and TFIs. For more than 5 factor experiments, regular resolution V designs are not df-efficient and regular resolution III or IV designs are not suitable, for full MEs and TFIs model.
In pursuit of economical designs for MEs and TFIs model, irregular fractions have been tried for maximizing df-efficiency at the expense of orthogonality. Now, almost all the MEs and TFIs are estimable because they are not fully confounded and hence, one look for designs that confound MEs with TFIs only partially, that is, irregular designs of resolution between III and IV. Since there can be more than one such design, criteria for selecting the best suitable design is applied. Generalized resolution criteria selects the one having highest generalized resolution value, however, there can be more than one design of equal generalized resolution value. In this situation, the MMA criterion is useful in selecting the best design, because it attaches unique value to each design. MMA selects design that protects most MEs from aliasing, in hierarchy with I, with MEs and with TFIs. Thus, it assures better estimation of some MEs but does not ensure least confounding (non- orthogonality) among MEs and TFIs. This gets reflected in the A-efficiency values, because it is based on $(X'X)^{-1}$. Therefore, a criterion that can asses designs in terms of least confounding and higher A efficiency would be useful.
EEC index criterion selects a design of 3.2^{n-m} design which estimates MEs and TFIs with minimum average variance-covariance, that is, the highest A-efficiency. Alike the MMA criterion, it provides ranking for designs with equal generalized resolution, but unlike MMA, it gives ranks in terms of maximum unconfounding of factorial effects simply based on aliased pair counts (n_{12}^{d}). EEC index is most suitable for 3.2^{n-m} irregular designs because, MMA gives best ranking to designs of higher generalized resolution which is ineffective in improving estimability of factorial effects in irregular designs. Logically speaking, EEC index adopts effect hierarchy principle and MMA applies effect heredity principle in selection of 3.2^{n-m} design.

3.2. Selection of the best design for 3.2^{n-m} *designs*

There are two advantages of 3.2^{n-m} irregular design as compared to saturated designs for MEs and TFIs model. Firstly, this class of designs embeds within an orthogonal design, a regular 2^{n-m+1} fraction suitable for fitting models in MEs and $2^{n-m+1}-1-n$ selected TFIs. Thus, a single experiment data can be analyzed for 3.2^{n-m} design data as well as that for the embedded regular design. The estimation index by Chen and Cheng (2004) for the embedded designs can be used to select the best among those and the extra 2^{n-m} degrees of freedom can be used for the analysis of variance (ANOVA) of the embedded design model. Secondly, unlike saturated designs which correlate every factorial effect estimate with all the others in the model, 3.2^{n-m} designs correlate only aliased factorial effects that is, each factorial effect estimates is correlated with only one or two other factorial effects.

In order to select the best design for the model (1) we consider all possible generators including those involving MEs and TFIs. The procedure involves considering alias set of each design and counting the number of aliased pair of MEs and TFIs and aliased triplets of MEs and TFIs.

Table 2 shows values of five different design statistics, namely, EEC index, A-efficiency, df-efficiency, generalized resolution and MMA ranking of irregular fractional factorials shown in Table 1. It is observed that EEC index is proportional to A-efficiency but inversely proportional to df-efficiency because, the total number of estimable factorial effects is lesser in designs with higher number of paired aliases. For example, EEC index is highest for d_1^6 having lower df-efficiency, while it is lowest for d_6^6, having highest df-efficiency. This implies that economical designs are generally less variance efficient. Among 3.2^{6-3} design, d_2^6, d_5^6 and d_6^6 have the same generalized resolution of 2.67 with MMA ranks 3, 4 and 2 respectively. However, as per variance-covariance based EEC index they receive ranks 2, 4 and 5. The reason for this contradiction is that, half of the MEs and TFIs are least confounded in design d_2^6 while only one third in d_6^6 (see Table 1). Two designs, d_3^5 and d_4^6 are ranked 1 by MMA, respectively for better estimability of A, B, D, E at the cost of higher confounding of C and TFIs, and better estimability of E, F at the cost of higher confounding of A,B,C,D and TFIs (see Table 1).

Table 3 illustrates existence of orthogonal designs embedded in 3.2^{n-m} designs having estimation index 2 or 3. They can be used as orthogonal main effect design for estimation of MEs and/or design for estimation of TFIs not estimable from 3.2^{n-m}, design due to full confounding.

Table 2: EEC index, A-efficiency, EEC and MMA based rank, Average variance covariance, Generalized Resolution of 3.2^{5-2}, 3.2^{6-3}, 3.2^{7-3}, 3.2^{8-4} and 3.2^{9-4} designs

Design	EEC index	A-efficiency	EEC Rank	Average Variance	Average Covariance	df-Efficiency	Generalized Resolution	MMA Rank
d_1^5	16	88.89	1	0.0469	0.0156	66.67	2.67	3
d_2^5	16	88.89	1	0.0469	0.0156	66.67	1.67	5
d_3^5	13	81.93	2	0.0498	0.0176	66.67	3.67	1
d_4^5	6	72.38	3	0.0557	0.0215	66.67	2.67	2
d_5^5	6	72.38	3	0.0557	0.0215	66.67	2.67	4
d_1^6	12	78.79	1	0.0521	0.0208	75.00	1.67	6
d_2^6	10	74.07	2	0.0543	0.0230	79.17	2.67	3
d_3^6	10	74.07	2	0.0543	0.0230	79.17	1.67	5
d_4^6	7	71.11	3	0.0567	0.0247	79.17	3.67	1
d_5^6	6	70.89	4	0.0573	0.0243	75.00	2.67	4
d_6^6	6	69.84	5	0.0580	0.0268	87.5	2.67	2
d_1^7	23	81.38	1	0.0251	0.0081	60.42	3.67	1
d_2^7	20	78.21	2	0.0259	0.0086	60.42	2.67	2
d_1^8	19	75.12	1	0.0269	0.0101	70.83	2.67	2
d_2^8	19	74.29	2	0.0272	0.0124	77.08	3.67	1
d_1^9	40	84.54	1	0.0112	0.0031	47.92	3.67	1
d_2^9	37	81.86	2	0.0125	0.0036	47.92	3.67	2

Here average covariance show absolute covariance values.

Table 3: Embedded 2^{6-2} design in 3.2^{6-3} design with estimation index

Design	d_1^6	d_2^6	d_3^6	d_4^6	d_5^6	d_6^6
Resolution III (*Estimation Index 2*)	1	1	1	-	-	1
Resolution IV (*Estimation Index 3*)	-	-	-	1	1	-

1 indicates existence of design

4. Characterization of 3.2^{n-m} Designs

4.1. A variance balance design

It is known that variance balanced designs are useful for fitting full MEs and TFIs model. The designs d_1^5 and d_2^5 are variance-covariance balanced 5-factor designs alike 2^{5-1} design (see Table 4 in Appendix). All MEs and TFIs are estimated with equal variance and covariance values because they get distributed uniformly in alias set as paired aliases. These designs are listed because unlike variance balanced regular 2^{5-1} fraction, they are suitable when estimation must be complemented with model analysis of variance.

From Theorems 1 and 2, it is easy to see that a necessary and sufficient condition for a 3.2^{n-m} design to be variance and covariance balanced is $n_2^d{=}n_e^d$.

4.2. Two mixed orthogonal designs

A design which has few MEs, estimable orthogonally to remaining partially confounded MEs and TFIs is termed as a mixed orthogonal design. Such designs would be useful in experiments, where, few MEs are of special interest and it is desirable to estimate them independently and with higher precision. Among designs listed in Table 1, a 5-factor design d_4^5 and 6-factor design d_5^6 are such designs. The design d_4^5 estimates three MEs B, C and D independently of remaining MEs and TFIs, using all 24 runs. Similarly d_5^6 estimates two MEs C and D independently of other MEs and TFIs using all 24 runs. It is further important because, if desired, all six MEs can be estimated orthogonally to TFIs from the embedded 16 run resolution IV design defined by generator I=ABCD=CDEF.

4.3. A 24 run df-efficient design in six factors

In literature, 22-run saturated design in 6-factors by Rechtschaffner (1967) is recommended for MEs and TFIs model (Box and Draper 2007). The design d_6^6 (see Table 1) is competitive to this design. It estimates all lower order factorial effects except one TFI (AF) with higher, 70% A-efficiency, and AF is estimable from an embedded 16-run design defined by $d_{e_1}^6$ (see Appendix).

5. Conclusion

The proposed EEC indexing method for selection of the best design does not require any tedious computations, just from alias pattern of 2^{n-m}, one can select the most A-efficient 3.2^{n-m} design. EEC index based rank is indicator of maximum unconfounding among MEs and TFIs while MMA rank is indicator of maximum unconfounding among MEs. Thus, highest EEC ranked 3.2^{n-m} design would be suitable for MEs and TFIs model. As a by product of proposed method, one can identify minimum moment aberration 3.2^{n-m} design from study of alias set, as well as, one can identify a 3.2^{n-m} design that can be extended into a response surface design from the study of alias sets of embedded 2^{n-m} designs.

6. Appendix

Table 4: Variance and covariance of factorial effects of 3.2^{5-2} and 3.2^{6-3} designs

	d_1^5		d_2^5		d_3^5		d_4^5		d_5^5	
Effects	Var	Cov	Var	Cov	Var	Cov	Var	Cov	Var	Cov
I	0.047	0.016	0.047	0.016	0.047	0	0.047	0.016	0.047	0.016
A	0.047	0.016	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031
B	0.047	0.016	0.047	0.016	0.047	0.016	0.047	0	0.063	0.031
C	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0	0.047	0.000
D	0.047	0.016	0.047	0.016	0.047	0.016	0.047	0	0.063	0.031
E	0.047	0.016	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031
AB	0.047	0.016	0.047	0.016	0.063	0.031	0.063	0.031	0.063	0.031
AC	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0.000
AD	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.063	0.031
AE	0.047	0.016	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031
BC	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0.016
BD	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.063	0.031
BE	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0.016
CD	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0.000
CE	0.047	0.016	0.047	0.016	0.047	0.016	0.063	0.031	0.047	0.016
DE	0.047	0.016	0.047	0.016	0.063	0.031	0.063	0.031	0.063	0.031

Here covariance show absolute values.

Table 4: Conti..

d_1^6			d_2^6			d_3^6			d_4^6			d_5^6			d_6^6		
FE	Var	Cov	FE	Var	Cov	FE	Var	Cov	FE	Var	Cov	FE	Var	Cov	FE	Var	Cov
I	0.063	0.031	I	0.047	0.016	I	0.047	0.016	I	0.047	0.000	I	0.063	0.031	I	0.047	0.016
E	0.063	0.031	BE	0.047	0.016	A	0.047	0.016	A	0.063	0.031	AE	0.063	0.031	AE	0.047	0.016
BF	0.063	0.031	A	0.063	0.031	B	0.047	0.016	BE	0.063	0.031	BF	0.063	0.031	A	0.063	0.031
A	0.047	0.016	EF	0.063	0.031	AB	0.047	0.016	CF	0.063	0.031	A	0.047	0.016	BF	0.063	0.031
AE	0.047	0.016	BF	0.063	0.031	C	0.063	0.031	B	0.063	0.031	E	0.047	0.016	E	0.063	0.031
B	0.063	0.031	B	0.063	0.031	EF	0.063	0.031	AE	0.063	0.031	B	0.047	0.016	B	0.063	0.031
BE	0.063	0.031	E	0.063	0.031	AC	0.063	0.031	DF	0.063	0.031	F	0.047	0.016	AF	0.063	0.031
F	0.063	0.031	AF	0.063	0.031	D	0.047	0.016	C	0.063	0.031	C	0.047	0	CD	0.063	0.031
C	0.047	0.016	C	0.047	0.016	BC	0.047	0.016	DE	0.063	0.031	D	0.047	0	C	0.047	0.016
CE	0.047	0.016	DF	0.047	0.016	E	0.063	0.031	AF	0.063	0.031	BE	0.063	0.031	BD	0.047	0.016
D	0.047	0.016	D	0.047	0.016	CF	0.063	0.031	D	0.063	0.031	EF	0.063	0.031	D	0.047	0.016
DE	0.047	0.016	CF	0.047	0.016	AE	0.063	0.031	CE	0.063	0.031	AF	0.063	0.031	BC	0.047	0.016
CD	0.047	0.016	F	0.063	0.031	F	0.063	0.031	BF	0.063	0.031	BD	0.063	0.031	F	0.063	0.031
AF	0.047	0.016	AE	0.063	0.031	CE	0.063	0.031	E	0.047	0.016	CE	0.063	0.031	AB	0.063	0.031
AD	0.047	0.016	AB	0.063	0.031	AF	0.063	0.031	CD	0.047	0.016	DF	0.063	0.031	BE	0.063	0.031
CF	0.047	0.016	AC	0.047	0.016	BE	0.047	0.016	F	0.047	0.016	BC	0.063	0.031	AC	0.063	0.031
BD	0.047	0.016	DE	0.047	0.016	DF	0.047	0.016	BD	0.047	0.016	DE	0.063	0.031	CE	0.063	0.031
DF	0.047	0.016	BC	0.047	0.016	BF	0.047	0.016	BC	0.047	0.016	CF	0.063	0.031	DF	0.063	0.031
			CE	0.047	0.016	DE	0.047	0.016	EF	0.047	0.016				AD	0.063	0.031
															DE	0.063	0.031
															CF	0.063	0.031

Alias sets of 2^{5-2} fractional factorial designs

d_1^5

I	=ABCDE	=ABC	=DE
A	=BCDE	=BC	=ADE
B	=ACDE	=AC	= BDE
C	=ABDE	=AB	=CDE
D	=ABCE	=ABCD	=E
AD	=BCE	=BCD	=AE
BD	=ACE	=ACD	=BE
CE	=ABD	=ABE	=CD

d_2^5

I	=ABCDE	=ACDE	=B
A	=BCDE	=CDE	=AB
C	=ABDE	=ADE	=BC
D	=ABCE	=ACE	=BD
E	=ABCD	=ACD	=BE
AC	=BDE	=DE	=ABC
AD	=BCE	=CE	=ABD
AE	=BCD	=CD	=ABE

d_3^5

I	=ABC	=CDE	=ABDE
A	=BC	=ACDE	=BDE
B	=AC	=BCDE	=ADE
C	=AB	=DE	=ABCDE
D	=ABCD	=CE	=ABE
E	=ABCE	=CD	=ABD
AD	=BCD	=ACE	=BE
AE	=BCE	=ACD	=BD

d_4^5

I	=ABCD	=BCDE	=AE
A	=BCD	=ABCDE	=E
B	=ACD	=CDE	=ABE
C	=ABD	=BDE	=ACE
D	=ABC	=BCE	=ADE
AB	=CD	=ACDE	=BE
AC	=BD	=ABDE	=CE
AD	=BC	=ABCE	=DE

d_5^5

I	=ABD	=BE	=ADE
A	=BD	=ABE	=DE
B	=AD	=E	=ABDE
C	=ABCD	=BCE	=ACDE
D	=AB	=BDE	=AE
AC	=BCD	=ABCE	=CDE
BC	=ACD	=CE	=ABCDE
CD	=ABC	=BCDE	=ACE

Alias sets of 2^{6-3} fractional factorial designs

d_1^6

I	=ABCD	=ABCDE	=ACDEF	=E	=BEF	=BF	=ACDF
A	=BCD	=BCDE	=CDEF	=AE	=ABEF	=ABF	=CDF
B	=ACD	=ACDE	=ABCDEF	$=BE^1$	$=EF^1$	=F	=ABCDF
C	=ABD	=ABDE	=ADEF	=CE	=BCEF	=BCF	=ADF
D	=ABC	=ABCE	=ACEF	=DE	=BDEF	=BDF	=ABCDF
AB^2	$=CD^2$	=CDE	=BCDEF	=ABE	=AEF	=AF	=BCDF
AD^3	$=BC^3$	=BCE	=CEF	=ADF	=ABDEF	=ABDF	=CF
BD^4	$=AC^4$	=ACE	=ABCEF	=BDE	=BEF	=DF	=ABCF

1, 2, 3, 4: Only one of the commonly superscripted alias effects is not estimable.

d_2^6

I	=ABCD	=BCDEF	=ACDE	=AEF	=BE	=ABF	=CDF
A	=BCD	=ABCDEF	=CDE	=EF	=ABE	=BF	=ACDF
B	=ACD	=CDEF	=ABCDE	=ABEF	=E	=AF	=BCDF
C	=ABD	=BDEF	=ADE	=ACEF	=BCE	=ABCF	=DF
D	=ABC	=BCEF	=ACE	=ADEF	=BDE	=ABDE	=CF
F	=ABCDF	=BCDE	=ACDF	=AE	=BEF	$=AB^1$	$=CD^1$
AC^2	$=BD^2$	=ABDEF	=DE	=CEF	=ABCF	=BCF	=ADF
AD^3	$=BC^3$	=ABCEF	=CE	=DEF	=ABDE	=BDE	=ACF

1, 2, 3: Only one of the commonly superscripted alias effects is not estimable.

d_3^6

I	=ABCD	=BDEF	=ABDEF	=ACEF	=CEF	=A	=BCD
B	=ACD	=DEF	=ADEF	=ABCEF	=BCEF	$=AB^1$	$=CD^1$
C	=ABD	=BCDEF	=ABCDEF	=AEF	=EF	$=AC^2$	$=BD^2$
D	=ABC	=BEF	=ABEF	=ACDEF	=CDEF	$=AD^3$	$=BC^3$
E	=ABCDE	=BDF	=ABDF	=ACF	=CE	=AE	=BCDE
F	=ABCDF	=BCE	=ABDE	=ACE	=CE	=AF	=BCDF
BE	=ACDE	=DF	=ADF	=ABCF	=BCF	=ABF	=CDE
BF	=ACDF	=DE	=ABF	=ABCE	=BCE	=ABF	=CDE

1, 2, 3: Only one of the commonly superscripted alias effects is not estimable.

d_4^6

I	=ABCD	=BCEF	=CDE	=ADEF	=ABE	=BDF	=ACF
A	=BCD	=ABCEF	=ACDE	=DEF	=BE	=ABDF	=CF
B	=ACD	=CEF	=BCDE	=ABDEF	=AE	=DF	=ABCF
C	=ABD	=BEF	=DE	=AEF	=ACBE	=BCDF	=AF
D	=ABC	=BCDEF	=CE	=AEF	=ABDE	=BF	=ACDF
E	=ABCDE	=BCF	$=CD^1$	=ADF	$=AB^1$	=BDEF	=ACEF
F	=ABCDF	=BCE	=CDEF	=ADE	=ABEF	$=BD^2$	$=AC^2$
AD^3	$=BC^3$	=ABCDEF	=ACE	=EF	=BDE	=ABF	=CDF

1, 2, 3: Only one of the commonly superscripted alias effects is not estimable.

d_5^6

I	=ABCD	=BCDE	=CDEF	=AE	=ABEF	=BF	=ACDF
A	=BCD	=ABCDE	=ACDEF	=E	=BEF	=ABF	=CDF
B	=ACD	=CDE	=BCDEF	=ABE	=AEF	=F	=ABCDF
C	=ABD	=BDE	=DEF	=ACE	=ABCEF	=BCF	=ADE
D	=ABC	=BCE	=CEF	=ADE	=ABDEF	=ADF	=ACF
$AB^\dagger$	$=CD^\dagger$	=ACDE	=ABCDEF	=BE	=EF	=AF	=BCDF
AC^1	$=BD^1$	=ABDE	=ADEF	=CE	=BCEF	=ABCF	=DF
AD^2	$=BC^2$	=ABCE	=ACEF	=DE	=BDEF	=ABDF	=CF

1, 2: Only one of the commonly superscripted alias effects is not estimable.
† indicates factorial effect is not estimable.

d_6^6

I	=ABCDE	=ABF	=AE	=CDEF	=BCD	=ACDF	=BEF
A	=BCDE	=BF	=E	= ACDEF	=ABCD	=CDF	=ABEF
B	=ACDE	$=AF^\dagger$	=ABE	=BCDEF	$=CD$	=ABCDF	$=EF$
C	=ABDE	=ABCF	=ACE	=DEF	=BD	=ADF	=BCEF
D	=ABCE	=ABDF	=ADE	=CEF	=BC	=ACF	=BDEF
F	=ABCDEF	=AB	=AEF	=CDE	=BCDF	=ACD	=BE
AC	=BDE	=BCF	=CE	=ADEF	=ABD	=DF	=ABCEF
AD	=BCE	=BDF	=DE	=ACEF	=ABC	=CF	=ABDEF

† indicates factorial effect is not estimable.

Alias set of embedded 2^{6-2} design in d_6^6

$d_{e_1}^6$

I	= ABCD	= ACDEF	= BEF
A	= BCD	= CDEF	= ABEF
B	= ACD	= ABCDEF	= EF
C	= ABD	= ADEF	= BCEF
D	= ABC	= ACEF	= BDEF
E	= ABCDE	= ACDF	= BF
F	= ABCDF	= ACDE	= BE
AB	= CD	= BCDEF	= AEF
AC	= BD	= DEF	= ABCEF
AD	= BC	= CEF	= ABDEF
AE	= BCDE	= CDF	= ABF
AF	= BCDF	= CDE	= ABE
CE	= ABDE	= ADF	= BCF
CF	= ABDF	= ADE	= BCE
DE	= ABCE	= ACF	= BDF
DF	= ABCF	= ACE	= BDE

Alias sets of 2^{7-3} fractional factorial designs

d_1^7

I	= ABCD	= BDFG	= DEG	= ACFG	= ABCEG	= BEF	= ACDEF
A	= BCD	= ABDFG	= ADEG	= CFG	= BCEG	= ABEF	= CDEF
B	= ACD	= DFG	= BDEG	= ABCFG	= ACEG	= EF	= ABCDEF
C	= ABD	= BCDFG	= CDEG	= AFG	= ABEG	= BCEF	= ADEF
D	= ABC	= BFG	= EG	= ACDFG	= ABCDEG	= BDEF	= ACEF
E	= ABCDE	= BDEFG	= DG	= ACEFG	= ABCG	= BF	= ACDF
F	= ABCDF	= BDG	= DEFG	= ACG	= ABCEFG	= BE	= ACDE
G	= ABCDG	= BDF	= DE	= ACF	= ABCE	= BEFG	= ACDEFG
AB	= CD	= ADFG	= ABDEG	= BCFG	= CEG	= AEF	= BCDEF
AC	= BD	= ABCDFG	= ACDEG	= FG	= BEG	= ABCEF	= DEF
AD	= BC	= ABFG	= AEG	= CDFG	= BCDEG	= ABDEF	= CEF
AE	= BCDE	= ABDEFG	= ADG	= CEFG	= BCG	= ABF	= CDF
AF	= BCDF	= ABDG	= ADEFG	= CG	= BCEFG	= ABE	= CDE
AG	= BCDG	= ABDF	= ADE	= CF	= BCE	= ABEFG	= CDEFG
BG	= ACDG	= DF	= BDE	= ABCF	= ACE	= EFG	= ABCDEFG
CE	= ABDE	= BCDEFG	= CDG	= AEFG	= ABG	= BCF	= ADF

d_2^7

I	= ABCDE	= CDEFG	= ADFG	= ABFG	= BCEFG	= ACE	= BD
A	= BCDE	= ACDEFG	= DFG	= BFG	= ABCEFG	= CE	= ABD
B	= ACDE	= BCDEFG	= ABDFG	= AFG	= CEFG	= ABCE	= D
C	= ABDE	= DEFG	= ACDFG	= ABCFG	= BEFG	= AE	= BCD
E	= ABCD	= CDFG	= ADEFG	= ABEFG	= BCFG	= AC	= BDE
F	= ABCDEF	= CDEG	= ADG	= ABG	= BCEG	= ACEF	= BDF
G	= ABCDEG	= CDEF	= ADF	= ABF	= BCEF	= ACEG	= BDG
AB	= CDE	= ABCDEFG	= BDFG	= FG	= ACEFG	= BCE	= AD
AF	= BCDEF	= ACDEG	= DG	= BG	= ABCEG	= CEF	= ABDF
AG	= BCDEG	= ACDEF	= DF	= BF	= ABCEF	= CEG	= ABDG
BC	= ADE	= BDEFG	= ABCDFG	= ACFG	= EFG	= ABE	= CD
BE	= ACD	= BCDFG	= ABDEFG	= AEFG	= CFG	= ABC	= DE
CF	= ABDEF	= DEG	= ACDG	= ABCG	= BEG	= AEF	= BCDF
CG	= ABDEG	= DEF	= ACDF	= ABCF	= BEF	= AEG	= BCDG
EF	= ABCDF	= CDG	= ADEG	= ABEG	= BCG	= ACF	= BDEF
EG	= ABCDG	= CDF	= ADEF	= ABEF	= BCF	= ACG	= BDEG

Alias sets of 2^{8-4} fractional factorial designs

d_1^8

I	= ABCDE	= ABCFG	= CDGH	= BEFH	= DEFG	= ABEGH	= ACDFH
A	= BCDE	= BCFG	= ACDGH	= ABEFH	= ADEFG	= BEGH	= CDFH
B	= ACDE	= ACFG	= BCDGH	= EFH	= BDEFG	= AEGH	= ABCDFH
D	= ABCE	= ABCDFG	= CGH	= BDEFH	= EFG	= ABDEGH	= ACFH
E	= ABCD	= ABCEFG	= CDEGH	= BFH	= DFG	= ABGH	= ACDEFH
F	= ABCDEF	= ABCG	= CDFGH	= BEH	= DEG	= ABEFGH	= ACDH
G	= ABCDEG	= ABCF	= CDH	= BEFGH	= DEF	= ABEH	= ACDFGH
H	= ABCDEH	= ABCFGH	= CDG	= BEF	= DEFGH	= ABEG	= ACDF
AB	= CDE	= CFG	= ABCDGH	= AEFH	= ABDEFG	= EGH	= BCDFH
AH	= BCDEH	= BCFGH	= ACDG	= ABEF	= ADEFGH	= BEG	= CDF
BD	= ACE	= ACDFG	= BCGH	= DEFH	= BEFG	= ADEGH	= ABCFH
BE	= ACD	= ACEFG	= BCDEGH	= FH	= BDFG	= AGH	= ABCDEFH
BF	= ACDEF	= ACG	= BCDFGH	= EH	= BDEG	= AEFGH	= ABCDH
BG	= ACDEG	= ACF	= BCDH	= EFGH	= BDEF	= AEH	= ABCDFGH
CH	= ABDEH	= ABFGH	= DG^2	= BCEF	= CDEFGH	= ABCEG	= ADF
DF^3	= ABCEF	= ABCDG	= CFGH	= BDEH	= EG^3	= ABDEFGH	= ACH
ABDFH	= ACEGH	= BCDEFG	= CEFH	= BDGH	= AFG	= ADE	= BC
BDFH	= CEGH	= ABCDEFG	= ACEFH	= ABDGH	= FG^1	= DE^1	= ABC
ADFH	= ABCEGH	= CDEFG	= BCEFH	= DGH	= ABFG	= ABDE	= C
ABFH	= ACDEGH	= BCEFG	= CDEFH	= BGH	= ADFG	= AE	= BCD
ABDEFH	= ACGH	= BCDFG	= CFH	= BDEGH	= AEFG	= AD	= BCE
ABDH	= ACEFGH	= BCDEG	= CEH	= BDFGH	= AG	= ADEF	= BCF
ABDFGH	= ACEH	= BCDEF	= CEFGH	= BDH	= AF	= ADEG	= BCG
ABDF	= ACEG	= BCDEFGH	= CEF	= BDG	= AFGH	= ADEH	= BCH
DFH	= BCEGH	= ACDEFG	= ABCEFH	= ADGH	= BFG	= BDE	= AC
BDF	= CEG	= ABCDEFGH	= ACEF	= ABDG	= FGH	= DEH	= ABCH
AFH	= ABCDEGH	= CEFG	= BCDEFH	= GH	= ABDFG	= ABE	= CD
ADEFH	= ABCGH	= CDFG	= BCFH	= DEGH	= ABEFG	= ABD	= CE
ADH	= ABCEFGH	= CDEG	= BCEH	= DFGH	= ABG	= ABDEF	= CF
ADFGH	= ABCEH	= CDEF	= BCEFGH	= DH	= ABF	= ABDEG	= CG
ABCDF	= AEG	= BDEFGH	= EF^2	= BCDG	= ACFGH	= ACDEH	= BH
ABH	= ACDEFGH	= BCEG	= CDEH	= BFGH	= ADG	= AEF	= BCDF

1, 2 , 3: Only one of the commonly superscripted alias effects is not estimable.

d_2^8

I	= ABCDE	= ABFGH	= ACF	= BEG	= CDEFGH	= BDEF	= ACDG
A	= BCDE	= BFGH	= CF	= ABEG	= ACDEFGH	= ABDEF	= CDG
B	= ACDE	= AFGH	= ABCF	= EG	= BCDEFGH	= DEF	= ABCDG
C	= ABDE	= ABCFGH	= AF	= BCEG	= DEFGH	= BCDEF	= ADG
D	= ABCE	= ABDFGH	= ACDF	= BDEG	= CEFGH	= BEF	= ACG
E	= ABCD	= ABEFGH	= ACEF	= BG	= CDFGH	= BDF	= ACDEG
F	= ABCDEF	= ABGH	= AC	= BEFG	= CDEGH	= BDE	= ACDFG
G	= ABCDEG	= ABFH	= ACFG	= BE	= CDEFH	= BDEFG	= ACD
H	= ABCDEH	= ABFG	= ACFH	= BEGH	= CDEFG	= BDEFH	= ACDGH
AB	= CDE	= FGH	= BCF	= AEG	= ABCDEFGH	= ADEF	= BCDG
AD	= BCE	= BDFGH	= CDF	= ABDEG	= ACEFGH	= ABEF	= CG
AE	= BCD	= BEFGH	= CEF	= ABG	= ACDFGH	= ABDF	= CDEG
AG	= BCDEG	= BFH	= CFG	= ABE	= ACDEFH	= ABDEFG	= CD
AH	= BCDEH	= BFG	= CFH	= ABEGH	= ACDEFG	= ABDEFH	= CDGH
BC	= ADE	= ACFGH	= ABF	= CEG	= BDEFGH	= CDEF	= ABDG
BF	= ACDEF	= AGH	= ABC	= EFG	= BCDEGH	= DE	= ABCDFG
BCGH	= AEFH	= ABCEFG	= ADEGH	= BCDFH	= DFG	= CEH	= ABDH
ABCGH	= EFH	= BCEFG	= DEGH	= ABCDFH	= ADFG	= ACEH	= BDH
CGH	= ABEFH	= ACEFG	= ABDEGH	= CDFH	= BDFG	= BCEH	= ADH
BGH	= ACEFH	= ABEFG	= ACDEGH	= BDFH	= CDFG	= EH	= ABCDH
BCDGH	= ADEFH	= ABCDEFG	= AEGH	= BCFH	= FG	= CDEH	= ABH
BCEGH	= AFH	= ABCFG	= ADGH	= BCDEFH	= DEFG	= CH	= ABDEH
BCFGH	= AEH	= ABCEG	= ADEFGH	= BCDH	= DG	= CEFH	= ABDFH
BCH	= AEFGH	= ABCEF	= ADEH	= BCDFGH	= DF	= CEGH	= ABDGH
BCG	= AEF	= ABCEFGH	= ADEG	= BCDF	= DFGH	= CE	= ABD
ACGH	= BEFH	= CEFG	= BDEGH	= ACDFH	= ABDFG	= ABCEH	= DH
ABCDGH	= DEFH	= BCDEFG	= EGH	= ABCFH	= AFG	= ACDEH	= BH
ABCEGH	= FH	= BCFG	= DGH	= ABCDEFH	= ADEFG	= ACH	= BDEH
ABCH	= EFGH	= BCEF	= DEH	= ABCDFGH	= ADF	= ACEGH	= BDGH
ABCG	= EF	= BCEFGH	= DEG	= ABCDF	= ADFGH	= ACE	= BD
GH	= ABCEFH	= AEFG	= ABCDEGH	= DFH	= BCDFG	= BEH	= ACDH
CFGH	= ABEH	= ACEG	= ABDEFGH	= CDH	= BDG	= BCEFH	= ADFH

Alias sets of 2^{9-4} fractional factorial designs

d_1^9

I	= ABCDE	= ABFGH	= ACF	= BEGJ	= CDEFGH	= BDEF	= ACDGJ
A	= BCDE	= BFGH	= CF	= ABEGJ	= ACDEFGH	= ABDEF	= CDGJ
B	= ACDE	= AFGH	= ABCF	= EGJ	= BCDEFGH	= DEF	= ABCDGJ
C	= ABDE	= ABCFGH	= AF	= BCEGJ	= DEFGH	= BCDEF	= ADGJ
D	= ABCE	= ABDFGH	= ACDF	= BDEGJ	= CEFGH	= BEF	= ACGJ
E	= ABCD	= ABEFGH	= ACEF	= BGJ	= CDFGH	= BDF	= ACDEGJ
F	= ABCDEF	= ABGH	= AC	= BEFGJ	= CDEGH	= BDE	= ACDFGJ
G	= ABCDEG	= ABFH	= ACFG	= BEJ	= CDEFH	= BDEFG	= ACDJ
H	= ABCDEH	= ABFG	= ACFH	= BEGHJ	= CDEFG	= BDEFH	= ACDGHJ
J	= ABCDEJ	= ABFGHJ	= ACFJ	= BEG	= CDEFGHJ	= BDEFJ	= ACDG
AB	= CDE	= FGH	= BCF	= AEGJ	= ABCDEFGH	= ADEF	= BCDGJ
AD	= BCE	= BDFGH	= CDF	= ABDEGJ	= ACEFGH	= ABEF	= CGJ
AE	= BCD	= BEFGH	= CEF	= ABGJ	= ACDFGH	= ABDF	= CDEGJ
AG	= BCDEG	= BFH	= CFG	= ABEJ	= ACDEFH	= ABDEFG	= CDJ
AH	= BCDEH	= BFG	= CFH	= ABEGHJ	= ACDEFG	= ABDEFH	= CDGHJ
AJ	= BCDEJ	= BFGHJ	= CFJ	= ABEG	= ACDEFGHJ	= ABDEFJ	= CDG
BC	= ADE	= ACFGH	= ABF	= CEGJ	= BDEFGH	= CDEF	= ABDGJ
BD	= ACE	= ADFGH	= ABCDF	= DEGJ	= BCEFGH	= EF	= ABCGJ
BE	= ACD	= AEFGH	= ABCEF	= GJ	= BCDFGH	= DF	= ABCDEGJ
BF	= ACDEF	= AGH	= ABC	= EFGJ	= BCDEGH	= DE	= ABCDFGJ
BG	= ACDEG	= AFH	= ABCFG	= EJ	= BCDEFH	= DEFG	= ABCDJ
BH	= ACDEH	= AFG	= ABCFH	= EGHJ	= BCDEFG	= DEFH	= ABCDGHJ
BJ	= ACDEJ	= AFGHJ	= ABCFJ	= EG	= BCDEFGHJ	= DEFJ	= ABCDG
CD	= ABE	= ABCDFGH	= ADF	= BCDEGJ	= EFGH	= BCEF	= AGJ
CE	= ABD	= ABCEFGH	= AEF	= BCGJ	= DFGH	= BCDF	= ADEGJ
CJ	= ABDEJ	= ABCFGHJ	= AFJ	= BCEG	= DEFGHJ	= BCDEFJ	= ADG
DG	= ABCEG	= ABDFH	= ACDFG	= BDEJ	= CEFH	= BEFG	= ACJ
DH	= ABCEH	= ABDFG	= ACDFH	= BDEGHJ	= CEFG	= BEFH	= ACGHJ
DJ	= ABCEJ	= ABDFGHJ	= ACDFJ	= BDEG	= CEFGHJ	= BEFJ	= ACG
FH	= ABCDEFH	= ABG	= ACH	= BEFGHJ	= CDEG	= BDEH	= ACDFGHJ
BCJ	= ADEJ	= ACFGHJ	= ABFJ	= CEG	= BDEFGHJ	= CDEFJ	= ABDG
BDG	= ACEG	= ADFH	= ABCDFG	= DEJ	= BCEFH	= EFG	= ABCJ
BCGH	= AEFHJ	= ABCEFGJ	= ADEGH	= BCDFHJ	= DFGJ	= CEHJ	= ABDHJ
ABCGH	= EFHJ	= BCEFGJ	= DEGH	= ABCDFHJ	= ADFGJ	= ACEHJ	= BDHJ
CGH	= ABEFHJ	= ACEFGJ	= ABDEGH	= CDFHJ	= BDFGJ	= BCEHJ	= ADHJ
BGH	= ACEFHJ	= ABEFGJ	= ACDEGH	= BDFHJ	= CDFGJ	= EHJ	= ABCDHJ
BCDGH	= ADEFHJ	= ABCDEFGJ	= AEGH	= BCFHJ	= FGJ	= CDEHJ	= ABHJ
BCEGH	= AFHJ	= ABCFGJ	= ADGH	= BCDEFHJ	= DEFGJ	= CHJ	= ABDEHJ
BCFGH	= AEHJ	= ABCEGJ	= ADEFGH	= BCDHJ	= DGJ	= CEFHJ	= ABDFHJ
BCH	= AEFGHJ	= ABCEFJ	= ADEH	= BCDFGHJ	= DFJ	= CEGHJ	= ABDGHJ
BCG	= AEFJ	= ABCEFGHJ	= ADEG	= BCDFJ	= DFGHJ	= CEJ	= ABDJ
BCGHJ	= AEFH	= ABCEFG	= ADEGHJ	= BCDFH	= DFG	= CEH	= ABDH
ACGH	= BEFHJ	= CEFGJ	= BDEGH	= ACDFHJ	= ABDFGJ	= ABCEHJ	= DHJ
ABCDGH	= DEFHJ	= BCDEFGJ	= EGH	= ABCFHJ	= AFGJ	= ACDEHJ	= BHJ
ABCEGH	= FHJ	= BCFGJ	= DGH	= ABCDEFHJ	= ADEFGJ	= ACHJ	= BDEHJ
ABCH	= EFGHJ	= BCEFJ	= DEH	= ABCDFGHJ	= ADFJ	= ACEGHJ	= BDGHJ
ABCG	= EFJ	= BCEFGHJ	= DEG	= ABCDFJ	= ADFGHJ	= ACEJ	= BDJ
ABCGHJ	= EFH	= BCEFG	= DEGHJ	= ABCDFH	= ADFG	= ACEH	= BDH
GH	= ABCEFHJ	= AEFGJ	= ABCDEGH	= DFHJ	= BCDFGJ	= BEHJ	= ACDHJ
CDGH	= ABDEFHJ	= ACDEFGJ	= ABEGH	= CFHJ	= BFGJ	= BCDEHJ	= AHJ
CEGH	= ABFHJ	= ACFGJ	= ABDGH	= CDEFHJ	= BDEFGJ	= BCHJ	= ADEHJ
CFGH	= ABEHJ	= ACEGJ	= ABDEFGH	= CDHJ	= BDGJ	= BCEFHJ	= ADFHJ
CH	= ABEFGHJ	= ACEFJ	= ABDEH	= CDFGHJ	= BDFJ	= BCEGHJ	= ADGHJ
CG	= ABEFJ	= ACEFGHJ	= ABDEG	= CDFJ	= BDFGHJ	= BCEJ	= ADJ
CGHJ	= ABEFH	= ACEFG	= ABDEGHJ	= CDFH	= BDFG	= BCEH	= ADH
BDGH	= ACDEFHJ	= ABDEFGJ	= ACEGH	= BFHJ	= CFGJ	= DEHJ	= ABCHJ
BEGH	= ACFHJ	= ABFGJ	= ACDGH	= BDEFHJ	= CDEFGJ	= HJ	= ABCDEHJ
BGHJ	= ACEFH	= ABEFG	= ACDEGHJ	= BDFH	= CDFG	= EH	= ABCDH
BCDH	= ADEFGHJ	= ABCDEFJ	= AEH	= BCFGHJ	= FJ	= CDEGHJ	= ABGHJ
BCDG	= ADEFJ	= ABCDEFGHJ	= AEG	= BCFJ	= FGHJ	= CDEJ	= ABJ
BCDGHJ	= ADEFH	= ABCDEFG	= AEGHJ	= BCFH	= FG	= CDEH	= ABH
BCFG	= AEJ	= ABCEGHJ	= ADEFG	= BCDJ	= DGHJ	= CEFJ	= ABDFJ
GHJ	= ABCEFH	= AEFG	= ABCDEGHJ	= DFH	= BCDFG	= BEH	= ACDH
CDH	= ABDEFGHJ	= ACDEFJ	= ABEH	= CFGHJ	= BFJ	= BCDEGHJ	= AGHJ

d_2^9

I	= ABCDE	= DEFGH	= GHJ	= BFGJ	= ABCFGH	= ABCDEGHJ	= ACDEFGJ
A	= BCDE	= ADEFGH	= AGHJ	= ABFGJ	= BCFGH	= BCDEGHJ	= CDEFGJ
B	= ACDE	= BDEFGH	= BGHJ	= FGJ	= ACFGH	= ACDEGHJ	= ABCDEFGJ
C	= ABDE	= CDEFGH	= CGHJ	= BCFGJ	= ABFGH	= ABDEGHJ	= ADEFGJ
D	= ABCE	= EFGH	= DGHJ	= BDFGJ	= ABCDFGH	= ABCEGHJ	= ACEFGJ
E	= ABCD	= DFGH	= EGHJ	= BEFGJ	= ABCEFGH	= ABCDGHJ	= ACDFGJ
F	= ABCDEF	= DEGH	= FGHJ	= BGJ	= ABCGH	= ABCDEFGHJ	= ACDEGJ
G	= ABCDEG	= DEFH	= HJ	= BFJ	= ABCFH	= ABCDEHJ	= ACDEFJ
H	= ABCDEH	= DEFG	= GJ	= BFGHJ	= ABCFG	= ABCDEGJ	= ACDEFGHJ
J	= ABCDEJ	= DEFGHJ	= GH	= BFG	= ABCFGHJ	= ABCDEGH	= ACDEFG
AB	= CDE	= ABDEFGH	= ABGHJ	= AFGJ	= CFGH	= CDEGHJ	= BCDEFGJ
AD	= BCE	= AEFGH	= ADGHJ	= ABDFGJ	= BCDFGH	= BCEGHJ	= CEFGJ
AE	= BCD	= ADFGH	= AEGHJ	= ABEFGJ	= BCEFGH	= BCDGHJ	= CDFGJ
AF	= BCDEF	= ADEGH	= AFGHJ	= ABGJ	= BCGH	= BCDEFGHJ	= CDEGJ
AH	= BCDEH	= ADEFG	= AGJ	= ABFGHJ	= BCFG	= BCDEGJ	= CDEFGHJ
AJ	= BCDEJ	= ADEFGHJ	= AGH	= ABFG	= BCFGHJ	= BCDEGH	= CDEFG
BC	= ADE	= BCDEFGH	= BCGHJ	= CFGJ	= AFGH	= ADEGHJ	= ABDEFGJ
BD	= ACE	= BEFGH	= BDGHJ	= DFGJ	= ACDFGH	= ACEGHJ	= ABCEFGJ
BE	= ACD	= BDFGH	= BEGHJ	= EFGJ	= ACEFGH	= ACDGHJ	= ABCDFGJ
BG	= ACDEG	= BDEFH	= BHJ	= FJ	= ACFH	= ACDEHJ	= ABCDEFJ
BJ	= ACDEJ	= BDEFGHJ	= BGH	= FG	= ACFGHJ	= ACDEGH	= ABCDEFG
CD	= ABE	= CEFGH	= CDGHJ	= BCDFGJ	= ABDFGH	= ABEGHJ	= AEFGJ
CE	= ABD	= CDFGH	= CEGHJ	= BCEFGJ	= ABEFGH	= ABDGHJ	= ADFGJ
CF	= ABDEF	= CDEGH	= CFGHJ	= BCGJ	= ABGH	= ABDEFGHJ	= ADEGJ
DF	= ABCEF	= EGH	= DFGHJ	= BDGJ	= ABCDGH	= ABCEFGHJ	= ACEGJ
DH	= ABCEH	= EFG	= DGJ	= BDFGHJ	= ABCDFG	= ABCEGJ	= ACEFGHJ
DJ	= ABCEJ	= EFGHJ	= DGH	= BDFG	= ABCDFGHJ	= ABCEGH	= ACEFG
EH	= ABCDH	= DFG	= EGJ	= BEFGHJ	= ABCEFG	= ABCDGJ	= ACDFGHJ
ADF	= BCEF	= AEGH	= ADFGHJ	= ABDGJ	= BCDGH	= BCEFGHJ	= CEGJ
ADH	= BCEH	= AEFG	= ADGJ	= ABDFGHJ	= BCDFG	= BCEGJ	= CEFGHJ
ADJ	= BCEJ	= AEFGHJ	= ADGH	= ABDFG	= BCDFGHJ	= BCEGH	= CEFG
AEH	= BCDH	= ADFG	= AEGJ	= ABEFGHJ	= BCEFG	= BCDGJ	= CDFGHJ
DEFJ	= BDEHJ	= BFH	= ABCFJ	= ACHJ	= ACDEFH	= BDEG	= ACG
ADEFJ	= ABDEHJ	= ABFH	= BCFJ	= CHJ	= CDEFH	= ABDEG	= CG
BDEFJ	= DEHJ	= FH	= ACFJ	= ABCHJ	= ABCDEFH	= DEG	= ABCG
CDEFJ	= BCDEHJ	= BCFH	= ABFJ	= AHJ	= ADEFH	= BCDEG	= AG
EFJ	= BEHJ	= BDFH	= ABCDFJ	= ACDHJ	= ACEFH	= BEG	= ACDG
DFJ	= BDHJ	= BEFH	= ABCEFJ	= ACEHJ	= ACDFH	= BDG	= ACEG
DEJ	= BDEFHJ	= BH	= ABCJ	= ACFHJ	= ACDEH	= BDEFG	= ACFG
DEFGJ	= BDEGHJ	= BFGH	= ABCFGJ	= ACGHJ	= ACDEFGH	= BDE	= AC
DEFHJ	= BDEJ	= BF	= ABCFHJ	= ACJ	= ACDEF	= BDEGH	= ACGH
DEF	= BDEH	= BFHJ	= ABCF	= ACH	= ACDEFHJ	= BDEGJ	= ACGJ
ABDEFJ	= ADEHJ	= AFH	= CFJ	= BCHJ	= BCDEFH	= ADEG	= BCG
AEFJ	= ABEHJ	= ABDFH	= BCDFJ	= CDHJ	= CEFH	= ABEG	= CDG
ADFJ	= ABDHJ	= ABEFH	= BCEFJ	= CEHJ	= CDFH	= ABDG	= CEG
ADEJ	= ABDEFHJ	= ABH	= BCJ	= CFHJ	= CDEH	= ABDEFG	= CFG
ADEFHJ	= ABDEJ	= ABF	= BCFHJ	= CJ	= CDEF	= ABDEGH	= CGH
ADEF	= ABDEH	= ABFHJ	= BCF	= CH	= CDEFHJ	= ABDEGJ	= CGJ
BCDEFJ	= CDEHJ	= CFH	= AFJ	= ABHJ	= ABDEFH	= CDEG	= ABG
BEFJ	= EHJ	= DFH	= ACDFJ	= ABCDHJ	= ABCEFH	= EG	= ABCDG
BDFJ	= DHJ	= EFH	= ACEFJ	= ABCEHJ	= ABCDFH	= DG	= ABCEG
BDEFGJ	= DEGHJ	= FGH	= ACFGJ	= ABCGHJ	= ABCDEFGH	= DE	= ABC
BDEF	= DEH	= FHJ	= ACF	= ABCH	= ABCDEFHJ	= DEGJ	= ABCGJ
CEFJ	= BCEHJ	= BCDFH	= ABDFJ	= ADHJ	= AEFH	= BCEG	= ADG
CDFJ	= BCDHJ	= BCEFH	= ABEFJ	= AEHJ	= ADFH	= BCDG	= AEG
CDEJ	= BCDEFHJ	= BCH	= ABJ	= AFHJ	= ADEH	= BCDEFG	= AFG
EJ	= BEFHJ	= BDH	= ABCDJ	= ACDFHJ	= ACEH	= BEFG	= ACDFG
EFHJ	= BEJ	= BDF	= ABCDFHJ	= ACDJ	= ACEF	= BEGH	= ACDGH
EF	= BEH	= BDFHJ	= ABCDF	= ACDH	= ACEFHJ	= BEGJ	= ACDGJ
DFHJ	= BDJ	= BEF	= ABCEFHJ	= ACEJ	= ACDF	= BDGH	= ACEGH
AEJ	= ABEFHJ	= ABDH	= BCDJ	= CDFHJ	= CEH	= ABEFG	= CDFG
AEFHJ	= ABEJ	= ABDF	= BCDFHJ	= CDJ	= CEF	= ABEGH	= CDGH
AEF	= ABEH	= ABDFHJ	= BCDF	= CDH	= CEFHJ	= ABEGJ	= CDGJ
ADFHJ	= ABDJ	= ABEF	= BCEFHJ	= CEJ	= CDF	= ABDGH	= CEGH

Treatment combinations of 24 runs 3.2^{5-2} and 3.2^{6-3} designs

d_1^5	d_2^5	d_3^5	d_4^5	d_5^5	d_1^6	d_2^6	d_3^6	d_4^6	d_5^6	d_6^6
1	1	1	1	1	b	a	a	a	a	**b**
ab	b	a	a	a	abc	b	c	b	b	c
ac	ab	b	ab	ab	abd	abc	abc	abc	abc	**abc**
bc	ac	ab	ac	c	bcd	abd	abd	d	abd	d
d	bc	ac	bc	ac	ae	acd	acd	acd	acd	**abd**
ad	abc	bc	abc	abc	be	bcd	bcd	bcd	bcd	**bcd**
bd	ad	d	ad	d	ce	ae	ae	be	be	**e**
abd	bd	abd	bd	ad	abce	be	be	ce	ce	abe
cd	abd	cd	abd	bd	de	ce	abce	abce	abce	**ace**
acd	cd	acd	cd	cd	abde	de	de	de	de	**ade**
bcd	bcd	bcd	acd	acd	acde	acde	abde	abde	abde	**cde**
abcd	abcd	abcd	abcd	bcd	bcde	bcde	acde	acde	bcde	abcde
e	ae	e	e	e	af	bf	af	af	af	**af**
ae	be	abe	be	be	cf	cf	bf	bf	cf	bf
be	abe	ce	abe	abe	df	abcf	abcf	cf	abcf	**cf**
abe	ce	ace	ce	ce	acdf	df	df	abdf	df	**df**
ce	bce	bce	ace	bce	aef	abdf	abdf	acdf	abdf	**acdf**
ace	abce	abce	bce	abce	bef	bcdf	acdf	bcdf	acdf	bcdf
bce	de	de	de	ade	cef	aef	aef	aef	aef	**abef**
abce	bde	ade	ade	bde	abcef	cef	cef	cef	bef	acef
de	abde	bde	bde	abde	def	abcef	abcef	abcef	cef	**bcef**
abde	acde	abde	cde	acde	abdef	def	abdef	def	def	adef
acde	bcde	acde	bcde	bcde	acdef	abdef	acdef	abdef	acdef	**bdef**
bcde	abcde	bcde	abcde	abcde	bcdef	acdef	bcdef	bcdef	bcdef	**abcdef**

Treatment combinations in bold represent embedded designs, 2^{6-2} in d_6^6

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