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# Semi parametric estimation for autoregressive process with infinite variance

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**Abstract.** In this paper we consider a positive stable AR(1) process. We focus on the tail behavior and we based on the extreme quantile approach to derive an asymptotically normal estimator for the autoregression parameter. A simulation study illustrates the main results.

## 1. Introduction

Suppose $X_1, \ldots, X_n$ be a sequence of dependent random variables with common heavy tail distribution function $F_X$, i.e.,

$$\lim_{v \to \infty} \frac{1 - F_X(vx)}{1 - F_X(v)} = x^{-\alpha}, \tag{1}$$

for all $x > 0$, where $0 < \alpha < 2$ is the so-called tail index. Heavy tailed distributions including the Pareto, Frchet and $\alpha$-stable distributions have been employed in various fields such as internet traffic data, finance, insurance, etc.; see De Haan and Ferreira (12), Embrechts et al. (9) for more applications. Under a slightly stricter condition than (1), one can show that

$$1 - F_X(x) = cx^{-\alpha}(1 + o(1)), \tag{2}$$

for some $c > 0$ as $x \to \infty$. Defining the tail quantile function of $F_X$ as $U_X(t) := F_X^{\leftarrow}(1 - 1/t) = \inf\{s,\ F_X(s) \geq 1 - 1/t\}$, for $t > 1$, so the condition (1) is equivalent to

$$\lim_{t \to \infty} \frac{U_X(tx)}{U_X(t)} = x^{1/\alpha}, \tag{3}$$

for any $x > 0$.
In order to be able to correctly assess the asymptotic non-degenerate behavior of semi parametric estimators of extreme event parameters, we need more than just the first order condition (3). A convenient refinement can be found in the assumption that there exists a constant $\rho < 0$ and a function $g(t)$ with constant sign for large values of $t$, such that

$$\lim_{t \to \infty} \frac{\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}}{g(t)} = x^{-\alpha} \frac{x^\rho - 1}{\rho}, \tag{4}$$

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for all $x > 0$ (see de Haan and Stadtmüller (11)).
A well known estimator for $\alpha$ is the so-called Hill's estimator (15) defined as

$$\widehat{\alpha}_X = \left\{ \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n} \right\}^{-1} \tag{5}$$

where $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$. Solving $1 - F_n(X_{n-k,n}) = cX_{n-k,n}^{-\widehat{\alpha}}$, where $F_n$ is the empirical distribution of $F$, one obtains an estimator for $c$ as $\widehat{c} = \frac{k}{n} X_{n-k,n}^{\widehat{\alpha}}$.
Estimating the tail index for dependent data has received much attention, especially Hill's estimator has been studied (see Hsing (14), Rootzen et al. (21), Resnick and Stărică, (19; 20)).

Lévy-stable distributions are a rich class verified (1) and (4) of was introduced and characterized by Paul Lévy, about 1925 in his study of normalized sums of independent random variables. It is a class of distributions that allow skewness and fat tails; it includes those of Gaussian and Cauchy and has many intriguing mathematical properties. They were suggested like models for many types of physical and economic systems.

The drawback for these distributions is the lack of explicit formulas for their densities allowing their use, except three cases, in which, one knows their formulas (Gaussian, Cauchy and Lévy distributions). Luckily, now there are reliable computer programs to compute Lévy-stable distribution functions, densities and quantiles see for example [33] and [35] . Thus, it is possible to use Lévy-stable models in various practical fields.

We can say that $X \sim S(\alpha, \beta, \gamma, \mu)$ is a stable law if and only if its characteristic function has the form

$$\log \psi(t) = \begin{cases} it\mu - \gamma |t|^{\alpha} \left\{ 1 - i\beta \frac{t}{|t|} \tan \frac{\pi}{2}\alpha \right\}, & \alpha \neq 1 \\ \\ it\mu - \gamma |t| \left\{ 1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \log |t| \right\}, & \alpha = 1, \end{cases}$$

where the characteristic exponent (index of stability, tail exponent) $\alpha \in ]0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a dispersion parameter $\gamma > 0$, a location parameter $\mu \in \mathbf{R}$.

The family of stable laws $S(\alpha, 1, 1, \mu)$ with $0 < \alpha < 1,\ \mu \geq 0$ define positive random variables with support $(\mu, \infty[$, such distributions are used to model the non-negative quantity with impulsive property in finance and physics.

Most applications in statistics need time dependence, let the AR(1) process

$$X_t = a\ X_{t-1} + \varepsilon_t, \tag{6}$$

where $0 < a < 1$ and $\sum_{j=0}^{\infty} a^{j\delta} < \infty$ for $0 < \delta < \alpha$ and $\{\varepsilon_t\} \sim$ i.i.d. which, for simplicity, we take to be positive stable $S(\alpha, 1, 1, \mu),\ 0 < \alpha < 1,\ \mu \geq 0$. From Samorodnitsky and Taqqu (22) these random variables have the following approximate of the tail distribution for $x \to \infty$

$$\mathbf{P}(\varepsilon_t > x) \sim \frac{2}{\pi} \Gamma(\alpha) \sin(\frac{\alpha\pi}{2}) x^{-\alpha}, \tag{7}$$

and $X \sim S(\alpha, 1, 1/(1 - a^{\alpha}), \mu/(1 - a))$. If $k \to \infty$ , $k/n \to 0$ Resnick and Stărică (19) proved that applying the Hill estimator (11) to the observed time series $X_1, X_2, \ldots, X_n$ from the model AR(1) in (6) yields a consistent estimator of $\alpha$. Other estimator for $\alpha$ which is the Pickands estimator (18) is given by :

$$\widehat{\alpha}_X^P = \left[ \frac{1}{\log 2} \log \left( \frac{X_{(n-k+1,n)} - X_{(n-2k+1,n)}}{X_{(n-2k+1,n)} - X_{(n-4k+1,n)}} \right) \right]^{-1} \tag{8}$$

this estimator is consistent for dependent data (see Drees (6–8)). We can estimate the extreme index $\alpha$ by the t-Hill estimator given by :

$$\widehat{\alpha}_X^{t-H} = \left[ \left( \frac{1}{k} \sum_{j=1}^{k} \frac{X_{n-k,n}}{X_{n-j+1,n}} \right)^{-1} - 1 \right]^{-1}, \tag{9}$$

introduced in Fabián and Stehlík (10) which is robust in the i.i.d. case, its consistency for dependent data was proven in Jordanova *et al* (16).

In Davis and Resnick (4), the authors establish the weak limit behavior for the sample analogue to autocorrelation function under an assumption that the innovations have a regularly varying tail with index $\alpha \in ]0, 2[$, so that variance do not exist. These results may then be applied to obtain estimates for $a$. Barlett and McCormick (1) provide estimates for the autoregression parameter $a$ based on the ratio of two sample values chosen with respect to an extreme value criteria, but the two estimators cited above have not a normal limiting behavior. The rest of this paper is organized as follows. In Section 2, the new semi parametric of $a$ is introduced and its properties examined. In Section 3, we compute confidence bounds for $a$ by some simulations and we present a discussion on its robustness. Section 4 is devoted to the proofs.

## 2. Defining the estimator and the main result

We consider the AR(1) process in (6), from Mikosch and Samorodnitsky (17) we have

$$\lim_{x \to \infty} \frac{\mathbf{P}(X_t > x)}{\mathbf{P}(\varepsilon_t > x)} = (1 - a^\alpha)^{-1},$$

thus we have the following approximation

$$\mathbf{P}(X_t > x) \sim \frac{2}{\pi} \Gamma(\alpha) \sin(\frac{\alpha\pi}{2})(1 - a^\alpha)^{-1} x^{-\alpha}.$$

Hence we can estimate $\frac{2}{\pi}\Gamma(\alpha)\sin(\frac{\alpha\pi}{2})(1-a^\alpha)^{-1}$ by $\frac{k}{n} X_{n-k,n}^{\widehat{\alpha}_X}$, where $k = k(n) \to \infty, k/n \to 0$ and

$$\widehat{\alpha}_X = \left[ \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n} \right]^{-1}.$$

It follows that

$$\widehat{a}_n = \left( 1 - \frac{2}{\pi} \Gamma(\widehat{\alpha}) \sin(\frac{\widehat{\alpha}\pi}{2}) \frac{n}{k} X_{n-k,n}^{\widehat{-\alpha_X}} \right)^{1/\widehat{\alpha}_X} \tag{10}$$

We note that from Resnick and Starica (20), we have

$$\sqrt{k}(\widehat{\alpha}_X - \alpha) \xrightarrow{D} N\left(0, \alpha^2 \frac{(1 + a^\alpha)}{(1 - a^\alpha)}\right), \tag{11}$$

provided that as $n \to \infty$

$$\begin{cases} k \to \infty, k/n \to 0 \\ \sqrt{k} g(U(n/k)) \to 0 \\ \text{either} \quad \limsup_{n\to\infty} n/k^{3/2} < \infty \text{ or } \liminf_{n\to\infty} n/k^{3/2} > 0, \end{cases} \tag{12}$$

with $\xrightarrow{D}$ stands for convergence in distribution.

The asymptotic normality of $\widehat{a}_n$ is established in the following theorem.

**Theorem 2.1.** *Suppose (6) and (12) hold then*

$$\frac{\sqrt{k}}{\log(n/k)} (\widehat{a}_n - a) \xrightarrow{D} N\left(0, \frac{\alpha^2(1 + a^\alpha) a^{2-2\alpha} (1 - a^\alpha)^3}{\left(\frac{2}{\pi}\Gamma(\alpha)\sin(\frac{\alpha\pi}{2})\right)^2}\right)$$

## 3. Simulation study

Tail index estimation depends for its accuracy on a precise choice of the sample fraction, i.e., the number of extreme order statistics on which the estimation is based. The most common methods of adaptive choice of the threshold $k$ are based on the minimization of some kind of MSE's estimates :

$$k_{opt} = \arg\min_{k} E(\widehat{\alpha} - \alpha)^2 \tag{13}$$

We mention the pioneering papers by Hall and Welsh (13), Danielsson et al (3) and Beirlant et al (2).

In the first part, to illustrate the performance of our estimator $\widehat{a}_n$, we generate 100 replications of the time series $(X_1, \ldots, X_n)$ of sizes 1000 and 3000, where $X_t$ is an AR(1) process satisfying

$$X_t = aX_{t-1} + \varepsilon_t,\ 1 < t < n, \tag{14}$$

with $0 < a < 1$, and $\varepsilon_t \sim S(\alpha, 1, 1, 4),\ 0 < \alpha < 1$ , note that we use (13) for compute the values of the optimal fraction integer $k_{opt}$, the results are presented in Table 1 and Table 2, where $lb$ and $ub$ stand respectively for lower bound and upper bound of the confidence interval.

Table 1: Performance and 95% confidence intervals for $a = 0.2$

| $\alpha$ | 0.4 | | 0.5 | |
|---|---|---|---|---|
| $n$ | 1000 | 3000 | 1000 | 3000 |
| $\widehat{a}_n$ | 0.2186259 | 0.1863074 | 0.2145797 | 0.2098868 |
| Bias | 0.01862593 | -0.01369255 | 0.01457966 | 0.009886798 |
| RMSE | 0.1479794 | 0.08530847 | 0.09854816 | 0.07335559 |
| $lb$ | 0.1849442 | 0.1673934 | 0.1395198 | 0.1603074 |
| $ub$ | 0.2523076 | 0.2052215 | 0.2896395 | 0.2594662 |
| length | 0.06736343 | 0.03782804 | 0.1501196 | 0.09915873 |

Table 2: Performance and 95% confidence intervals for $a = 0.3$

| $\alpha$ | 0.4 | | 0.5 | |
|---|---|---|---|---|
| $n$ | 1000 | 3000 | 1000 | 3000 |
| $\widehat{a}_n$ | 0.3321577 | 0.3202625 | 0.3047179 | 0.2990464 |
| Bias | 0.03215773 | 0.02026246 | 0.004717884 | -0.0009535607 |
| RMSE | 0.1628001 | 0.1478351 | 0.09987891 | 0.08413913 |
| $lb$ | 0.280616 | 0.2763420 | 0.2348950 | 0.2548524 |
| $ub$ | 0.3836995 | 0.3641830 | 0.3745407 | 0.3432405 |
| length | 0.1030835 | 0.08784099 | 0.1396457 | 0.08838803 |

In the second part in this study, we generate 100 replicates of sizes 1000 from the AR(1) in (14) with $a = 0.2$, we compare the bias and the root mean squared error (RMSE) of the three estimators of $a$ (our estimator $\widehat{a}_n$, $\widehat{a}_n^P$ which we use the Pickands estimator in (8) for estimating the tail index $\alpha$ and $\widehat{a}_n^{t-H}$ which we use the t-Hill estimator in (9)). The results are presented in Table 3. We remark that our estimator $\widehat{a}_n$ has the smallest bias and the $\widehat{a}_n^{t-H}$ estimator has the smallest rmse.

Table 3: Comparison of the estimators of autoregressive parameter

| $\alpha$ | 0.4 | | 0.5 | |
|---|---|---|---|---|
| | Bias | RMSE | Bias | RMSE |
| $\widehat{a}_n$ | 0.00348134 | 0.1261292 | 0.0003560953 | 0.09668266 |
| $\widehat{a}_n^P$ | 0.01371989 | 0.1582915 | 0.02393854 | 0.1773766 |
| $\widehat{a}_n^{t-H}$ | -0.03986046 | 0.05898246 | -0.000885142 | 0.06662949 |

In the third part in this study, we made 100 samples of n = 1000 observations from the AR(1) in (14) for $\alpha = 0.5,\ a = 0.2$. Then we plotted in figure 1 the Hill and the t-Hill plots of the averages of the corresponding estimators together of $\alpha$ for different $k$, we remark that both estimators have similar behavior for fixed number of upper order statistics and show deviations from the true value of the extreme index $\alpha = 0.5$ as $k$ is increased.

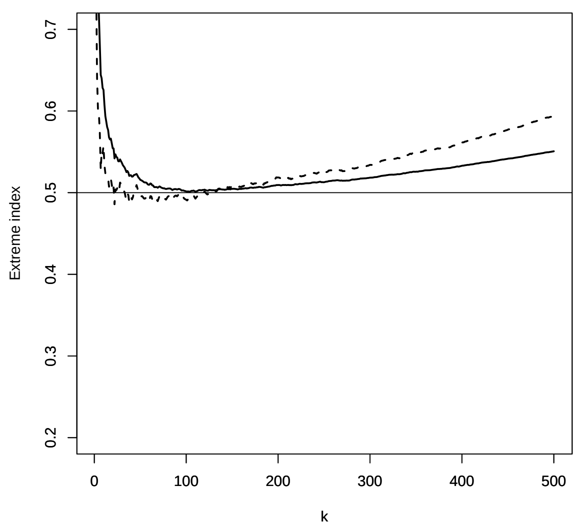


Figure 1: Extreme index estimation by Hill's estimator (solid line) and t-Hill estimator (dotted line), horizontal line is the true value $\alpha = 0.5$.

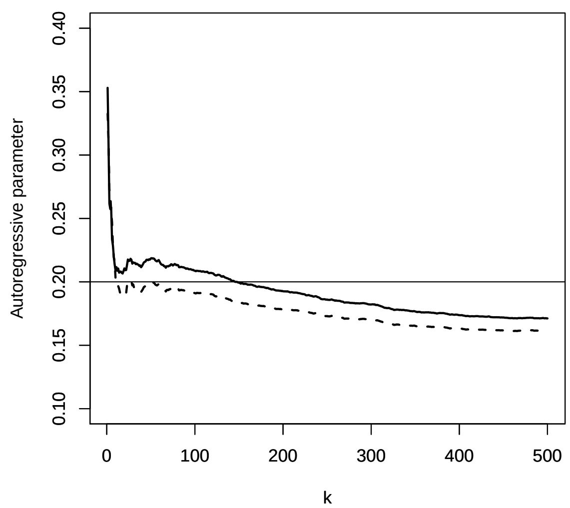


Figure 2: Autoregressive parameter estimation using Hill's estimator (solid line) and t-Hill estimator (dotted line), horizontal line is the true value $a = 0.2$.

In figure 2 we see that both plots of the $\widehat{a}_n$ estimator are not resistant to large observations. Hence our estimator $\widehat{a}_n$ for the autoregressive parameter it not robust using the two estimators Hill and t-Hill of extreme index.

## 4. Proof

Note that

$$\begin{aligned}\tfrac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}-\tfrac{2}{\pi}\Gamma(\alpha)\sin(\tfrac{\alpha\pi}{2})(1-a^{\alpha})^{-1} &= \left(\tfrac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}-\tfrac{k}{n}X_{n-k,n}^{\alpha}\right)\\ &+\tfrac{k}{n}U^{\alpha}(n/k)\left(\tfrac{X_{n-k,n}^{\alpha}}{U^{\alpha}(n/k)}-1\right)\\ &+\tfrac{k}{n}U^{\alpha}(n/k)-\tfrac{2}{\pi}\Gamma(\alpha)\sin(\tfrac{\alpha\pi}{2})(1-a^{\alpha})^{-1}\end{aligned}$$

Using Mean-Value Theorem we find

$$\begin{aligned}\tfrac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}-\tfrac{2}{\pi}\Gamma(\alpha)\sin(\tfrac{\alpha\pi}{2})(1-a^{\alpha})^{-1} &= \left(\tfrac{k}{n}X_{n-k,n}^{\alpha}(\widehat{\alpha}_X-\alpha)\log X_{n-k,n}\right)(1+o_P(1))\\ &+\tfrac{k}{n}U^{\alpha}(n/k)\left(\tfrac{X_{n-k,n}^{\alpha}}{U^{\alpha}(n/k)}-1\right)\\ &+\tfrac{k}{n}U^{\alpha}(n/k)-\tfrac{2}{\pi}\Gamma(\alpha)\sin(\tfrac{\alpha\pi}{2})(1-a^{\alpha})^{-1}\end{aligned}$$

From Drees (8) we have $\dfrac{X_{n-k,n}^{\alpha}}{U^{\alpha}(n/k)}=1+O_P(1/\sqrt{k})$ and using (11) we obtain

$$\frac{\sqrt{k}}{\log(n/k)}\left(\frac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}-\frac{2}{\pi}\Gamma(\alpha)\sin(\frac{\alpha\pi}{2})(1-a^{\alpha})^{-1}\right)\xrightarrow{D}N\left(0,\alpha^4\frac{(1+a^{\alpha})}{(1-a^{\alpha})}\right)$$

By application the delta method, it follows that the estimator $\widehat{a}_n$ defined in (10) satisfies the following result

$$\frac{\sqrt{k}}{\log(n/k)}(\widehat{a}_n-a)\xrightarrow{D}N\left(0,\alpha^4\frac{(1+a^{\alpha})}{(1-a^{\alpha})}\left[f'\left(\frac{2}{\pi}\Gamma(\alpha)\sin(\frac{\alpha\pi}{2})(1-a^{\alpha})^{-1}\right)\right]^2\right),$$

where $f(x)=\left(1-\frac{\frac{2}{\pi}\Gamma(\alpha)\sin(\frac{\alpha\pi}{2})}{x}\right)^{1/\alpha}$. This completes the proof of Theorem 2.1.

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