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Reduction of cost of experimentation using sequential rotatable designs

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Abstract. Box and Hunter (1957) introduced the concept of rotatability as a desirable condition. They also introduced the concept of sequential rotatable designs to reduce the number of design points. Later sequential rotatable designs in non-orthogonal blocks using mixed order rotatable designs were developed by many authors. In this paper it has been shown that the cost of experiment can be reduced by using sequential rotatable designs. The reduction of cost of the designs, one of Myers and Montgomery (2002) and other of Cochran and Cox (1957) is respectively 65% and 30%.

1. Introduction

Box and Hunter (1957) introduced the concept of rotatability and also derived the conditions of rotatable designs of order 'd'. They also stated as the requirement of a good design that a design of order 'd' must be augmented to form a design of order '(d+1)'. Das and Narasimham (1962) constructed sequential rotatable designs in orthogonal blocks which requires too many additional central points. In these designs all the factors appear in the same order in any block. But after the end of each experiment the experimenter may have one of the following decisions:

(a) He may come to a conclusion and no further experimentation is necessary.
(b) The order of some of the factors may be increased.
(c) The interaction terms of some factors may be included without increasing the order of the factors.

To cope with these situations and also to reduce the number of design points, Adhikary and Panda (1982) introduced the mixed-order response surface designs in which the factors are not of the same order. Adhikary and Panda (1983, 1990) constructed sequential designs using non-orthogonal blocks mainly from theoretical point of view, without taking into account of the actual design points of performed experiment. In this study conducted experiments, one of Myers and Montgomery (2002) and other of Cochran and Cox (1957) have been taken into considerations.

Myers and Montgomery (2002) conducted a central composite design (CCD) with SORD on 3 factors in 20 design points. It has been shown that a first-order rotatable design (FORD) could be conducted in 7 design points from the above 20 design points and the lack of fit is insignificant at the first stage. So the experiment could be terminated at this stage and the reduction in cost is 65% (assuming the cost of experiment at any point is constant). Whereas the experiment considered by Cochran and Cox (1957) could be considered in two stages with 7 points in each stage and the experiment could be terminated after second stage by saving 30% of the cost of the experiment.

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2. Illustrations of cost reduction for real examples

The current section presents two real examples, how the cost of an experiment can be reduced by using sequential rotatable designs. For illustrations, two real examples, one from Myers and Montgomery (2002) and the other from Cochran and Cox (1959) have been considered.

Table 2.1.1: The Bread-wrapper experimental data

Trail No.	x_1	x_2	x_3	y
1	-1	-1	-1	6.6
2	1	-1	-1	6.9
3	1	1	-1	7.9
4	-1	-1	1	6.1
5	1	-1	1	9.2
6	-1	1	1	6.8
7	1	1	1	10.4
8	-1	1	-1	7.3
9	-1.682	0	0	9.8
10	1.682	0	0	5.0
11	0	-1.682	0	6.9
12	0	1.682	0	6.3
13	0	0	-1.682	4.0
14	0	0	1.682	8.6
15	0	0	0	10.1
16	0	0	0	9.9
17	0	0	0	12.2
18	0	0	0	9.7
19	0	0	0	9.7
20	0	0	0	9.6

2.1. The Bread-wrapper experiment

An experiment (Myers and Montgomery, 2002) was conducted to study the response surface relating the strength of bread-wrapper stock in grams per square inch (y) to sealing temperature (X_1), cooling bar temperature (X_2), and percent polythylene additive (X_3). The levels of these factors are X_1 = (temp. $-$ 255^{0}F)/30; X_2 = (temp. $-$ 55^{0}F)/9; X_3 = (polythylene $-$ 1.1%)/0.6. Five levels of each factor were involved in the design. The coded and the natural levels are given below.

Coded levels		- 1.682	- 1.000	0.000	1.000	1.682
Natural	X_1	204.5	225	255	285	305.5
levels	X_2	39.9	46	55	64	70.1
(for factors)	X_3	0.09	0.5	1.1	1.7	2.11

The layout of the experiment and the responses are given in (Myers and Montgomery, 2002, p. 324–326). For ready reference, it is displayed in Table 2.1.1.

Table 2.1.2: Estimated parameters of the factors and corresponding t-value

Parameter	Estimated value	t-value
b_0	10.165	
b_1	-1.1036	4.09466
b_2	0.0872	0.35705
b_3	1.020	0.36419
b_{11}	-0.760	2.8969
b_{22}	-1.042	3.9718
b_{33}	-1.148	4.37583*
b_{12}	-0.350	0.4393
b_{13}	-0.500	0.49696
b_{23}	0.1302	0.07454

The fitted second-order response model is
$y = 10.165 - 1.1036x_1 + 0.0872x_2 + 1.020x_3 - 0.760x_1^2 - 1.042x_2^2 - 1.148x_3^2 - 0.350x_1x_2 - 0.500x_1x_3 + 0.130x_2x_3$

Table 2.1.3: ANOVA TABLE

Source of variation	Sum of squares	Degrees of freedom	Mean square	F
Regression (linear and quadratic)	70.3056	9	7.8117	
Lack of fit	6.9044	5	1.3809	1.39
Experimental Error	4.96	5	0.9920	
Total	82.17	19		

At 5% level of significance, F is insignificant. So we conclude that lack of fit is insignificant and b_{33} is significant.

FORD: At first we decide to use a FORD. For this, out of these above 20 design points we take only 7 points where 4 design points are given by $\frac{1}{2}$ of 2^3 experiment with the defining equation I $= -123$ and 3 central points. The design matrix with the response are given below.

x_1	x_2	x_3	y
-1	-1	-1	6.6
-1	1	1	6.8
1	1	-1	7.9
1	-1	1	9.2
0	0	0	9.9
0	0	0	12.2
0	0	0	9.7

Table 2.1.4: ANOVA Table for Bread-wrapper experiment on 7 points using FORD

Source of variation	Sum of squares	Degrees of freedom	Mean square	F
First-order term	7.1675	3	2.3892	1.2379
Lack of fit	21.2411	1	21.2411	11.0058
Error	3.86	2	1.93	
Total	32.2686	6		

Tabulated $F_{0.05;1,2}$=18.51. So the lack of fit is insignificant.

Parameters	b_0	b_1	b_2	b_3
Estimated values	8.8143	-1.025	0.775	1.125

So our required fitted FORD model is
$y = 8.8143 - 1.025x_1 + 0.775x_2 + 1.125x_3$.

Conclusion: For this experiment, a FORD could be conducted with 7 design points from 20 design points. As the lack of fit is insignificant, so the experiment could be terminated at this stage. The reduction of cost is 65% (assuming the cost of experiment at any point is constant).

2.2. Experiment for examining the minor element (Cu, Mo, Fe) effects on the growth of lettuce in water culture (Cochran and Cox, 1957)

An experiment is given in (Cochran and Cox, 1957), to study the effects of the minor elements, namely, copper (Cu), Molybdenum (Mo) and Iron (Fe) on the growth of lettuce in water culture.

A preliminary step in any experiment of this type is to set up the relations between the coded X-scales and the original scales in which the levels are recorded. Effects of minor elements copper (X_1), Molybdenum (X_2) and Iron (X_3) are represented by the models below:
X_1 = 1.472 + 0.84 ln (concentration)
X_2 = 1.472 + 0.84 ln (concentration)
X_3 = 0.5057 + 0.84 ln (concentration).

Five levels of each factor were involved in the design. The coded and natural levels are given below:

Coded levels		- 1.682	- 1.000	0.000	1.000	1.682
Natural	X_1	0.02469	0.52714	0.18286	0.60135	1.35437
levels	X_2	0.02469	0.52714	0.18286	0.60135	1.35437
(for factors)	X_3	0.07395	0.16654	0.54770	1.80119	4.05665

The design matrix (D) and the vector (Y) of responses for 20 points are given in (Cochran and Cox, 1957, p. 350–351, Tables 8A.8, 8A.9). A SORD was fitted with these 20 design points and the lack of fit was found to be insignificant.

The fitted second-order response surface model is
$y = 24.0401 - 4.7420X_1 - 1.1885X_2 + 0.2285X_3 - 5.4262X_1^2 - 0.8837X_2^2 - 5.1593X_3^2 - 1.6738X_1X_2 - 1.1462X_1X_3 - 0.9712X_2X_3$.

2.2.1. FORD

At first we decide to conduct a FORD. For this we take 7 design points in Block 1 of which 4 design points are obtained from a $\frac{1}{2}$ of 2^3 experiment with defining equation I=−123 and remaining 3 points are central points.

Block 1:	X_1	X_2	X_3	Y
	-1	-1	-1	16.44
	-1	1	1	19.90
	1	1	-1	6.92
	1	-1	1	7.83
	0	0	0	22.22
	0	0	0	19.45
	0	0	0	22.76

Parameters	b_0	b_1	b_2	b_3
Estimated values	16.50857	-5.39753	0.6375	1.0925
t-value		6.15815*	0.72734	1.24646

* Significant at 5% level.

Table 2.2.1: ANOVA Table for the growth of lettuce in water calture on 7 points using FORD

Source of variation	Sum of squares	Degrees of freedom	Mean square	F
First-order terms	122.9324	3	40.9774	
Lack of fit	130.2767	1	130.2767	42.3954
Error	6.1458	2	3.0729	
Total	259.3549	6		

So the lack of fit is highly significant. Therefore, the FORD gives a very bad (poor) fit.

2.2.2. FORD-SORD

In Block 1, it is observed that b_1 is significant, but b_2 and b_3 are insignificant. So, first factor X_1 is more important. Thus, it is decided to construct a FORD-SORD design in which the second and third factor will appear in first-order but the first factor will occur in second-order. That is, G_1 = (2,3), G_21 = (1).

BLOCK 2 7 points are taken in Block 2 so that 14 design points will form a FORD-SORD of Adhikary and Panda (1982, 1983). In this block 4 design points are given by $\frac{1}{2}$ of 2^3 with defining equation I = 123 and remaining 3 points are given by

X_1	X_2	X_3
-1.682	0	0
1.682	0	0
0	0	0

Here p^*_{11} = 0.59945 (please see Adhikary and Panda, 1983).

Table 2.2.2: ANOVA Table for the growth of lettuce in water calture on 14 points using FORD-SORD

Source of variation	Sum of squares	Degrees of freedom	Mean square	F
First-order terms	311.8341	3	103.9447	26.1122
Second-order terms	281.4053	3	98.8077	23.5656
Lack of fit	94.8471	4	23.7118	5.9567
Error	11.9421	3	3.9807	
Total	700.0286	13		

Tabulated $F_{0.05;4,3}$ = 9.12, so the lack of fit is insignificant.

Table 2.2.3: Estimated parameters of the factors and corresponding t-value

Parameter	Estimated value	t-value	P-value
b_0	19.4515		
b_1	-4.7415	8.7828*	
b_2	0.5088	0.7212	
b_3	-0.5812	0.8240	
b_{11}	-4.8660	7.9686*	
b_{12}	-1.6737	2.3728	
b_{13}	-1.1462	1.6250	

Tabulated $t_{0.025;3} = 4.303$ and * denotes significant value.

The fitted FORD-SORD response model is
$y = 19.4515 - 4.7415X_1 + 0.5088X_2 - 0.5812X_3 - 4.8660X_1^2 - 1.6737X_1X_2 - 1.1462X_1X_3.$

3. Conclusions

For this experiment a FORD-SORD could be conducted in two blocks with 7 design points in each stage and the experiment could be terminated after 2nd block by saving 30% of the cost of experiment.

Suggestions: Instead of conducting non-sequential SORD or TORD, it is advised to conduct the experiment sequentially with more than one blocks if necessary, starting from a FORD and using mixed order designs in different blocks.

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