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Characterization of Dirichlet distribution in terms of conditional distributions

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Abstract. In this paper, a necessary and sufficient condition is given in terms of conditional distributions, for an n-component random vector to be Dirichlet.

1. Introduction

Dirichlet distribution has applications in many areas of science like Biology, Geology, Management Science etc. It is also considered as a prior distribution in Baysian analysis. As a result, the study of this distribution, through various characterizations has attracted the statisticians over decades. Darroch and Ratcliff (1971) and Fabius (1973) have characterized it based on the independence of some functions of the underlying random vector. James and Mosimann (1980) introduced the concept of neutrality, using which they characterized Dirichlet distribution. Rao and Sinha (1988) have characterized the distribution, assuming the linearity of regressions. In this paper, we will be characterizing a multivariate Dirichlet distribution, specifying some conditional distributions. This technique of characterization has become popular over the years, see for example, Arnold *et al.* (1992), Sreehari (2005), Sreehari and Vasudeva (2012).

. A random vector $X' = (X_1, X_2, \dots, X_n)$ is said to have Dirichlet distribution if its probability density function (pdf) is given by

$$f(x_1, x_2, \dots, x_n) = \frac{\Gamma(p_1 + p_2 + \cdots + p_{n+1})}{\prod_{i=1}^{n+1} \Gamma(p_i)} \prod_{i=1}^{n} x_i^{p_i - 1} (1 - x_1 - x_2 - \cdots - x_n)^{p_{n+1} - 1}, \tag{1}$$

$x_i \geq 0$, $i = 1, 2, \dots, n$, $x_1 + x_2 + \cdots + x_n < 1$, where p_i, $i = 1, 2, \dots, n+1$ are positive constants (for details, see Kotz et al. (2000)). The above distribution is denoted by Dirichlet $(p_1, p_2, \dots, p_{n+1})$.

. It is well known that the marginals are Dirichlet. In particular, the univariate marginals are Beta. The conditional distributions are not Dirichlet. However, if $Y_i = \frac{X_i}{1-X_n}$, $i = 1, 2, \dots, (n-1)$, then $(Y_1, Y_2, \dots, Y_{n-1})$ given X_n is Dirichlet. We give below, a brief proof of this result in the form of a lemma, as it plays a crucial role in our paper. We notice that the conditional distribution is invariant of the conditioning random variable (rv). The next section contains the lemma and the characterization theorem.

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2. Lemma and Theorem

Lemma 2.1. *Let $X' = (X_1, X_2, \ldots, X_n)$ be Dirichlet and let $Y_i = \frac{X_i}{1-X_n}$, $i = 1, 2, \ldots, (n-1)$. Then $(Y_1, Y_2, \ldots, Y_{n-1})$ is Dirichlet and is independent of X_n.*

Proof. The pdf of X is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{\Gamma(p_1 + p_2 + \cdots + p_{n+1})}{\prod_{i=1}^{n+1} \Gamma(p_i)} \prod_{i=1}^{n} x_i^{p_i-1}(1 - x_1 - x_2 - \cdots - x_n)^{p_{n+1}-1},$$

$x_i \geq 0$, $i = 1, 2, \ldots, n$, $x_1 + x_2 + \cdots + x_n < 1$, where $p_i > 0$, $i = 1, 2, \ldots, n+1$ are constants. One can show that X_n is Beta$(p_n, p_1 + p_2 + \cdots + p_{n-1} + p_{n+1})$. Consequently, the conditional pdf of $(X_1, X_2, \ldots, X_{n-1})$ given X_n is

$$\begin{aligned} g_1(x_1, x_2, \ldots, x_{n-1}|x_n) &= \frac{\Gamma(p_1 + p_2 + \cdots + p_{n-1} + p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_{n-1})\Gamma(p_{n+1})} \\ &\quad \times \frac{\prod_{i=1}^{n-1} x_i^{p_i-1}(1 - x_1 - x_2 - \cdots - x_n)^{p_{n+1}-1}}{(1 - x_n)^{p_1+p_2+\cdots+p_{n+1}-1}}. \end{aligned}$$

By the transformation, $Y_i = \frac{X_i}{1-X_n}$, $i = 1, 2, \ldots, (n-1)$, one can obtain the pdf of $(Y_1, Y_2, \ldots, Y_{n-1})$ given X_n as

$$\begin{aligned} h(y_1, y_2, \ldots, y_{n-1}|x_n) &= \frac{\Gamma(p_1 + p_2 + \cdots + p_{n-1} + p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_{n-1})\Gamma(p_{n+1})} \\ &\quad \times \prod_{i=1}^{n-1} y_i^{p_i-1}(1 - y_1 - y_2 - \cdots - y_{n-1})^{p_{n+1}-1}, \end{aligned} \tag{2}$$

$y_i > 0$, $i = 1, 2, \ldots, n-1$ with $y_1 + y_2 + \cdots + y_{n-1} < 1$.
Since the conditional distribution of $(Y_1, Y_2, \ldots, Y_{n-1})$ is independent of the conditioning variable X_n, we note that $(Y_1, Y_2, \ldots, Y_{n-1})$ and X_n are independent. Further, by (2) we note that $(Y_1, Y_2, \ldots, Y_{n-1})$ is Dirichlet$(p_1, p_2, \ldots, p_{n-1}, p_{n+1})$. □

Theorem 2.2. *Let $X' = (X_1, X_2, \ldots, X_n)$ be a random vector with continuous non-negative components and satisfying $X_1 + X_2 + \cdots + X_n < 1$. Let $Y_i = \frac{X_i}{1-X_n}$, $i = 1, 2, \ldots, (n-1)$ and $Z_i = \frac{X_i}{1-X_{n-1}}$, $i = 1, 2, \ldots, (n-2), n$. Then X is Dirichlet$(p_1, p_2, \ldots, p_n, p_{n+1})$ if and only if*

(i) $Y' = (Y_1, Y_2, \ldots, Y_{n-1})$ is Dirichlet$(p_1, p_2, \ldots, p_{n-1}, p_{n+1})$ and is independent of X_n and

(ii) $Z' = (Z_1, Z_2, \ldots, Z_{n-2}, Z_n)$ is Dirichlet$(p_1, p_2, \ldots, p_{n-2}, p_n, p_{n+1})$ and is independent of X_{n-1}.

Proof. Suppose that X is Dirichlet$(p_1, p_2, \ldots, p_n, p_{n+1})$. By the above lemma, Conditions (i) and (ii) follow. We now establish the converse.

. Suppose that the conditions (i) and (ii) hold. Let $h_1(\cdot)$ and $h_2(\cdot)$ denote respectively the pdf's of Y and Z. Then we have

$$\begin{aligned} h_1(y_1, y_2, \ldots, y_{n-1}) &= h_1(y_1, y_2, \ldots, y_{n-1}|x_n) \\ &= \frac{\Gamma(p_1 + p_2 + \cdots + p_{n-1} + p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_{n-1})\Gamma(p_{n+1})} \\ &\quad \times \prod_{i=1}^{n-1} y_i^{p_i-1}(1 - y_1 - y_2 - \cdots - y_{n-1})^{p_{n+1}-1} \end{aligned} \tag{3}$$

$y_i > 0$, $i = 1, 2, \ldots, (n-1)$, $y_1 + y_2 + \cdots + y_{n-1} < 1$.
Let $g_1(\cdot)$ denote the conditional pdf of $(X_1, X_2, \ldots, X_{n-1})$ given X_n. From (3), one can get

$$g_1(x_1, x_2, \ldots, x_{n-1}|x_n) = \frac{\Gamma(p_1+p_2+\cdots+p_{n-1}+p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_{n-1})\Gamma(p_{n+1})} \times \frac{\prod_{i=1}^{n-1} x_i^{p_i-1}(1-x_1-x_2-\cdots-x_n)^{p_{n+1}-1}}{(1-x_n)^{p_1+p_2+\cdots+p_{n-1}+p_{n+1}-1}}. \tag{4}$$

On similar lines, one can get the pdf $g_2(\cdot)$ of $(X_1, X_2, \ldots, X_{n-2}, X_n)$ given X_{n-1} as

$$g_2(x_1, x_2, \ldots, x_{n-2}, x_n|x_{n-1}) = \frac{\Gamma(p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_{n-2})\Gamma(p_n)\Gamma(p_{n+1})} \times \frac{\prod_{i=1, i\neq(n-1)}^{n} x_i^{p_i-1}(1-x_1-x_2-\cdots-x_n)^{p_{n+1}-1}}{(1-x_{n-1})^{p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1}-1}}. \tag{5}$$

From the relation,

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= g_1(x_1, x_2, \ldots, x_{n-1}|x_n) f_n(x_n) \\ &= g_2(x_1, x_2, \ldots, x_{n-2}, x_n|x_{n-1}) f_{n-1}(x_{n-1}), \end{aligned} \tag{6}$$

where $f_{n-1}(x_{n-1})$ and $f_n(x_n)$ are the pdfs of X_{n-1} and X_n and from (4) and (5) we have

$$\begin{aligned} f_{n-1}(x_{n-1}) &= \frac{g_1(x_1, x_2, \ldots, x_{n-1}|x_n)}{g_2(x_1, x_2, \ldots, x_{n-2}, x_n|x_{n-1})} f_n(x_n) \\ &= \frac{\Gamma(p_1+p_2+\cdots+p_{n-1}+p_{n+1})\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1})\Gamma(p_{n-1})} \\ &\quad \times \frac{x_{n-1}^{(p_{n-1}-1)}(1-x_{n-1})^{(p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1}-1)}}{x_n^{(p_n-1)}(1-x_n)^{(p_1+p_2+\cdots+p_{n-1}+p_{n+1}-1)}} f_n(x_n). \end{aligned}$$

Integrating both sides with respect to x_{n-1}, one gets

$$\begin{aligned} 1 &= \frac{\Gamma(p_1+p_2+\cdots+p_{n-1}+p_{n+1})\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1})\Gamma(p_{n-1})} \\ &\quad \times \frac{\Gamma(p_{n-1})\Gamma(p_1+p_2+\cdots+p_{n-2}+p_n+p_{n+1})}{\Gamma(p_1+p_2+\cdots+p_n+p_{n+1})} \\ &\quad \times \frac{f_n(x_n)}{x_n^{p_n-1}(1-x_n)^{p_1+p_2+\cdots+p_{n-1}+p_{n+1}-1}} \end{aligned}$$

In turn, one gets

$$f_n(x_n) = \frac{\Gamma(p_1+p_2+\cdots+p_n+p_{n+1})}{\Gamma(p_1+p_2+\cdots+p_{n-1}+p_{n+1})\Gamma(p_n)} \times x_n^{p_n-1}(1-x_n)^{p_1+p_2+\cdots+p_{n-1}+p_{n+1}-1}, \ 0 < x_n < 1, \tag{7}$$

which is Beta$(p_n, p_1+p_2+\cdots+p_{n-1}+p_{n+1})$.
Recalling from (6), the relation

$$f(x_1, x_2, \ldots, x_n) = g_1(x_1, x_2, \ldots, x_{n-1}|x_n) f_n(x_n)$$

and substituting for $g_1(\cdot)$ and $f_n(\cdot)$ from (4) and (7), one gets

$$f(x_1, x_2, \ldots, x_n) = \frac{\Gamma(p_1 + p_2 + \cdots + p_n + p_{n+1})}{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)\Gamma(p_{n+1})} \times \prod_{i=1}^{n} x_i^{p_i - 1}(1 - x_1 - x_2 - \cdots - x_n)^{p_{n+1}-1},$$

$x_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} x_i < 1$, i.e., X is Dirichlet$(p_1, p_2, \ldots, p_n, p_{n+1})$. Hence the proof is complete. □

Corollary 2.3. *A random vector (X_1, X_2) with continuous non-negative components and satisfying $X_1 + X_2 < 1$ is bivariate Beta with parameters p_1, p_2 and p_3 if and only if*

(i) $\frac{X_1}{1-X_2}$ *is Beta (p_1, p_3) and is independent of X_2*

(ii) $\frac{X_2}{1-X_1}$ *is Beta(p_2, p_3) and is independent of X_1.*

Remark 2.4. *Conditions (i) and (ii) of Theorem 2.2 can be equivalently worded as*

(i) $Y' = (Y_1, Y_2, \ldots, Y_{n-1})$ given X_n is Dirichlet$(p_1, p_2, \ldots, p_{n-1}, p_{n+1})$ and

(ii) $Z' = (Z_1, Z_2, \ldots, Z_{n-2}, Z_n)$ given X_{n-1} is Dirichlet$(p_1, p_2, \ldots, p_{n-2}, p_n, p_{n+1})$.

Remark 2.5. *In characterizing a n-variate Dirichlet, Darroch and Ratcliff (1971) need n conditions on the independence of certain functions of the random variables. Rao and Sinha (1988) impose the condition of linearity of all the n regressions. Further, they need the pdf in a certain form. Our result is based on only two conditions, irrespective of n, which may be of interest to note. However, when $n = 2$ our conditions turn out to be much stronger than Theorem 1 of Darroch and Ratcliff (1971).*

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