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Asymptotic behavior of non-identical multivariate mixture

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Abstract. Finite mixture models (FMM) have received increasing attention in recent years and have proven to be useful in modeling heterogeneous data with a finite number of unobserved sub-population. FMM are a powerful and flexible tool for modeling complex data. In this article, we study the asymptotic distribution of the appropriately linear normalized coordinatewise maximum and minimum under multivariate FMM from independent, but not obligatory identically distributed random vectors. We obtain sufficient conditions for this weak convergence, as well as the limit forms. Sufficient conditions for this convergence when the components of the mixture have different linear normalization have been derived. Illustrative examples are provided, which lend further support to our theoretical results.

1. Introduction

In the last few years, much attention has been paid to the study of the order statistics from independent, but non-identically distributed (INID) random variables (rvs). An earlier result and perhaps the most important one for the asymptotic behaviour of order statistics based on INID rvs is due to Mejzler [1949, 1953], who proved that a non-degenerate distribution function (df) $H(x)$ will be, under some uniformity assumptions (UAs), a limiting distribution of the suitably normalized maximum from some sequence of independent rvs if and only if

$$\begin{cases} \log H(x) \text{ is concave, or} \\ \omega(H) = \sup\{x : H(x) < 1\} < \infty \text{ and } \log H(\omega(H) - e^{-x}) \text{ is concave}, x > 0, \text{ or} \\ \alpha(H) = \inf\{x : H(x) > 0\} > -\infty \text{ and } \log H(\alpha(H) + e^{x}) \text{ is concave}, \ x > 0. \end{cases} \tag{1.1}$$

And a non-degenerate df $L(x)$ will be, under some UAs, a limiting distribution of the suitably normalized minimum from some sequence of independent rvs if and only if

$$\begin{cases} \log[1 - L(x)] \text{ is concave, or} \\ \omega(L) \text{ is finite and } \log\{1 - L[\omega(L) - e^{-x}]\} \text{ is concave}, x > 0, \text{ or} \\ \alpha(L) \text{ is finite and } \log\{1 - L[\alpha(L) + e^{x})]\} \text{ is concave}, \ x > 0. \end{cases} \tag{1.2}$$

The reader can refer to Weissman (1975a, 1975b) and Galambos (1978, 1987).

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Consider n m-dimensional distributed random vectors $X_j = (X_{1,j}, X_{2,j}, ..., X_{m,j}), j = 1, 2, ..., n$, with respective dfs $F_j(x) = F_j(x_1, x_2, ..., x_m)$. The coordinatewise maximum vector Z_n is defined by $Z_n = (Z_{1,n}, Z_{2,n}, ..., Z_{m,n})$, where $Z_{s,n} = \max\{X_{s,1}, X_{s,2}, ..., X_{s,n}\},\ s = 1, 2, ..., m$. The df of the vector Z_n can be explicitly written by

$$H_n(x) = P(Z_n \leq x) = \prod_{j=1}^{n} F_j(x).$$

Moreover, the df of the marginal vector $Z_{s_k,n} = (Z_{s_1,n}, Z_{s_2,n}, ..., Z_{s_k,n})$, where $s_k = (s_1, s_2, ..., s_k)$, $1 \leq s_1 < s_2 < ... < s_k \leq m$, $1 \leq k \leq m$, is given by $H_{s_k,n}(x_{s_k}) = \prod_{j=1}^{n} F_{s_k,j}(x_{s_k})$, where $F_{s_k,j}(x_{s_k}) = P(X_{s_1,j} \leq x_{s_1}, X_{s_2,j} \leq x_{s_2}, ..., X_{s_k,j} \leq x_{s_k})$ is the corresponding marginal of the df $F_j(x)$. We adopt here, and throughout this paper, the convention that the components of the numerical vectors $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$ are signified by a subscript and basic arithmetical operations are always meant componentwise. Thus, $x \leq y$ means $x_s \leq y_s,\ s = 1, 2, ..., k,\ x + y = (x_1 + y_1, ..., x_k + y_k),\ x\ y = (x_1 y_1, ..., x_k y_k)$ and $x/y = (x_1/y_1, ..., x_k/y_k)$. Let α be the left end vector and ω be the right end vector. On the other hand, the special vectors $0 = (0, 0, ..., 0)$ and $\pm\infty = (\pm\infty, \pm\infty, ..., \pm\infty)$ will be used. Finally, we use the abbreviations $F_{s_m,j}(x_{s_m}) = F_j(x)$ and $F_{s_1,j}(x_{s_1}) = F_{s,j}(x_s)$, if $s_1 = (s),\ 1 \leq s \leq m$.

Recently, Barakat *et al.* (2012) discussed the limit behaviour of multivariate maximum from n m-dimensional INID random vectors, who proved that, under some UAs, which are the kind of restrictions on the individual terms $F_1(x), F_2(x), ...$, as well as on some sequences $g_n(x) = a_n x + b_n$ of linear transformations, where $\underline{a}_n = (a_{1,n}, a_{2,n}, ..., a_{m,n}) > 0$ and $\underline{b}_n = (b_{1,n}, b_{2,n}, ..., b_{m,n})$, the limiting df for the vector $\underline{Z}_n$ is given by

$$H(x) = \exp(-u(t; x)),$$

where

$$u(t; x) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{1 \leq s_1 < s_2 < ... < s_k \leq m} u_{s_k}(t; x_{s_k})$$

and

$$u_{s_k}(t; x_{s_k}) = \lim_{n \to \infty} \sum_{j=1}^{[nt]} [1 - F_{s_k,j}(g_n(x_{s_k}))],\ k = 1, 2, ..., m,$$

exists and is finite for all $0 < t \leq 1$, whenever it is finite for $t = 1$.

1.1. Finite mixture models (FMM)

Most statistical methods assume that you have a sample of observations, all of which come from the same distribution, and that you are interested in modeling that one distribution. If you actually have data from more than one distribution with no information to identify which observation goes with which distribution, standard models wont help you. However, FMM might come to the rescue. They use a mixture of parametric distributions to model data, estimating both the parameters for the separate distributions and the probabilities of component membership for each observation.

Consider finite r different populations with multivariate dfs $F_{j,i}(x),\ F_{j,2}(x), ...,\ F_{j,r}(x)$ from which we have collected the sample data, then the combined r components sample data has the following multivariate df

$$F_j(x) \stackrel{d}{=} \sum_{i=1}^{r} p_i F_{j,i}(x), \tag{1.3}$$

where

$$F_{j,i}(x) = 1 - \sum_{k=1}^{m} (-1)^{k+1} \sum_{1 \leq s_1 < s_2 < ... < s_k \leq m} G_{s_k,j,i}(x_{s_k}),$$

$G_{s_k,j,i}(x_{s_k}) = P(X_{s_1,j,i} > x_{s_1}, X_{s_2,j,i} > x_{s_2}, ..., X_{s_k,j,i} > x_{s_k})$ is the survival function of the df $F_{s_k,j,i}(x_{s_k})$. In this mixture, the ith component (df of the ith subpopulation) is $F_{j,i}(x)$, the mixing proportions $p_i > 0, i = 1, ..., r$ are such that $\sum_{i=1}^{r} p_i = 1$ and " $\stackrel{d}{=}$ " denotes the equality in distribution.

In addition, FMM provide the following features:

- FMM provide a flexible framework for analyzing a variety of data.
- FMM provide parametric alternatives that describe the unknown distributions in terms of mixtures of known distributions.
- FMM enable you to assess the probabilities of events or simulate draws from the unknown distribution the same way you do when your data are from a known distribution.
- FMM provide a parametric modeling approach to one-dimensional cluster analysis.
- FMM provide a mechanism that can account for unobserved heterogeneity in the data.
- FMM are often used to study data from a population that is suspected to be composed of a number of homogeneous subpopulations. For example, mixture distributions are used routinely to accommodate the genetic heterogeneity thought to underlie many human diseases.
- FMM have been not only widely applied to classification, clustering, and pattern identification problems for independent data, but could also be used for longitudinal data to describe differences

The reader can refer to Everitt and Hand (1981), Titterington et al. (1985), Lindsay (1995) and McLachlan and Basford (1988). Finally, AL-Hussaini and El-Adll (2004) obtained the asymptotic distribution of normalized maximum under finite mixture models.

1.2. UAs under FMM (UAM)

In this subsection, we extend the UAs of Barakat et al. (2012) from multivariate to finite multivariate mixture. We say that a sequence $\{F_{j,i}(x)\}$ of dfs and the sequences $g_n(x)$ of normalizing constants satisfy the UAM for maximum vector Z_n if, for $i = 1, 2, ..., r$

$$\lim_{n\to\infty} \max_{1\leq j\leq n} \{[1 - F_{s_k,j,i}(g_n(x_{s_k}))],\ g_n(x_{s_k}) > \alpha_{s_k}(F_{s_k,j,i})\} = 0,\ k = 1, 2, ..., m, \qquad (C_1)$$

and for any fixed number $0 < t \leq 1$,

$$\lim_{n\to\infty} \sum_{j=1}^{[nt]} [1 - F_{s_k,j,i}(g_n(x_{s_k}))] = u_{s_k,i}(t; x_{s_k}),\ k = 1, 2, ..., m, \qquad (C_2)$$

exists and is finite for all $0 < t \leq 1$, whenever it is finite for $t = 1$. Under UAMs (C_1, C_2) for maximum the limit df of the maximum for the ith subpopulation of the mixture is given by $H_i(x) = \exp(-u_i(t; x))$. Here, $x > \alpha(H_i(x))$.

Remark 1.1. *For a sequence $T_n(x_{s_k})$ of linear transformations, the UAM for the multivariate minimum (W_n) under finite mixture is similarly defined for the maximum except that $F_{s_k,j,i}(x_{s_k})$ is to be replaced by $1 - F_{s_k,j,i}(x_{s_k})$ in both limit relations (C_1, C_2), and $x < \omega(L_i(x))$, where $L_i(x)$ is the limiting distribution of $T_n^{-1}(W_n)$.*

Remark 1.2. *Since, any results of minimum can be easily deduced from the corresponding results of maximum, the emphasis of our study will be mainly on the maximum.*

We will close this section by the following important lemma, where the proof of our main results (given below) depends on the fact in the following lemma (cf. Barakat 2002).

Lemma 1.3. *For any $0 \leq y_j \leq \frac{1}{2}, j = 1, 2, ..., n,$ we have*

$$\exp\left(-(1 + \max_{1\leq j\leq n} y_j) \sum_{j=1}^{n} y_j\right) \leq \prod_{j=1}^{n}(1 - y_j) \leq \exp\left(-\sum_{j=1}^{n} y_j\right).$$

The right-hand inequality remains to hold for all $0 \leq y_j \leq 1$.

2. Main results

The results in the paper are formulated and obtained in analogous way to Pancheva (1985), who discusses monotone normalizations and works without mixing. The possible non-degenerate limit dfs of maximum and minimum based on a random sample drawn from multivariate finite mixture populations with r components normalized by the same sequence of linear normalization, as well as, the sufficient conditions for the existence of these limit dfs are given in Theorems 1 and 3. Theorems 2 and 4 are intended to compliment the results of Theorems 1 and 3 by considering multivariate finite mixture populations with r components normalized by different sequence of linear normalization.

2.1. Asymptotic maximum from INID random vectors under FMM

Theorem 2.1. *Let $X_1, X_2, ..., X_n$ be INID random vectors respect to dfs $F_j(x),\ j = 1, 2, ..., n$ given by (1.3). Then under the UAM for the maximum, we get*

$$P[Z_n \le g_n(x)] \ \xrightarrow[n]{w}\ H(x) = \prod_{i=1}^{r}[H_i(x)]^{p_i}, \ \ \max_{1\le i\le r} \alpha(H_i(x)) < x < \max_{1\le i\le r} \omega(H_i(x)), \tag{2.1}$$

where $\{g_n(x)\}_{n=1}^{\infty}$ is a sequence of linear transformations, $H_i(x), i = 1, 2, ..., r$ are a non-degenerate dfs and $\xrightarrow[n]{w}$ means converges weakly as $n \to \infty$.

Proof. Since

$$P[Z_n \le g_n(x)] = \prod_{j=1}^{n} F_j(g_n(x)),$$

then from (1.3) $F_j(g_n(x)) = \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))$, we get

$$P[Z_n \le g_n(x)] = \prod_{j=1}^{n}\sum_{i=1}^{r} p_i F_{j,i}(g_n(x)).$$

$$P[Z_n \le g_n(x)] = \prod_{j=1}^{n}\left(1 - \left[1 - \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))\right]\right).$$

By putting $y_j = 1 - \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))$ in Lemma 1, we have

$$\exp\left[-\left(1 + \max_{1\le j\le n}\left(1 - \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))\right)\right)\sum_{j=1}^{n}\left(1 - \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))\right)\right] \le$$

$$P[Z_n \le g_n(x)] \le \exp\left[-\sum_{j=1}^{n}\left(1 - \sum_{i=1}^{r} p_i F_{j,i}(g_n(x))\right)\right].$$

$$\exp\left[-\left(1 + \sum_{i=1}^{r} p_i \max_{1\le j\le n}[1 - F_{j,i}(g_n(x))]\right)\left(\sum_{i=1}^{r} p_i \sum_{j=1}^{n}[1 - F_{j,i}(g_n(x))]\right)\right] \le$$

$$P[Z_n \le g_n(x)] \le \exp\left[-\sum_{i=1}^{r} p_i \sum_{j=1}^{n}(1 - F_{j,i}(g_n(x)))\right].$$

By taking limit as $n \to \infty$ and apply UAM for the maximum (C_1, C_2), we get

$$P[Z_n \le g_n(x)] \ \xrightarrow[n]{w}\ \exp\left(-\sum_{i=1}^{r} p_i u_i(t; x)\right) = \prod_{i=1}^{r}[H_i(x)]^{p_i},$$

which has to be proved.
□

Remark 2.2. *From Theorem 1, we have the following cases of the df $H(x)$ in (2.1)*

$$H(x) = \begin{cases} 0 & \text{if}, \quad \underline{x} \leq \max_{1\leq i\leq r} \alpha(H_i(x)), \\ 1 & \text{if}, \quad \underline{x} \geq \max_{1\leq i\leq r} \omega(H_i(x)). \end{cases}$$

Remark 2.3. *From Theorem 1, if $H_i(x) = H(x)$ for $i = 1, 2, ..., r$, then*

$$P[Z_n \leq g_n(x)] \xrightarrow[n]{w} H(x) = \exp\left(\sum_{k=1}^{m} (-1)^k \sum_{1\leq s_1<s_2<...<s_k\leq m} u_{s_k}(t; x_{s_k})\right).$$

Theorem 2.4. *Let $X_1, X_2, ..., X_n$ be INID random vectors respect to dfs $F_j(x)$, $j = 1, 2, ..., n$ given by (1.3) and there exist sequences $\{g_n(x)\}_{n=1}^{\infty}$ and $\{g_{i,n}(x)\}_{n=1}^{\infty}$ of linear transformations such that, for $i = 1, 2, ..., r$,*

(C_1^*) UAM for the maximum is satisfied under $\{g_{i,n}(x)\}_{n=1}^{\infty}$,

(C_2^*) $\lim_{n\to\infty} \sum_{j=1}^{[nt]} [F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x))] = v_i(t; x)$.

Then

$$P[Z_n \leq g_n(x)] \xrightarrow[n]{w} \prod_{i=1}^{r} \exp\left[-p_i(\eta_i(t; x) - v_i(t; x))\right], \tag{2.2}$$

where $\eta_i(t; x) = \lim_{n\to\infty} \sum_{j=1}^{[nt]} [1 - F_{j,i}(g_{i,n}(x))]$.

Proof. Since

$$P[Z_n \leq g_n(x)] = \prod_{j=1}^{n} F_j(g_n(x))$$

and from (1.3) $F_j(g_n(x)) = \sum_{i=1}^{k} p_i F_{j,i}(g_n(x))$, then we have

$$P[Z_n \leq g_n(x)] = \prod_{j=1}^{n} \sum_{i=1}^{k} p_i \left[F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))\right]$$

$$P[Z_n \leq g_n(x)] = \prod_{j=1}^{n} \left[1 - \left(1 - \sum_{i=1}^{k} p_i [F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))]\right)\right].$$

By putting $y_j = 1 - \sum_{i=1}^{r} p_i \left[F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))\right]$ in Lemma 1, we have

$$\exp\left[-\left(1 + \max_{1\leq j\leq n}\left(1 - \sum_{i=1}^{k} p_i \left[F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))\right]\right)\right) \times \right.$$

$$\left.\sum_{j=1}^{n}\left(1 - \sum_{i=1}^{k} p_i \left[F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))\right]\right)\right] \leq P[Z_n \leq g_n(x)]$$

$$\leq \exp\left[-\sum_{j=1}^{n}\left(1 - \sum_{i=1}^{k} p_i \left[F_{j,i}(g_{i,n}(x)) + (F_{j,i}(g_n(x)) - F_{j,i}(g_{i,n}(x)))\right]\right)\right].$$

$$\exp\left[-\left(1+\sum_{i=1}^{k}p_i\max_{1\le j\le n}(1-[F_{j,i}(g_{i,n}(x))+(F_{j,i}(g_n(x))-F_{j,i}(g_{i,n}(x)))])\right)\times\right.$$
$$\left.\sum_{i=1}^{k}p_i\sum_{j=1}^{n}(1-[F_{j,i}(g_{i,n}(x))+(F_{j,i}(g_n(x))-F_{j,i}(g_{i,n}(x)))])\right]\le P[Z_n\le g_n(x)]$$
$$\le\exp\left[-\sum_{i=1}^{k}p_i\sum_{j=1}^{n}(1-[F_{j,i}(g_{i,n}(x))+(F_{j,i}(g_n(x))-F_{j,i}(g_{i,n}(x)))])\right].$$

By taking limit as $n\to\infty$ and apply UAM for the maximum (C_1, C_2), we get

$$P[Z_n\le g_n(x)]\ \xrightarrow[n]{w}\ \exp\left(-\sum_{i=1}^{r}p_i(\eta_i(t;x)-v_i(t;x))\right)$$

or, equivalently

$$P[Z_n\le g_n(x)]\ \xrightarrow[n]{w}\ \prod_{i=1}^{r}\exp\left(-p_i(\eta_i(t;x)-v_i(t;x))\right),$$

which completes the proof. □

2.2. Asymptotic minimum from INID random vectors under FMM

As Remark 2 pointed out, since, any results of minimum can be easily deduced from the corresponding results of maximum, the following results (Theorems 3 and 4) can be obtained in a simple way as Theorems 1 and 2.

Theorem 2.5. *Let $X_1, X_2, ..., X_n$ be INID random vectors respect to dfs $F_j(x),\ j=1,2,...,n$ given by (1.3). Then under UAM for the minimum we get*

$$P[W_n\le T_n(x)]\ \xrightarrow[n]{w}\ L(x)=1-\prod_{i=1}^{r}[1-L_i(x)]^{p_i},\ \max_{1\le i\le r}\alpha(L_i)<x<\max_{1\le i\le r}\omega(L_i) \tag{2.3}$$

where $\{T_n(x)\}_{n=1}^{\infty}$ is a sequence of linear transformations and $L_i(x),\ i=1,2,...,r$ are a non-degenerate dfs.

Remark 2.6. *From Theorem 3, we have the following cases of the df $L(x)$ in (2.3)*

$$L(x)=\begin{cases}0 & \text{if,}\quad x\le\max_{1\le i\le r}\alpha(L_i(x))\\ 1 & \text{if,}\quad x\ge\max_{1\le i\le r}\omega(L_i(x)).\end{cases}$$

Remark 2.7. *From Theorem 3, if $L_i(x)=L(x)$ for $i=1,2,...,r$, then*

$$P[W_n\le T_n(x)]\ \xrightarrow[n]{w}\ L(x).$$

Theorem 2.8. *Let $X_1, X_2, ..., X_n$ be INID random vectors respect to dfs $F_j(x),\ j=1,2,...,n$ given by (1.3) and there exist sequences $\{T_n(x)\}_{n=1}^{\infty}$ and $\{T_{i,n}(x)\}_{n=1}^{\infty}$ of linear transformation such that, for $i=1,2,...,r$,*

$(\mathfrak{C}_1^*)$ *UAM of the minimum is satisfied under $\{T_{i,n}(x)\}_{n=1}^{\infty}$,*

$(\mathfrak{C}_2^*)$ $\lim_{n\to\infty}\sum_{j=1}^{[nt]}[F_{j,i}(T_n(x))-F_{j,i}(T_{i,n}(x))]=\tau_i(t;x)$.

Then

$$P[W_n\le T_n(x)]\ \xrightarrow[n]{w}\ 1-\prod_{i=1}^{r}\exp(-p_i(\varphi_i(t;x)-\tau_i(t;x))), \tag{2.4}$$

where $\varphi_i(t;x)=\lim_{n\to\infty}\sum_{j=1}^{[nt]}F_{j,i}(T_{i,n}(x))$.

3. Applications

Example 3.1. *Let $X_j = (X_{1,j}, X_{2,j})$ be INID random vectors respect to dfs*

$$F_j(x_1, x_2) = pF_{j,1}(x_1, x_2) + (1-p)F_{j,2}(x_1, x_2),\ j = 1, 2, ..., n,\ 0 < p < 1,$$

where

$$F_{j,1}(x_1, x_2) = 1 - e^{-jx_1} - e^{-jx_2} + \frac{1}{e^{jx_1} + e^{jx_2} - 1},\ x_1,\ x_2 \geq 0,\ j = 1, 2, ..., n$$

and

$$F_{j,2}(x_1, x_2) = \frac{1}{e^{-jx_1} + e^{-jx_2} + 1},\ -\infty < x_1, x_2 < \infty,\ j = 1, 2, ..., n.$$

The normalizing transformations of the components may be chosen as,

$$g_n(x_1, x_2) = \left(\frac{1}{j}x_1 + \frac{1}{j}\log\frac{n}{2}, \frac{1}{j}x_2 + \frac{1}{j}\log\frac{n}{2}\right),$$

$$g_{1,n}(x_1, x_2) = \left(\frac{1}{j}x_1 + \frac{1}{j}\log n, \frac{1}{j}x_2 + \frac{1}{j}\log n\right)$$

and

$$g_{2,n}(x_1, x_2) = \left(\frac{1}{j}x_1 + \frac{1}{j}\log\frac{n}{4}, \frac{1}{j}x_2 + \frac{1}{j}\log\frac{n}{4}\right).$$

Firstly, we note that

$$\lim_{n\to\infty}\max_{1\leq j\leq n}[1 - F_{j,i}(g_{i,n}(x_1, x_2))] = \lim_{n\to\infty}\max_{1\leq j\leq n}[1 - F_{\ell,j,i}(g_{i,n}(x_\ell))] = 0,\ \ell = 1, 2,\ i = 1, 2,$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{\ell,j,1}(g_{1,n}(x_\ell))] = e^{-x_\ell},\ \lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{\ell,j,2}(g_{2,n}(x_1))] = 4e^{-x_\ell},\ \ell = 1, 2$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{j,1}(g_{1,n}(x_1, x_2))] = \frac{1}{e^{x_1} + e^{x_2}},\ \lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{j,2}(g_{2,n}(x_1, x_2))] = 0,$$

which are satisfies condition C_1^*, and

$$\lim_{n\to\infty}\sum_{j=1}^{n}[F_{j,1}(g_n(x_1, x_2)) - F_{j,1}(g_{1,n}(x_1, x_2))] = v_1(x_1, x_2) = -e^{-x_1} - e^{-x_2} + \frac{1}{e^{x_1} + e^{x_2}},$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}[F_{j,2}(g_n(x_1, x_2)) - F_{j,2}(g_{2,n}(x_1, x_2))] = v_2(x_1, x_2) = 2(e^{-x_1} + e^{-x_2}),$$

which are satisfies condition C_2^*. On the other hand, we have

$$\lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{j,1}(g_{1,n}(x_1, x_2))] = \eta_1(x_1, x_2) = e^{-x_1} + e^{-x_2} - \frac{1}{e^{x_1} + e^{x_2}}$$

and

$$\lim_{n\to\infty}\sum_{j=1}^{n}[1 - F_{j,2}(g_{2,n}(x_1, x_2))] = \eta_2(x_1, x_2) = 4(e^{-x_1} + e^{-x_2}).$$

Now, applying Theorem 2, we get

$$P[(Z_{1,n}, Z_{2,n}) \leq g_n(x_1, x_2)] \xrightarrow[n]{w} \exp\left[-2\left(e^{-x_1} + e^{-x_2} - \frac{p}{e^{x_1} + e^{x_2}}\right)\right].$$

Example 3.2. *Let $X_j = (X_{1,j}, X_{2,j})$ be INID random vectors respect to dfs*

$$F_j(x_1, x_2) = pF_{j,1}(x_1, x_2) + (1-p)F_{j,2}(x_1, x_2),\ j = 1, 2, ..., n,\quad 0 < p < 1,$$

where

$$F_{j,1}(x_1, x_2) = 1 - e^{-jx_1} - e^{-jx_2} + \frac{1}{e^{jx_1} + e^{jx_2} - 1},\ x_1 \geq 0, x_2 \geq 0,\ j = 1, 2, ..., n$$

and

$$F_{j,2}(x_1, x_2) = \frac{1}{e^{-jx_1} + e^{-jx_2} + 1},\ -\infty < x_1, x_2 < \infty,\ j = 1, 2, ..., n.$$

One can choose the linear normalization,

$$g_n(x_1, x_2) = \left(\frac{1}{j}x_1 + \frac{1}{j}\log n, \frac{1}{j}x_2 + \frac{1}{j}\log n\right).$$

Firstly, we note that

$$\lim_{n\to\infty} \max_{1\leq j\leq n} [1 - F_{j,i}(g_n(x_1, x_2))] = \lim_{n\to+\infty} \max_{1\leq j\leq n} [1 - F_{\ell,j,i}(g_n(x_\ell))] = 0,\ \ell = 1, 2,\ i = 1, 2,$$

which are satisfies condition C_1, and

$$\lim_{n\to\infty} \sum_{j=1}^{n} [1 - F_{\ell,j,i}(g_n(x_\ell))] = e^{-x_\ell},\ \ell = 1, 2,\ i = 1, 2,$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} [1 - F_{j,1}(g_n(x_1, x_2))] = \frac{1}{e^{x_1} + e^{x_2}},\ \lim_{n\to\infty} \sum_{j=1}^{n} [1 - F_{j,2}(g_n(x_1, x_2))] = 0,$$

which are satisfies condition C_2. Now, applying Theorem 1, we get

$$P[(Z_{1,n}, Z_{2,n}) \leq g_n(x_1, x_2)] \ \xrightarrow[n]{w}\ \exp\left(-e^{-x_1} - e^{-x_2} + \frac{p}{e^{x_1} + e^{x_2}}\right).$$

Example 3.3. *Let $X_j = (X_{1,j}, X_{2,j}, X_{3,j})$ be INID random vectors respect to dfs*

$$F_j(x_1, x_2, x_3) = \sum_{i=1}^{3} p_i F_{j,i}(x_1, x_2, x_3),\ j = 1, 2, ..., n,\ 0 < p_i < 1,$$

where

$$F_{j,1}(x_1, x_2, x_3) = 1 - e^{-jx_1} - e^{-jx_2} - e^{-jx_3} + \frac{1}{e^{jx_1} + e^{jx_2}} + \frac{1}{e^{jx_1} + e^{jx_3} - 1} + \frac{1}{e^{jx_2} + e^{jx_3} - 1}$$
$$- \frac{1}{e^{jx_1} + e^{jx_2} + e^{jx_3} - 1},\ x_1, x_2, x_3 \geq 0,\ j = 1, 2, ..., n,$$

$$F_{j,2}(x_1, x_2, x_3) = \frac{1}{e^{-jx_1} + e^{-jx_2} + e^{-jx_3} + 1},\ -\infty < x_1, x_2, x_3 < \infty,\ j = 1, 2, ..., n$$

and

$$F_{j,3}(x_1, x_2, x_3) = 1 - e^{-jx_1} - e^{-jx_2} - e^{-jx_3} + e^{-jx_1 - jx_2 - j^2 x_1 x_2} + e^{-jx_1 - jx_3 - j^2 x_1 x_3} + e^{-jx_2 - jx_3 - j^2 x_2 x_3}$$
$$- e^{-jx_1 - jx_2 - jx_3 - j^3 x_1 x_2 x_3},\ x_1, x_2, x_3 \geq 0,\ j = 1, 2, ..., n.$$

One can choose the linear normalization,

$$g_n(x_1, x_2, x_3) = \left(\frac{1}{j}x_1 + \frac{1}{j}\log n, \frac{1}{j}x_2 + \frac{1}{j}\log n, \frac{1}{j}x_3 + \frac{1}{j}\log n\right).$$

Firstly, we note that all functions are satisfies conditions C_1, C_2 and

$$H_1(x_1, x_2, x_3) = \exp\left[-e^{-x_1} - e^{-x_2} - e^{-x_3} + \frac{1}{e^{x_1} + e^{x_2} + e^{x_3}}\right],$$

$$H_2(x_1, x_2, x_3) = \exp\left(-e^{-x_1} - e^{-x_2} - e^{-x_3}\right),$$

$$H_3(x_1, x_2, x_3) = \exp\left(-e^{-x_1} - e^{-x_2} - e^{-x_3}\right).$$

Applying Theorem 1, we get

$$P[(Z_{1,n}, Z_{2,n}, Z_{3,n}) \le g_n(x_1, x_2, x_3)] \xrightarrow[n]{w} \exp\left(-e^{-x_1} - e^{-x_2} - e^{-x_3} + \frac{p_1}{e^{x_1} + e^{x_2} + e^{x_3}}\right).$$

Example 3.4. *Consider n m-dimensional random vectors $X_j = (X_{1,j}, X_{2,j}, ..., X_{m,j}), j = 1, 2, ..., n,$ with respective dfs $F_j(x) = pF_{j,1}(x) + (1-p)F_{j,2}(x),\ j = 1, 2, ..., n,\ 0 < p < 1,$ where*

$$F_{j,1}(x) = 1 - \exp(-\lambda_j \max(x_1, x_2, ..., x_m)),\ 0 \le x \le \infty,$$

$$F_{j,2}(x) = \exp\left[-\left(\sum_{s=1}^{m} e^{-x_s/\lambda_j}\right)^{\lambda_j}\right],\ -\infty \le x \le \infty$$

and

$$\lambda_j = \begin{cases} 1, & \text{if } j \text{ is odd,} \\ 2, & \text{if } j \text{ is even.} \end{cases}$$

Clearly, for any $1 \le k \le m$, $s_k = (s_1, s_2, ..., s_k)$ and $1 \le s_1 < s_2 < ... < s_k \le m$, we have

$$\sum_{j=1}^{n}(1 - F_{s_k,j,1}(x_{s_k})) = \begin{cases} \frac{n}{2}\left(e^{-2\max(x_{s_1}, x_{s_2}, ..., x_{s_k})} + e^{-\max(x_{s_1}, x_{s_2}, ..., x_{s_k})}\right), & \text{if } n \text{ is even,} \\ \frac{n-1}{2} e^{-2\max(x_{s_1}, x_{s_2}, ..., x_{s_k})} + \frac{n+1}{2} e^{-\max(x_{s_1}, x_{s_2}, ..., x_{s_k})}, & \text{if } n \text{ is odd} \end{cases}$$

and

$$\sum_{j=1}^{n}(1 - F_{s_k,j,2}(x_{s_k})) = \begin{cases} \frac{n}{2}(A + B), & \text{if } n \text{ is even,} \\ \frac{n-1}{2}A + \frac{n+1}{2}B, & \text{if } n \text{ is odd,} \end{cases}$$

where

$$A = \exp\left[-\left(e^{-x_{s_1}} + e^{-x_{s_2}} + ... + e^{-x_{s_k}}\right)\right] \text{ and } B = \exp\left[-\left(e^{\frac{-x_{s_1}}{2}} + e^{\frac{-x_{s_2}}{2}} + ... + e^{\frac{-x_{s_k}}{2}}\right)^2\right].$$

Therefore, by using the normalizing constants $a_{s,n} = 1$ and $b_{s,n} = \log \frac{n}{2},\ s = 1, 2, ..., m,$ we have

$$\lim_{n\to\infty} \max_{1\le j\le n}[1 - F_{s_k,j,1}(g_n(x_{s_k}))] = \lim_{n\to\infty}\left(\frac{2}{n} e^{-\lambda_j \max(x_{s_1}, x_{s_2}, ..., x_{s_k})}\right) = 0,$$

$$\lim_{n\to\infty} \max_{1\le j\le n}[1 - F_{s_k,j,2}(g_n(x_{s_k}))] = \lim_{n\to\infty}\left[1 - \exp\left(-\frac{2}{n}\left(e^{\frac{-x_{s_1}}{\lambda_j}} + e^{\frac{-x_{s_2}}{\lambda_j}} + ... + e^{\frac{-x_{s_k}}{\lambda_j}}\right)^{\lambda_j}\right)\right] = 0,$$

which implies that the uniformity assumption C_1 is satisfied. On the other hand, with the same normalizing constants, we get

$$\lim_{n\to\infty} \sum_{j=1}^{n}[1 - F_{s_k,j,1}(g_n(x_{s_k}))] = u_{s_k,1}(x_{s_k}) = e^{-\max(x_{s_1}, x_{s_2}, ..., x_{s_k})}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n}[1 - F_{s_k,j,2}(g_n(x_{s_k}))] = u_{s_k,2}(x_{s_k}) = e^{-x_{s_1}} + e^{-x_{s_2}} + ... + e^{-x_{s_k}},$$

which implies that the uniformity assumption C_2 is satisfied. From the above discussion, it is easy to show that

$$H_1(x) = \exp\left[\sum_{k=1}^{m} (-1)^k \sum_{1\le s_1<s_2<\ldots<s_k\le m} u_{s_k,1}(x_{s_k})\right] = e^{-\min(x_1,x_2,\ldots,x_m)}$$

and

$$H_2(x) = \exp\left[\sum_{k=1}^{m} (-1)^k \sum_{1\le s_1<s_2<\ldots<s_k\le m} u_{s_k,2}(x_{s_k})\right] = e^{-x_1} + e^{-x_2} + \ldots + e^{-x_m}.$$

Now, applying Theorem 1, we get

$$P[\underline{Z}_n \le \underline{g}_n(\underline{x})] \xrightarrow[n]{w} \exp\left[pe^{-\min(x_1,x_2,\ldots,x_m)} + (1-p)(e^{-x_1} + e^{-x_2} + \ldots + e^{-x_m})\right].$$

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