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Asymptotic behavior of record values with random indices

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Abstract. In this paper, we study the asymptotic behavior of a general sequence of upper record values, as well as lower record values, which is connected asymptotically with some regularly varying functions. Moreover, the limit distribution function of upper record values, as well as lower record values, with random indices, is studied under general conditions.

1. Introduction

Record values are natural phenomena that appear in many real life applications. We can record values in many forms in our life such as the hottest day ever, the lowest stock market, the highest rate of floating a river and highest score of players in many sports. Actually, no one can't be interested in record values. Let $\{X_n,\ n \geq 1\}$ be a sequence of mutually independent random variables (rv's) with common distribution function (df) $F(x)$. Then an upper record value X_j of $\{X_n, n \geq 1\}$ is recognized if $X_{j:j} > X_{j-1:j-1},\ j > 1$. An analogous definition deals with lower record values. By definition X_1 is an upper as well as lower record value. Thus, the upper and the lower record values in the sequence $\{X_n,\ n \geq 1\}$ are the successive maxima and the successive minima, respectively. On the other hand, the upper and the lower record time sequences $\{N_n,\ n \geq 1\}$ and $\{M_n,\ n \geq 1\}$ are defined by $N_n = \min\{j : j > N_{n-1}, X_j > X_{N_{n-1}}, n > 1\},\ N_1 = 1$ and $M_n = \min\{j : j > M_{n-1}, X_j < X_{M_{n-1}}, n > 1\},\ M_1 = 1$, respectively. Therefore, the upper and the lower record value sequences $\{R_n\}$ and $\{L_n\}$ are defined by $R_n = X_{N_n}$ and $L_n = X_{M_n}$, respectively. Moreover, the df's of R_n and L_n can be expressed in terms of the functions $h(x) = -\log(\bar{F}(x))$ and $\tilde{h}(x) = -\log(F(x))$, where $\bar{F}(x) = 1 - F(x)$, as $P(R_n \leq x) = \Gamma_n(h(x))$ and $P(L_n \leq x) = \Gamma_n(\tilde{h}(x)), n > 1$, respectively (see Ahsanullah, 1995 and Arnold et al., 1998), where $\Gamma_n(x) = \frac{1}{(n-1)!}\int_0^x t^{n-1}e^{-t}dt$ is the incomplete gamma ratio function. The well-known asymptotic relation $\Gamma_n(\sqrt{n}x + n) \xrightarrow[n]{w} \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the standard normal distribution and " $\xrightarrow[n]{w}$ "means weak convergence as $n \to \infty$, enables us to deduce the following basic result.

Lemma 1.1 (c.f. Tata, 1969, see also Corollary 6.4.1 in Galambos, 1987). *Let $a_n, \tilde{a}_n > 0$ and $b_n, \tilde{b}_n \in \Re$, are normalizing constants such that*

$$\Phi_{R_n}(a_n x + b_n) = P(R_n \leq a_n x + b_n) \xrightarrow[n]{w} \Phi_R(x) \tag{1.1}$$

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and

$$\Phi_{L_n}(\tilde{a}_n x + \tilde{b}_n) = P(L_n \leq \tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Phi_L(x), \tag{1.2}$$

where $\Phi_R(x)$ *and* $\Phi_L(x)$ *are non-degenerate df's if and only if* $h_n(a_n x + b_n) = \frac{h(a_n x + b_n) - n}{\sqrt{n}} \to V(x)$ *and* $\tilde{h}_n(\tilde{a}_n x + \tilde{b}_n) = \frac{\tilde{h}(\tilde{a}_n x + \tilde{b}_n) - n}{\sqrt{n}} \to V(-x)$, *as* $n \to \infty$, *respectively, where* $V(x)$ *is finite on an interval and has at least two growth points. In this case we have* $\Phi_R(x) = \mathcal{N}(V(x))$ *and* $\Phi_L(x) = 1 - \mathcal{N}(V(-x))$.

Resnick (1973) showed that the function $V(.)$ can only take three possible types (denoted by $V_j(x;\gamma), j = 1,2,3, \gamma > 0$), or equivalently there are only three kinds of distributions that could arise as limiting distributions of suitably normalized upper (lower) record values. Namely, the only possible limiting distributions of suitably normalized upper record value are $\mathcal{N}(V_j(x;\gamma)), j = 1,2,3$, where

$$V_1(x;\gamma) = \begin{cases} -\infty, & x < 0, \\ \gamma \log x, & x \geq 0; \end{cases} \qquad V_2(x;\gamma) = \begin{cases} -\gamma \log \mid x \mid, & x < 0, \\ \infty, & x \geq 0; \end{cases}$$

$$V_3(x;\gamma) = V_3(x) = x, \forall\ x.$$

Barakat et al. (2012) studied the limit df of the random record model. This study gives the corresponding results for upper and lower record values, with random sample size under general conditions. The following two theorems summarize the results of Barakat et al. (2012).

Theorem 1.1. *Let* ν_n *be a nonnegative integer-valued rv. Furthermore, let* $a_n > 0$ *and* $b_n \in \Re$ *be suitable normalizing constants such that*

$$\Phi_{R_{\nu_n}}(a_n x + b_n) = P(R_{\nu_n} \leq a_n x + b_n) \xrightarrow[n]{w} \Psi_R(x), \tag{1.3}$$

where $\Psi_R(x)$ *is a non-degenerate df. Furthermore, let* $\mathcal{K}_r$ *be the class of all non-degenerate limit df's defined by (1.3). Then, for any non-degenerate df* $\Psi_R(x), \Psi_R(x) \in \mathcal{K}_r$ *if and only if*

$$\begin{cases} (i) \;\; \Phi_{R_n}(a_n x + b_n) = P(R_n \leq a_n x + b_n) \xrightarrow[n]{w} \Phi_R(x) = \mathcal{N}(V_j(x;\gamma)), \\ (ii) \;\; P(\frac{\nu_n - n}{\sqrt{n}} \leq z) \xrightarrow[n]{w} A(z) = P(\tau \leq z). \end{cases}$$

The limit df $\Psi_R(x)$ *has the form* $\Psi_R(x) = \int_{-\infty}^{\infty} \mathcal{N}(V_j(x;\gamma) - z) dA(z), j \in \{1,2,3\}$.

Theorem 1.2. *Let* ν_n *be a nonnegative integer-valued rv. Furthermore, let* $\tilde{a}_n > 0$ *and* $\tilde{b}_n \in \Re$, *be suitable normalizing constants such that*

$$\Phi_{L_{\nu_n}}(\tilde{a}_n x + \tilde{b}_n) = P(L_{\nu_n} \leq \tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Psi_L(x), \tag{1.4}$$

where $\Psi_L(x)$ *is a non-degenerate df. Furthermore, let* $\mathcal{K}_\ell$ *be the class of all non-degenerate limit df's defined by (1.4). Then, for any non-degenerate df* $\Psi_L(x), \Psi_L(x) \in \mathcal{K}_\ell$ *if and only if*

$$\begin{cases} (i) \;\; \Phi_{L_n}(\tilde{a}_n x + \tilde{b}_n) = P(L_n \leq \tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Phi_L(x) = 1 - \mathcal{N}(V_j(-x;\gamma)), \\ (ii) \;\; P(\frac{\nu_n - n}{\sqrt{n}} \leq z) \xrightarrow[n]{w} A(z) = P(\tau \leq z). \end{cases}$$

The limit df $\Psi_L(x)$ *has the form* $\Psi_L(x) = 1 - \int_{-\infty}^{\infty} \mathcal{N}(V_j(-x;\gamma) - z) dA(z), j \in \{1,2,3\}$.

Remark 1.1. It is natural to look for the limitations on ν_n, under which we get the relations $\Phi_R(x) = \Psi_R(x)$ and $\Phi_L(x) = \Psi_L(x)$, $\forall\ x$. In view of Theorems 1.1 and 1.2, the last equalities are satisfied if and only if the df $A(z) = P(\tau \leq z)$ is degenerate at zero, which means the asymptotically almost randomlessness of ν_n.

Let $\{X_i\}$ and $\{Y_i\}$ be two sequences of rv's defined on the same probability space (Ω, F, P), where $\{X_i\}$ is i.i.d with common df $F(x)$. Furthermore, let $\{R_n^\star = Y_{N_n}\}(\{L_n^\star = Y_{M_n}\})$ and $\{R_n = X_{N_n}\}(\{L_n = X_{M_n}\})$ be the upper (lower) record values corresponding the sequences $\{Y_i\}$ and $\{X_i\}$, respectively. When the sample size itself is assumed to be a positive integer-valued rv ν_n and $\{R_n^\star\}(\{L_n^\star\})$ is assumed to be connected asymptotically with $\{R_n\}(\{L_n\})$ and some regularly varying functions, we derive the limit df's of upper (lower) record values $\{R_{\nu_n}^\star = Y_{N_{\nu_n}}\}(\{L_{\nu_n}^\star = Y_{M_{\nu_n}}\})$. More specifically, we study the weak convergence of the df of the general sequence $\{R_{\nu_n}^\star\}(\{L_{\nu_n}^\star\})$, where $\{R_n^\star\}(\{L_n^\star\})$ belongs to a restricted class

$\mathcal{L}(R_n;\phi|\tau)(\mathcal{L}(L_n;\phi|\tau))$, in which the elements are connected asymptotically with the rv τ, the sequence $\{R_n\}(\{L_n\})$ and some regularly varying functions, where the following definition characterizes the class $\mathcal{L}(R_n;\phi|\tau)(\mathcal{L}(L_n;\phi|\tau))$.

Definition 1.1. Let $[t]$ denote the greatest integer less than or equal to t. Let τ, be a positive rv and $\phi(n)$ is regularly varying function at $(+\infty)$ with index $\rho > 0$ (that is $\lim_{n\to\infty} \frac{\phi(nx)}{\phi(n)} = x^\rho,\ x \in (0,\infty)$, written $\phi(n) \in \mathrm{RV}_\rho$, for more details on the regularly varying functions, see Haan, 1970). If for any sequence u_n of real numbers, we have

$$\left|P\left(R_n^\star \le u_n|\tau = z\right) - P\left(R_{[\phi(n)]} \le u_n|\tau = z\right)\right| \underset{n}{\longrightarrow} 0$$

and

$$\left|P\left(L_n^\star \le u_n|\tau = z\right) - P\left(L_{[\phi(n)]} \le u_n|\tau = z\right)\right| \underset{n}{\longrightarrow} 0,$$

where " $\underset{n}{\longrightarrow}$ "stands for convergence, as $n \to \infty$, for all $z > 0$, following Xie Shengrong (1997), we call the sequence $\{R_n^*\}(\{L_n^*\})$ dependent on τ in connecting with $\{R_n\}(\{L_n\})$ and $\phi(n) \in \mathrm{RV}_\rho$. Moreover, in this case we write $\{R_n^*\} \in \mathcal{L}(R_n;\phi|\tau)(\{L_n^*\} \in \mathcal{L}(L_n;\phi|\tau))$.

2. Main results

The following two theorems extend the results of Barakat and El-Shandidy (2004) to any upper and lower record values. These simple theorems, besides they will be needed in obtaining the limit df's of $\{R_{\nu_n}^*\}$ and $\{L_{\nu_n}^*\}$, where $\{R_n^\star\} \in \mathcal{L}(R_n;\phi|\tau)$ and $\{L_n^*\} \in \mathcal{L}(L_n;\phi|\tau)$, they are of independent interest.

Theorem 2.1. *Let $a_n > 0$ and b_n be normalizing constants for which*

$$\Phi_{R_{[\phi(n)]}}(a_n x + b_n) = P(R_{[\phi(n)]} \le a_n x + b_n) \underset{n}{\overset{w}{\longrightarrow}} \Phi_R(x), \tag{2.1}$$

where $\phi(n) \in RV_\rho$, $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \underset{n}{\longrightarrow} 0$, $0 < \rho < 2$ and $\Phi_R(x)$ is a non-degenerate df. Then, we get $\Phi_R(x) = \mathcal{N}(V_j(x;\gamma)), j \in \{1,2,3\}$.

Proof. We observe that (2.1) is satisfied if and only if

$$h_{[\phi(n)]}(a_n x + b_n) = \frac{h(a_n x + b_n) - [\phi(n)]}{\sqrt{[\phi(n)]}} \underset{n}{\longrightarrow} V(x), \tag{2.2}$$

where $\Phi_R(x) = \mathcal{N}(V(x))$. The fact that $\phi(n) \underset{n}{\longrightarrow} \infty$, yields $[\phi(n)] \sim \phi(n)$. Moreover, the fact that $0 \le F(a_n x + b_n) \le 1$, for all x and n, leads to

$$h_{[\phi(n)]}(a_n x + b_n) \sim h_{\phi(n)}(a_n x + b_n) \underset{n}{\longrightarrow} V(x). \tag{2.3}$$

Now, for any $m, \ell, \ell' > 0$ it is not difficult to verify that

$$h_\ell(a_m x + b_m) = \sqrt{\frac{\ell'}{\ell}} \frac{h(a_m x + b_m) - \ell'}{\sqrt{\ell'}} + \frac{\ell' - \ell}{\sqrt{\ell}}. \tag{2.4}$$

By choosing $\ell = \phi(n)$ and $\ell' = \phi(m_n(s))$, where $m_n(s) = n + [ns\phi^{-\frac{1}{2}}(n)] = n\in_n(s)$, (2.4) can be written in the form

$$h_{\phi(n)}(a_{m_n(s)} x + b_{m_n(s)}) = \mathcal{A}_n h_{\phi(m_n(s))}(a_{m_n(s)} x + b_{m_n(s)}) + \mathcal{B}_n. \tag{2.5}$$

where

$$\mathcal{A}_n = \sqrt{\frac{\phi(m_n(s))}{\phi(n)}}$$

and

$$\mathcal{B}_n = \frac{\phi(m_n(s)) - \phi(n)}{\sqrt{\phi(n)}}.$$

Since $m_n(s) \xrightarrow[n]{} \infty$, the relation (2.3) yields

$$h_{\phi(m_n(s))}(a_{m_n(s)}x + b_{m_n(s)}) \xrightarrow[n]{} V(x). \tag{2.6}$$

We now estimate the limit of the sequences $\mathcal{A}_n$ and $\mathcal{B}_n$, as $n \to \infty$. Since, $\frac{\phi(nx)}{\phi(n)} \xrightarrow[n]{} x^\rho$ uniformly in x and $\in_n(s) \xrightarrow[n]{} 1$, we get $\frac{\phi(m_n(s))}{\phi(n)} = \frac{\phi(n\in_n(s))}{\phi(n)} \xrightarrow[n]{} 1$. Therefore,

$$\mathcal{A}_n = \sqrt{\frac{\phi(m_n(s))}{\phi(n)}} \xrightarrow[n]{} 1. \tag{2.7}$$

On the other hand, in view of the condition $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \xrightarrow[n]{} 0$, we get $\phi(m_n(s)) = \phi(n\in_n(s)) = \phi(n)\in_n^\rho(s) + o(\phi^{\frac{1}{2}}(n))$, where $\in_n(s) = \frac{m_n(s)}{n} = 1 + s\phi^{-\frac{1}{2}}(n) + \frac{\alpha}{n}, 0 < \alpha < 1$. Therefore,

$$\phi(m_n(s)) - \phi(n) = \phi(n)\in_n^\rho(s) + o(\phi^{\frac{1}{2}}(n)) - \phi(n) = \phi(n)\left(s\rho\phi^{-\frac{1}{2}}(n) + \frac{\rho\alpha}{n} + o(\phi^{-\frac{1}{2}}(n))\right). \tag{2.8}$$

Now, since $\phi(n) \in \mathrm{RV}_\rho$, we can write $\phi(n) = n^\rho S(n)$, where $S(x)$ is slowly varying function (see, Subsection 6.3 of Galambos, 1995). Moreover, since $\rho < 2$, we get

$$\frac{\sqrt{\phi(n)}}{n} = \sqrt{n^{\rho-2}S(x)} \xrightarrow[n]{} 0. \tag{2.9}$$

Combining this fact with (2.8), we get

$$\mathcal{B}_n \sim \sqrt{\phi(n)}\left(s\rho\phi^{-\frac{1}{2}}(n) + \frac{\rho\alpha}{n} + o(\phi^{-\frac{1}{2}}(n))\right) = s\rho + \frac{\rho\alpha\sqrt{\phi(n)}}{n} + o(1) \xrightarrow[n]{} s\rho. \tag{2.10}$$

Combining now (2.10), (2.7), (2.6) with (2.5), we get

$$h_{\phi(n)}(a_{m_n(s)}x + b_{m_n(s)}) \xrightarrow[n]{} V(x) + s\rho. \tag{2.11}$$

Moreover, in view of the result of Resnick (1973) and by applying Khinchine type theorem on (2.11) and (2.3), we deduce that there exist measurable functions $\alpha(s) > 0$ and $\beta(s)$, such that

$$V(x) = V(\alpha(s)x + \beta(s)) - s\rho. \tag{2.12}$$

By solving the functional equation (2.12) (see Resnick, 1973), as $\rho > 0$, the conclusion of the theorem will be obtained. ■

Theorem 2.2. *Let $\tilde{a}_n > 0$ and $\tilde{b}_n$ be normalizing constants for which*

$$\Phi_{L_{[\phi(n)]}}(\tilde{a}_n x + \tilde{b}_n) = P(L_{[\phi(n)]} \leq \tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Phi_L(x),$$

where $\phi(n) \in RV_\rho$, $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \xrightarrow[n]{} 0$, $0 < \rho < 2$ and $\Phi_L(x)$ is a non-degenerate df. Then, $\Phi_L(x) = 1 - \mathcal{N}(V_j(-x;\gamma)), j \in \{1,2,3\}$.

Proof. The method of the proof of Theorem 2.2 is the same as that Theorem 2.1, except only the obvious changes. Hence, for brevity the details of the proof are omitted. ■

Now, the main results concerning the limit df's of the sequences $\{R^\star_{\nu_n}\}$ and $\{L^\star_{\nu_n}\}$ are given in the following two theorems.

Theorem 2.3. *Suppose that $R^\star_n \in \mathcal{L}(R_n;\phi|\tau)$. Let $a_n > 0$ and $b_n \in \Re, n \geq 1$, be suitable normalizing constants for which*

$$P(R^\star_n \leq a_n x + b_n) \xrightarrow[n]{w} \Phi_R(x),$$

where $\Phi_R(x)$ is a non-degenerate df. Furthermore, let ν_n be a sequence of nonnegative integer-valued rv's, which satisfies

$$\frac{\nu_n - n}{n\phi^{-\frac{1}{2}}(n)} \xrightarrow[n]{p} \tau, \tag{2.13}$$

where " $\xrightarrow[n]{p}$ "stands for convergence in probability, as $n \to \infty$, $\phi(n) \in RV_\rho$, $0 < \rho < 2$, $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \xrightarrow[n]{} 0$ and the df $A(z) = P(\tau \le z)$ is symmetric and continuous at zero, i.e. $A(0-) = A(0) = A(0+) = 0$ (the condition $A(0-) = A(0)$ is not considered in the original Theorem 3 of Xie Shengrong, 1997, for the maximum order statistics, but as we shall see in the proof of Theorem 2.3, it seems to be indispensable, see also Remark 2.2). Then,

$$P(R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) \xrightarrow[n]{w} \int_{-\infty}^{\infty} \mathcal{N}(V_j(x;\gamma) - \rho z)dA(z), j \in \{1,2,3\}.$$

Proof. Let $d > 0$ be a positive real number. Clearly, for any $\varepsilon \in (0, d)$, (2.13) implies

$$P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) \le \nu_n \le n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon)) \xrightarrow[n]{} 1. \tag{2.14}$$

On the other hand, we can write

$$\begin{gathered}
P(R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) = \\
P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) \le \nu_n \le n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon), R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) + \\
P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) > \nu_n \text{ or } \nu_n > n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon), R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) \\
= T_n^{(1)}(\varepsilon) + T_n^{(2)}(\varepsilon).
\end{gathered}$$

According to (2.14), $T_n^{(2)}(\varepsilon) \xrightarrow[n]{} 0$, so calculating $T_n^{(1)}(\varepsilon)$ is only necessary. However,

$$\begin{gathered}
T_n^{(1)}(\varepsilon) = P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) \le \nu_n \le n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon), R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) \\
= \int_{-\infty}^{\infty} P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) \le \nu_n \le n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon), R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}|\tau = z)dA(z) \\
\le P(-d \le \tau \le d) + 2\int_d^{\infty} P(n + n\phi^{-\frac{1}{2}}(n)(\tau - \varepsilon) \le \nu_n \le n + n\phi^{-\frac{1}{2}}(n)(\tau + \varepsilon), \\
R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}|\tau = z)dA(z).
\end{gathered}$$

By using the following well-known inequality

$$R^\star_{n'} \le R^\star_{n''}, \quad n' \le n'',$$

we get

$$\begin{aligned}
T_n^{(1)}(\varepsilon) &\le P(-d \le \tau \le d) + 2\int_d^{\infty} P(R^\star_{[m_n(\tau-\varepsilon)]} \le a_{[\phi(n)]}x + b_{[\phi(n)]}|\tau = z)dA(z) \\
&= P(-d \le \tau \le d) + 2\int_d^{\infty} P(R^\star_{[m_n(z-\varepsilon)]} \le a_{[\phi(n)]}x + b_{[\phi(n)]}|\tau = z)dA(z),
\end{aligned}$$

where $m_n(z - \varepsilon) = n + [n(z - \varepsilon)\phi^{-\frac{1}{2}}(n)]$. Since $\{R^\star_n\} \in \mathcal{L}(R_n; \phi|\tau)$, we get

$$T_n^{(1)}(\varepsilon) \le P(-d \le \tau \le d) + 2\int_d^{\infty} P(R_{[\phi([m_n(z-\varepsilon)])]} \le a_{[\phi(n)]}x + b_{[\phi(n)]}|\tau = z)dA(z) + o(1). \tag{2.15}$$

Now, let us introduce

$$\tau_d = \begin{cases} \tau, & \tau > d, \\ d, & \tau \le d. \end{cases}$$

Clearly,

$$P(\tau_d \le z) = \begin{cases} 0, & z < d, \\ P(\tau \le z), & z \ge d. \end{cases}$$

Therefore, the relation (2.15) yields

$$T_n^{(1)}(\varepsilon) \le P(-d \le \tau \le d) + 2P(R_{[\phi([m_n(\tau_d-\varepsilon)])]} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) + o(1).$$

On the other hand, since $\tau_d - \varepsilon > 0$, in view of the conditions $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \underset{n}{\longrightarrow} 0$ and $\rho < 2$, we get

$$\frac{\phi(m_n(\tau_d - \varepsilon)) - \phi(n)}{\phi^{\frac{1}{2}}(n)} \underset{n}{\longrightarrow} \rho(\tau_d - \varepsilon).$$

Consequently, by using Theorem 1.1 (the upper record values case) and Theorem 2.1, after letting $d \longrightarrow 0$, we get

$$\overline{\lim_{n\to\infty}}\ P(R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) \le \int_{-\infty}^{\infty} \mathcal{N}(V_j(x;\gamma) - \rho z)dA(z), j \in \{1,2,3\}. \tag{2.16}$$

By applying a similar argument (with only the obvious modification) we can prove that

$$\underline{\lim_{n\to\infty}}\ P(R^\star_{\nu_n} \le a_{[\phi(n)]}x + b_{[\phi(n)]}) \ge \int_{-\infty}^{\infty} \mathcal{N}(V_j(x;\gamma) - \rho z)dA(z), j \in \{1,2,3\}. \tag{2.17}$$

Thus, the claimed result for upper record values case follows by combining (2.16) and (2.17). This completes the proof of the theorem. ■

Theorem 2.4. *Suppose that* $L^\star_n \in \mathcal{L}(L_n;\phi|\tau)$. *Let* $\tilde{a}_n > 0$ *and* $\tilde{b}_n \in \Re, n \ge 1$, *be suitable normalizing constants for which*

$$P(L^\star_n \le \tilde{a}_n x + \tilde{b}_n) \underset{n}{\overset{w}{\longrightarrow}} \Phi_L(x),$$

where $\Phi_L(x)$ *is a non-degenerate df. Furthermore, let* ν_n *be a sequence of nonnegative integer-valued rv's, which satisfies*

$$\frac{\nu_n - n}{n\phi^{-\frac{1}{2}}(n)} \underset{n}{\overset{p}{\longrightarrow}} \tau,$$

where $\phi(n) \in RV_\rho$, $0 < \rho < 2$, $\phi^{\frac{1}{2}}(n)(\frac{\phi(nx)}{\phi(n)} - x^\rho) \underset{n}{\longrightarrow} 0$ *and the df* $A(z) = P(\tau \le z)$ *is symmetric and continuous at zero. Then,*

$$P(L^\star_{\nu_n} \le \tilde{a}_{[\phi(n)]}x + \tilde{b}_{[\phi(n)]}) \underset{n}{\overset{w}{\longrightarrow}} 1 - \int_{-\infty}^{\infty} \mathcal{N}(V_j(-x;\gamma) - \rho z)dA(z), j \in \{1,2,3\}.$$

Proof. Without significant modifications, the method of the proof of Theorem 2.4 is the same as that Theorem 2.3, except only the obvious changes. Hence, for brevity the details of the proof are omitted. ■

Remark 2.1. By taking $\phi(x) = x, \{X_i\} = \{Y_i\}, i \ge 1$, Theorem 2.3 and Theorem 2.4 will immediately return to Theorem 1.1 and Theorem 1.2 for upper and lower record values, respectively.

Remark 2.2. Clearly, in upper and lower record values we can replace the symmetry condition of the df of τ by the condition $A(0-) = 0$.

3. Applications

Let $\{Y_n, n \ge 1\}$ represent an infinite sequence of independent continuous rv's. Suppose Y_n represents the maximum of $\alpha(n)$ i.i.d rv's each having the df F. Thus F_n, the df of Y_n, is given by $F_n(y) = F^{\alpha(n)}(y)$. In the international athletic events, $\alpha(n)$ may represent the size of the population of athletes at the time of the nth event. Nevzorov (1985) refers to the model of associated record statistics of F_n as the F^α-model. This model is originally suggested by Yang (1975) by assuming that observations, while being independent, are non-identical distributed in a special way. Clearly, if we assume that F is exponential, then in view of Example 2.3.1 in Arnold et al. (1998), we have

$$\Phi_{R_n}(\sqrt{n}x + n) = P(R_n \le \sqrt{n}x + n) \underset{n}{\overset{w}{\longrightarrow}} \Phi_R(x) = \mathcal{N}(x).$$

Moreover, if we assume, as example, $\alpha(n) = Cn, C > 0$, we get $\alpha(n) \in RV_1$ and $\alpha^{\frac{1}{2}}(n)(\frac{\alpha(nx)}{\alpha(n)} - x) \underset{n}{\longrightarrow} 0$ (since, $\alpha^{\frac{1}{2}}(n)(\frac{\alpha(nx)}{\alpha(n)} - x) = 0$). Therefore, in view of Theorem 2.1 (by taking $\phi(n) = \alpha(n)$)

$$\Phi_{R_{[\alpha(n)]}}(\sqrt{n}x + n) \xrightarrow[n]{w} \mathcal{N}(x). \tag{3.1}$$

On the other hand, we can check that $p_n = \frac{\alpha(n)}{\sum_{j=1}^{n}\alpha(j)} = \frac{2}{n+1} \underset{n}{\longrightarrow} 0$ and $\sum_{j=1}^{n} p_j^2 = \circ((\sum_{j=1}^{n} p_j)^{\frac{1}{2}})$ (since, $\sum_{j=1}^{n} \frac{1}{(j+1)^2} \underset{n}{\longrightarrow} \frac{\pi^2}{6} - 1$ and $\sum_{j=1}^{n} \frac{1}{j+1} \underset{n}{\longrightarrow} \infty$). Therefore, in view of Nevzorov (1985) (see, also Arnold et al., 1998, Sec. 6.3.2), we get

$$\Phi_{R_n^\star}(\sqrt{n}x + n) \xrightarrow[n]{w} \mathcal{N}(x). \tag{3.2}$$

Now, if we assume that the rv's $X_1, X_2, ...$ are independent of τ, the relations (3.1) and (3.2) show that $R_n^\star \in \mathcal{L}(R_n; \alpha|\tau)$. Thus, Theorem 2.3 can be applied on $R_n^\star$ to get the following interesting relation:

$$P(R_{\nu_n}^\star \leq \sqrt{n}x + n) \xrightarrow[n]{w} \int_{-\infty}^{\infty} \mathcal{N}(x - z)dA(z) = \mathcal{N}(x) \star A(x).$$

This relation reveals an interesting property of the random F^α-model. Namely, $R_{\nu_n}^\star \sim R_n^\star + \nu_n$, as $n \to \infty$, in the sense that the df's of $R_{\nu_n}^\star$ and $R_n^\star + \nu_n$ converge weakly to the same df $\mathcal{N}(x) \star A(x)$. Finally, it is worth mentioning that this result may be obtained for many other choices of $\alpha(n)$, e.g., $\alpha(n) = Cn^\beta, C > 0, 0 < \beta < 2$. Moreover, we can choose any arbitrary df F, rather than the exponential, whenever (1.1) holds with $\mathcal{N}(x)$. Actually, the asymptotic normality of R_n holds in a broad spectrum of random phenomena. For example, F is a standard normal, or a Weibull, or a logistic, or an Extreme value, distribution.

4. Concluding remarks

The result of this paper (namely, Theorems 2.3 and 2.4) enables us to extend the results concerning the asymptotic behavior of the random record model, which is discussed by Barakat (2012), to some important record models, in which the observation process fails to consist of i.i.d rv's. Clearly, these generalized record models are more capable to explain a higher incidence of new records than the random record model, which was discussed by Barakat (2012). Example of such generalized model is F^α-model, suggested by Yang (1975), in which we assume that we are observing the maxima of a geometrically increasing population. As an application of Theorems 2.1 and 2.3, we obtained the asymptotic behavior of the random $F^\alpha-$model, where we assume that we are observing the maxima of a randomly geometrically increasing population.

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