# Relationships for Moments of Generalized Order Statistics from a General Class of Distributions

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**Abstract.** In this paper we derive some general recurrence relations between the moments of generalized order statistics from a general class of distributions, thus unifying the earlier results in this direction due to several authors.

## 1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Let $X_{j:n}$ denote the $j^{th}$ order statistic of a sample $(X_1, X_2, \ldots, X_n)$. Assume that $k > 0$, $n \in N$, $n \geq 2$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + n - r + M_r > 0 \ \forall \ r \in \{1, 2, \ldots, n-1\}$. If the random variables $U(r, n, \tilde{m}, k), r = 1, 2, \ldots, n$, possess a joint pdf of the form

$$f^{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1, u_2, \ldots, u_n) = k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\Big)(1-u_n)^{k-1}, \tag{1}$$

on the cone $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n < 1$ of $R^n$, then they are called uniform generalized order statistics.

Generalized order statistics (gos) based on some distribution function $F$ are defined by means of quantile transformation $X(r, n, \tilde{m}, k) = F^{-1}(U(r, n, \tilde{m}, k)), \ r = 1, 2, \ldots, n$. Then $X(r, n, \tilde{m}, k), \ r = 1, 2, \ldots, n$, are called generalized order statistics if their joint pdf is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n) = k\Big(\prod_{r=1}^{n-1} \gamma_r\Big)\Big(\prod_{i=1}^{n-1} \Big(1-F(x_i)\Big)^{m_i} f(x_i)\Big)\Big(1-F(x_n)\Big)^{k-1} f(x_n), \tag{2}$$

where $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

For convenience, let us define $X(0, n, \tilde{m}, k) = 0$. It can be seen that for $m_1 = \cdots = m_{n-1} = 0$, $k = 1, i.e., \gamma_i = n - i + 1; \ 1 \leq i \leq n-1$, we obtain the joint pdf of the ordinary order statistics. In a similar manner, choosing the parameters appropriately, some other models such as $k^{th}$ upper record

values ($m_1 = \ldots = m_{n-1} = -1$, $k \in N$, *i.e.*, $\gamma_i = k$, $1 \le i \le n - 1$), sequential order statistics ($m_r = (n-r+1)\alpha_r - (n-r)\alpha_{r+1} - 1$; $r = 1, \ldots, n-1$, $k = \alpha_n$; $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$, *i.e.*, $\gamma_i = (n-i+1)\alpha_i$; $1 \le i \le n-1$), order statistics with non-integral sample size ($m_1 = \ldots = m_{n-1} = 0$, $k = \alpha - n + 1$ *with* $n - 1 < \alpha \in R$, *i.e.*, $\gamma_i = \alpha - i + 1; 1 \le i \le n - 1$) [Rohatgi and Saleh (1988), Saleh et al. (1975)], Pfeifer's record values ($m_r = \beta_r - \beta_{r+1} - 1$, $r = 1, \ldots, n - 1$ *and* $k = \beta_n$; $\beta_1, \beta_2, \ldots, \beta_n > 0$, *i.e.*, $\gamma_i = \beta_i; 1 \le i \le n - 1$) and progressively type-II right censored order statistics ($m_i \in N_0$, $k \in N$) can be obtained [cf. Kamps (1995a,b), Kamps and Cramer (2001)].

The joint pdf of the first $r$ generalized order statistics is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1, x_2, \ldots, x_r) = c_{r-1}\Big(\prod_{i=1}^{r-1}\big(1 - F(x_i)\big)^{m_i} f(x_i)\Big) \cdot \big(1 - F(x_r)\big)^{k+n-r+M_r-1} f(x_r),$$

(3)

$$F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_r < F^{-1}(1).$$

We may consider two cases:
**Case I:** $m_1 = m_2 = \ldots = m_{n-1} = m$.
**Case II:** $\gamma_i \ne \gamma_j$, $i \ne j$, $i, j = 1, 2, \ldots, n - 1$.
For **Case I**, the $r^{th}$ generalized order statistic will be denoted by $X(r, n, m, k)$. The pdf of $X(r, n, m, k)$ is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}\big(1 - F(x)\big)^{\gamma_r - 1} f(x) g_m^{r-1}\big(F(x)\big), \qquad x \in R,$$

(4)

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \le r < s \le n$, is given by

$$f^{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\big[1 - F(x)\big]^m f(x) g_m^{r-1}\big(F(x)\big)$$
$$\cdot \big[h_m(F(y)) - h_m(F(x))\big]^{s-r-1}\big[1 - F(y)\big]^{\gamma_s - 1} f(y), \qquad x < y,$$

(5)

where

$$c_{r-1} = \prod_{j=1}^{r}\gamma_j, \qquad \gamma_j = k + (n - j)(m + 1), \qquad r = 1, 2, \ldots, n,$$

$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0, 1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1 - x)^{m+1}, & \text{if} \quad m \ne -1 \\ -\log(1 - x), & \text{if} \quad m = -1 \end{cases}$$

(cf. Kamps, 1995a,b).
Let $\mu_{(r,n,m,k)}^{(i)}$ denote the $i^{th}$ moment of the $r^{th}$ generalized order statistic $X(r, n, m, k)$. Similarly, $\mu_{(r,s,n,m,k)}^{(i,j)}$ denotes the $(i, j)^{th}$ product moment of the $r^{th}$ and $s^{th}$ generalized order statistics.
For **Case II**, the $r^{th}$ generalized order statistic will be denoted by $X(r, n, \tilde{m}, k)$. The pdf of $X(r, n, \tilde{m}, k)$ is given by

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x) \sum_{i=1}^{r} a_i(r)(1 - F(x))^{\gamma_i - 1}, \qquad x \in R,$$

(6)

and the joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k), 1 \le r < s \le n$, is given by

$$f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = c_{s-1}\Big\{ \sum_{i=r+1}^{s} a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}\Big\}\Big\{ \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\Big\}$$

$$\cdot\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}, \quad x < y, \tag{7}$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + n - i + M_i, \qquad r = 1, 2, \ldots, n,$$

$$a_i(r) = \prod_{j(\neq i)=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n$$

$$\text{and} \quad a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \le i \le s \le n,$$

(cf. Kamps and Cramer (2001)).

Further, it can be easily proved that

$$a_i(r) = (\gamma_{r+1} - \gamma_i)\, a_i(r+1),$$

$$c_{r-1} = \frac{c_r}{\gamma_{r+1}},$$

$$\text{and} \quad \sum_{i=1}^{r+1} a_i(r+1) = 0. \tag{8}$$

Also, for $m_1 = m_2 = \ldots = m_{n-1} = m$, it can be shown that

$$\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} = \frac{(1-F(x))^{\gamma_r}}{(r-1)!} g_m^{r-1}(F(x)) \tag{9}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i} = \frac{1}{(s-r-1)!}\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_s}\Big(\frac{1}{1-F(x)}\Big)^{(m+1)(s-r-1)}$$

$$\cdot\Big[h_m(F(y)) - h_m(F(x))\Big]^{s-r-1}. \tag{10}$$

In the above case II, let $\mu_{(r,n,\tilde{m},k)}^{(p)}$ denote the $p^{th}$ moment of the $r^{th}$ generalized order statistic $X(r, n, \tilde{m}, k)$. Similarly, $\mu_{(r,s,n,\tilde{m},k)}^{(p,q)}$ denotes the $(p, q)^{th}$ product moment of the $r^{th}$ and $s^{th}$ generalized order statistics.

Burkschat et al. (2003) defined generalized order statistics in an alternative way as:
$X(r, n, \tilde{m}, k) \sim F^{-1}(1 - W_r)$; $r = 1, 2, \ldots, n$, where $W_r = \prod_{j=1}^{r} B_j$, $B_j$ being independent random variables distributed as $Beta(\gamma_j, 1)$ having c.d.f. $F(t) = t^{\gamma_j}$, $t \in [0, 1]$.
Several authors like Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001), Ahmad and Fawzy (2003), Ahmad (2007), Khan, et al. (2007), Khan et al. (2010) and Saran and Pandey (2004, 2009) have done some work on generalized order statistics. In this paper, we have established some recurrence relations for single and product moments of generalized order statistics from a general class of distributions satisfying the characterizing differential equation:

$$\Big(\sum_{u=0}^{w} \alpha_u x^u\Big) f(x) = \Big(\sum_{v=0}^{z} \beta_v x^v\Big)(1 - F(x)), \quad -\infty < x < \infty, \tag{11}$$

where $w$ and $z$ are integers and $\alpha's$ and $\beta's$ are arbitrary real constants.

## 2. Recurrence Relations for Single Moments

In this section, we shall establish several recurrence relations for single moments of generalized order statistics from a general class of distributions satisfying the relation given in equation (11).
**Case I:** $m_1 = m_2 = \ldots = m_{n-1} = m$.

**Theorem 2.1.** *For a positive integer* $k \geq 1$, $n \in N$, $m \in Z, 1 \leq r \leq n$ *and* $i = 0, 1, 2, \ldots$,

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,m,k)}^{(i+u)} = \gamma_r \sum_{v=0}^{z} \frac{\beta_v}{(i+v+1)} \Big[ \mu_{(r,n,m,k)}^{(i+v+1)} - \mu_{(r-1,n,m,k)}^{(i+v+1)} \Big]. \tag{12}$$

*Proof.* For $1 \leq r \leq n$ and $i = 0, 1, 2, \ldots$, we have from eq. (4),

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,m,k)}^{(i+u)} = \frac{c_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} x^i [1 - F(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] \Big\{ \sum_{u=0}^{w} \alpha_u x^u f(x) \Big\} dx,$$

which on utilizing eq. (11), yields

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,m,k)}^{(i+u)} = \frac{c_{r-1}}{(r-1)!} \sum_{v=0}^{z} \beta_v \int_{-\infty}^{\infty} x^{i+v} [1 - F(x)]^{\gamma_r} g_m^{r-1}[F(x)] dx. \tag{13}$$

Integrating by parts in equation (13), treating $x^{i+v}$ for integration and rest of the integrand for differentiation, we get

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,m,k)}^{(i+u)} = \frac{c_{r-1}}{(r-1)!} \sum_{v=0}^{z} \frac{\beta_v}{(i+v+1)} \Big[ \gamma_r \int_{-\infty}^{\infty} x^{i+v+1} [1 - F(x)]^{\gamma_r - 1} \cdot$$

$$\cdot g_m^{r-1}[1 - F(x)] f(x) dx$$

$$-(r-1) \int_{-\infty}^{\infty} x^{i+v+1} [1 - F(x)]^{\gamma_r - 1 - 1} g_m^{r-2}[1 - F(x)] f(x) dx \Big].$$

Simplifying the above terms and rearranging them, we obtain the relation in (12) on using (4). □

**Remark 2.2.** *For* $r = 1$, *from the relation in eq. (12) we get the term* $\mu_{(0,n,m,k)}^{(i+v+1)}$, *the value of which will be zero as we have defined* $X(0, n, m, k) = 0$.

**Remark 2.3.** *Under the assumptions of Theorem 2.1, with* $k = 1, m = 0$, *we shall deduce the recurrence relation for single moments of ordinary order statistics from the general class of distributions (11) which is in agreement with the corresponding result obtained by Saran and Pushkarna (2010) as a particular case.*

**Remark 2.4.** *Putting* $k = 0, m = -1$ *in Theorem 2.1, we obtain the recurrence relation for single moments of record values from the general class of distributions (11).*

**Case II:** $\gamma_i \neq \gamma_j$, $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

**Theorem 2.5.** *For $n \in N$, $1 \le r \le n$, $k \ge 1$ and $p = 0, 1, 2, \ldots$,*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = \gamma_r \sum_{v=0}^{z} \frac{\beta_v}{p+v+1} \left[ \mu_{(r,n,\tilde{m},k)}^{(p+v+1)} - \mu_{(r-1,n,\tilde{m},k)}^{(p+v+1)} \right]. \tag{14}$$

*Proof.* For $1 \le r \le n$ and $p = 0, 1, 2, \ldots$, we have from (6),

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = c_{r-1} \int_{-\infty}^{\infty} x^p \sum_{i=1}^{r} a_i(r)[1 - F(x)]^{\gamma_i - 1} \left\{ \sum_{u=0}^{w} \alpha_u x^u f(x) \right\} dx,$$

which on using (11) gives

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = c_{r-1} \sum_{v=0}^{z} \beta_v \int_{-\infty}^{\infty} x^{p+v} \sum_{i=1}^{r} a_i(r)[1 - F(x)]^{\gamma_i} dx.$$

Integrating by parts treating $x^{p+v}$ for integration and rest of the integrand for differentiation, utilizing (8) and simplifying, we get the required result (14). $\square$

**Remark 2.6.** *For $r = 1$, from the relation in (14) we get the term $\mu_{(0,n,\tilde{m},k)}^{(p+v+1)}$ whose value will be zero as we have defined $X(0, n, \tilde{m}, k) = 0$.*

**Remark 2.7.** *Putting $m_i = m_j = m$ in (6) and using (9), the recurrence relation for single moments of generalized order statistics from the general class of distributions (11) for Case I, i.e., when $m_1 = m_2 = \ldots = m_{n-1} = m$, as obtained in Theorem 2.1, can be deduced from Theorem 2.5 with $\tilde{m}$ replaced by $m$.*

## 3. Recurrence Relations for Product Moments

**Case I:** $m_1 = m_2 = \ldots = m_{n-1} = m$.

**Theorem 3.1.** *For a positive integer $k \ge 1$, $n \in N$, $m \in Z$, $1 \le r \le s - 2 < n$ and $i, j = 0, 1, 2, \ldots$,*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,m,k)}^{(i,j+u)} = \gamma_s \sum_{v=0}^{z} \frac{\beta_v}{(j+v+1)} \left[ \mu_{(r,s,n,m,k)}^{(i,j+v+1)} - \mu_{(r,s-1,n,m,k)}^{(i,j+v+1)} \right], \tag{15}$$

*and, for $1 \le r \le n - 1$, $i, j = 0, 1, 2, \ldots$,*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,r+1,n,m,k)}^{(i,j+u)} = \gamma_{r+1} \sum_{v=0}^{z} \frac{\beta_v}{(j+v+1)} \left[ \mu_{(r,r+1,,n,m,k)}^{(i,j+v+1)} - \mu_{(r,n,m,k)}^{(i+j+v+1)} \right]. \tag{16}$$

*Proof.* From (5) we have, for $1 \le r \le s - 2 < n$ and $i, j = 0, 1, 2, \ldots$,

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,m,k)}^{(i,j+u)} = \frac{c_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} x^i [1 - F(x)]^m f(x) g_m^{r-1}[F(x)] I(x) dx, \tag{17}$$

where

$$I(x) = \int_x^{\infty} y^j \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [1 - F(y)]^{\gamma_s - 1} \left\{ \sum_{u=0}^{w} \alpha_u y^u f(y) \right\} dy.$$

Utilizing eq. (11), we get

$$I(x) = \sum_{v=0}^{z} \beta_v \int_x^{\infty} y^{j+v} \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [1 - F(y)]^{\gamma_s} dy. \tag{18}$$

Integrating by parts in (18) treating $y^{j+v}$ for integration and rest of the integrand for differentiation, and then substituting the resulting value of $I(x)$ in (17) and simplifying we get the relation in (15). In a similar manner, (16) can be easily established. $\square$

**Remark 3.2.** *Under the assumptions of Theorem 3.1, with $k = 1, m = 0$ we shall obtain the recurrence relations for product moments of ordinary order statistics from the general class of distributions* (11)*, which are in agreement with the corresponding results obtained by Saran and Pushkarna (2010).*

**Remark 3.3.** *Putting $k = 0, m = -1$ in Theorem 3.1, we obtain the recurrence relations for product moments of record values from the general class of distributions* (11)*.*

**Case II:** $\gamma_i \neq \gamma_j, i \neq j, \; i, j = 1, 2, \ldots, n - 1.$

**Theorem 3.4.** *For $n \in N, \; 1 \leq r < s \leq n, \; n \geq 2, \; k \geq 1$ and $p, q = 0, 1, 2, \ldots,$*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,\tilde{m},k)}^{(p,q+u)} = \gamma_s \sum_{v=0}^{z} \frac{\beta_v}{(q+v+1)} \left[ \mu_{(r,s,n,\tilde{m},k)}^{(p,q+v+1)} - \mu_{(r,s-1,n,\tilde{m},k)}^{(p,q+v+1)} \right]. \tag{19}$$

*Proof.* For $1 \leq r < s \leq n$ and $p, q = 0, 1, 2, \ldots,$ from (7) we have,

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,\tilde{m},k)}^{(p,q+u)} = c_{s-1} \int_{-\infty}^{\infty} x^p \left\{ \sum_{i=1}^{r} a_i(r) \left(1 - F(x)\right)^{\gamma_i} \right\} \frac{f(x)}{1 - F(x)} I(x) dx, \tag{20}$$

where

$$I(x) = \int_x^{\infty} y^q \left\{ \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i} \right\} \left\{ \sum_{u=0}^{w} \alpha_u y^u f(y) \right\} \frac{1}{1 - F(y)} dy.$$

Utilizing Eq. (11), we get

$$I(x) = \sum_{v=0}^{z} \beta_v \int_x^{\infty} y^{q+v} \left\{ \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i} \right\} dy. \tag{21}$$

Integrating by parts treating $y^{q+v}$ for integration and rest of the integrand for differentiation, we get

$$I(x) = \frac{1}{(q+v+1)} \int_x^{\infty} y^{q+v+1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \gamma_i \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i - 1} f(y) dy.$$

Substituting the above expression for $I(x)$ in (20), using (9) and simplifying, we derive the relation in (19). $\square$

**Remark 3.5.** *Putting $m_i = m_j = m$ in (6) and using (8) and (9), the recurrence relation for product moments of generalized order statistics from a general class of distributions* (11) *for Case I, i.e., when $m_1 = m_2 = \ldots = m_{n-1} = m$, as obtained in Theorem 3.1, can be deduced from Theorem 3.4 with $\tilde{m}$ replaced by $m$.*

**Remark 3.6.**

$$\text{Setting } \alpha_0 = \beta, \qquad\qquad \alpha_u \;\; (u \geq 1) = \begin{cases} 0, & if & u \neq \tau \\ 1, & if & u = \tau, \end{cases}$$

*and*

$$\beta_v = \begin{cases} 0, & if & v \neq \tau - 1 \\ \lambda\tau, & if & v = \tau - 1, \end{cases}$$

*we observe that* (11) *reduces to*

$$f(x)(\beta + x^\tau) = \lambda\tau(1 - F(x))x^{\tau-1}, \tag{22}$$

*which is the characterizing differential equation for Burr type XII distribution with p.d.f. in the form*

$$f(x) = \lambda \tau \beta^\lambda \frac{x^{\tau-1}}{(\beta + x^\tau)^{\lambda+1}}, \quad x > 0, \ \beta > 0, \ \lambda > 0, \ \tau > 0 \tag{23}$$

*(cf. Saran and Pandey (2009) and Khan and Khan (1987)).*

*Thus recurrence relations for single and product moments of generalized order statistics (for both cases) from Burr distribution defined above can be obtained as particular cases of the results obtained in Sections 2 and 3, which are in agreement with the results obtained by Pawlas and Szynal (2001) and Saran and Pandey (2009). It is worth mentioning here that similar recurrence relations for single and product moments of generalized order statistics (for both cases) for Lomax, Weibull, Weibull-gamma, Weibull-exponential, log logistic, exponential, generalized exponential, Rayleigh, generalized Rayleigh and generalized Pareto distributions can be obtained from the above results because these distributions are special cases of Burr type XII distribution by taking different values of the parameters involved as discussed in Tadikamalla (1980) and Saran and Pushkarna (2001).*

**Remark 3.7.**

*Setting*
$$\alpha_u \ \ (u \geq 1) = \begin{cases} 0, & if \quad u \neq 0 \\ 1, & if \quad u = 0, \end{cases}$$

$$\beta_0 = \lambda, \quad \beta_1 = \nu \ \ and \ \ \beta_v = 0 \ \forall \ v \geq 2,$$

*we observe that* (11) *reduces to*

$$f(x) = (\lambda + \nu x)(1 - F(x)),$$

*which is the characterizing differential equation for linear-exponential distribution with p.d.f. in the form*

$$f(x) = (\lambda + \nu x) \exp^{-\left(\lambda x + \frac{\nu x^2}{2}\right)}, \qquad 0 \leq x < \infty, \ \ \lambda, \nu > 0. \tag{24}$$

*One can deduce the recurrence relations for single and product moments of generalized order statistics from linear exponential distribution as particular cases of the results obtained in Sections 2 and 3, which are in agreement with the corresponding results of Saran and Pandey (2004, 2009).*

**Remark 3.8.**

*Setting*
$$\alpha_0 = (1 + \theta), \ \ \alpha_1 = \theta, \ \ \ \alpha_u = 0 \ \ \ \forall \ u \geq 2,$$

$$\beta_0 = \beta_1 = \theta^2 \ \ and \ \ \beta_v = 0 \ \forall \ v \geq 2,$$

*we observe that* (11) *reduces to*

$$(1 + \theta + \theta x)f(x) = \theta^2(1 + x)(1 - F(x)),$$

*which is the characterizing differential equation for Lindley distribution [cf. Lindley (1958), Athar et al. (2014)] with p.d.f. in the form*

$$f(x) = \frac{\theta^2}{1 + \theta}(1 + x)e^{-\theta x}, \ \ 0 \leq x < \infty, \ \ \theta > 0. \tag{25}$$

*Hence, one can deduce the recurrence relations for single and product moments of generalized order statistics from Lindley distribution as particular cases of the results derived in Sections 2 and 3.*

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