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Characterization of generalized uniform distribution based on lower record values

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Abstract. A probability distribution can be characterized through various methods. In this paper, generalized uniform distribution is characterized through the conditional expectation of lower record values. This characterization result can be applied for uniform and standard power-function distribution also.

1. Introduction

Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of independent and identically distributed (*iid*) random variables with cumulative distribution function (*cdf*) $F(.)$ and probability density function (*pdf*) $f(.)$. Set $Y_n = max(min)\{X_j \mid 1 \leq j \leq n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of $\{X_n \mid n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 1$. By definition, X_1 is an upper as well as a lower record value.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records, e.g. Olympic records. It is also useful in reliability theory, meteorology, hydrology, seismology, mining, medicine, traffic and industry among others.

For comprehensive accounts of the theory and applications of record values, we refer the readers to Ahsanullah (1995) and Arnold *et al.*, (1998).

Various characterization results on records have been established in different directions reported in literature. For recent reviews of the results on the topic, the reader is referred to Shawky and Bakoban (2008, 2009), Faizan and Khan (2011), Ahsanullah *et al.*, (2013) and Khan and Faizan (2013) amongst others.

2. Distribution of lower record values

The indices at which lower record values occure are given by the record times $\{L(n), n \geq 1\}$, where $L(n) = min\{j \mid j > L(n-1), X_j < X_{L(n-1)}, n \geq 1\}$ and $L(0) = 1$. The n^{th} lower record value will be

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denoted by $X_{L(n)}$. If we define $F_n(.)$ as the *cdf* for $X_{L(n)}$ for $n \geq 1$, then we have

$$F_n(x) = \int_{-\infty}^{x} \frac{(H(u))^{n-1}}{(n-1)!} dF(u), \qquad -\infty < x < \infty. \tag{1}$$

The *pdf* of $X_{L(n)}$ denoted by $f_n(.)$ is,

$$f_n(x) = \frac{(H(u))^{n-1}}{(n-1)!} f(x), \qquad -\infty < x < \infty. \tag{2}$$

The joint *pdf* of $X_{L(r)}$ and $X_{L(s)}$, $r < s$ is given by,

$$f_{L(r),L(s)}(x,y) = \frac{1}{(r-1)!(s-r-1)!} [H(x)]^{r-1} [H(y) - H(x)]^{s-r-1} h(x) f(y), \tag{3}$$

where $H(x) = -logF(x), \quad 0 < F(x) < 1 \quad$ and $\quad h(x) = -\frac{d}{dx} H(x) = \frac{f(x)}{F(x)}$.
The conditional *pdf* of $X_{L(s)}$ given $X_{L(r)} = x$, $1 \leq r < s$ is given by,

$$f_{L(s)}(y \mid X_{L(r)} = x) = \frac{1}{\Gamma(s-r)} [-lnF(y) + lnF(x)]^{s-r-1} \frac{f(y)}{F(x)}, \qquad y < x. \tag{4}$$

$$\Gamma(c, z) = \int_{z}^{\infty} t^{c-1} e^{-t} dt \quad c > 0,$$

denote the incomplete gamma function. For more details see Ahsanullah (1995).

3. Generalized Uniform Distribution

The *cdf* of generalized uniform distribution is as follows:

$$F(x : \alpha, \beta) = \left(\frac{x}{\beta} \right)^{\alpha+1}, \qquad (0 < x < \beta, \quad -1 < \alpha). \tag{5}$$

where α and β are referred to as the shape and the scale parameters, respectively. The generalized uniform distribution is a uniform distribution at $\alpha = 0$ and it is standard power-function distribution if $\beta = 1$.

The *pdf* is given by

$$f(x : \alpha, \beta) = \frac{(\alpha+1)}{\beta} \left(\frac{x}{\beta} \right)^{\alpha}, \qquad (0 < x < \beta, \quad -1 < \alpha). \tag{6}$$

For more details on this distribution and its applications one may refer to (Tiwari *et al.*,1996 and Lee, 2000).
Using (5) and (6), we have,

$$\frac{f(x)}{F(x)} = \left(\frac{\alpha+1}{x} \right). \tag{7}$$

4. Distribution of lower record values based on generalized uniform distribution

The *pdf* $f_n(x)$ and *df* $F_n(x)$ of the n^{th} lower record value $X_{L(n)}$ from generalized uniform distribution can easily be obtained by using equations (5) and (6) in (1) and (2), respectively as,

$$f_n(x) = \frac{[-ln(x/\beta)^{\alpha+1}]^{n-1} [((\alpha+1)/\beta)(x/\beta)^{\alpha}]}{\Gamma n}, \qquad (0 < x < \beta, \quad -1 < \alpha). \tag{8}$$

and

$$F_n(x) = \frac{[n, -ln(x/\beta)^{\alpha+1}]}{\Gamma n}, \quad (0 < x < \beta, \quad -1 < \alpha). \tag{9}$$

where $n \geq 1$. The k^{th} moment of the n^{th} lower record value $X_{L(n)}$ from generalized uniform distribution is easily given by,

$$E[X^k_{L(n)}] = \int_0^\beta x^k \frac{[-ln(x/\beta)^{\alpha+1}]^{n-1}[((\alpha+1)/\beta)(x/\beta)^\alpha]}{\Gamma n} dx. \tag{10}$$

setting $-ln(x/\beta)^{\alpha+1} = u$ in (10) and simplifying, we get,

$$E[X^k_{L(n)}] = \frac{\beta^k(\alpha+1)^n}{(\alpha+k+1)^n}. \tag{11}$$

Therefore,

$$\begin{aligned} E[X_{L(n)}] &= \frac{\beta(\alpha+1)^n}{(\alpha+2)^n}, \\ E[X^2_{L(n)}] &= \frac{\beta^2(\alpha+1)^n}{(\alpha+3)^n}, \\ Var[X_{L(n)}] &= \beta^2(\alpha+1)^n \left[\frac{1}{(\alpha+3)^n} - \frac{(\alpha+1)^n}{(\alpha+2)^n}\right]. \end{aligned} \tag{12}$$

5. Characterization of generalized uniform distribution based on lower record values

Theorem 5.1. *An absolutely continuous random variable X has a generalized uniform distribution with $F(x) = \left(\frac{x}{\beta}\right)^{\alpha+1}$, $(0 < x < \beta, \quad -1 < \alpha)$, if and only if,*

$$(\alpha+k+1)E[X^k_{L(s+1)} \mid X_{L(r)} = x] = (\alpha+1)E[X^k_{L(s)} \mid X_{L(r)} = x]. \tag{13}$$

Proof. Suppose that $F(x:\alpha,\beta) = \left(\frac{x}{\beta}\right)^{\alpha+1}$, $(0 < x < \beta, \quad -1 < \alpha)$ and $f(x:\alpha,\beta) = \frac{\alpha+1}{\beta}\left(\frac{x}{\beta}\right)^\alpha$.

$$\begin{aligned} E[X^k_{L(s+1)} \mid X_{L(r)} = x] &= \int_0^x \frac{y^k}{\Gamma(s-r+1)}[(\alpha+1)(lnF(x) - lnF(y))]^{s-r} \\ &\qquad \times(\alpha+1)\frac{y^{\alpha+1}}{x^{\alpha+1}}\frac{dy}{y}, \\ &= (\alpha+1)^{s-r+1}\int_0^x \frac{y^k}{\Gamma(s-r+1)}\left[-log\left(\frac{y}{x}\right)\right]^{s-r}\left(\frac{y}{x}\right)^{\alpha+1}\frac{dy}{y}. \end{aligned}$$

Let $-log\left(\frac{y}{x}\right) = u$ then $y = xe^{-u}$, $dy = -xe^{-u}du$,

$$\begin{aligned} E[X^k_{L(s+1)} \mid X_{L(r)} = x] &= \frac{(\alpha+1)^{s-r+1}x^k}{\Gamma(s-r+1)}\int_0^\infty u^{s-r}(-e^{-u})^{k+\alpha+1}du, \\ E[X^k_{L(s+1)} \mid X_{L(r)} = x] &= x^k\left(\frac{\alpha+1}{k+\alpha+1}\right)^{s-r+1}. \end{aligned} \tag{14}$$

Using the same line above, we can prove that

$$E[X^k_{L(s)} \mid X_{L(r)} = x] = x^k\left(\frac{\alpha+1}{k+\alpha+1}\right)^{s-r}. \tag{15}$$

Equating (14) and (15), the necessary part is proved.
For sufficiency part, we have

$$(\alpha+k+1)E[X^k_{L(s+1)} \mid X_{L(r)}=x] = (\alpha+1)E[X^k_{L(s)} \mid X_{L(r)}=x].$$

Then,

$$(\alpha+k+1)\int_0^x \frac{y^k}{\Gamma(s-r+1)}[lnF(x)-lnF(y)]^{s-r}\frac{f(y)}{F(x)}dy$$
$$= (\alpha+1)\int_0^x \frac{y^k}{\Gamma(s-r)}[lnF(x)-lnF(y)]^{s-r-1}\frac{f(y)}{F(x)}dy. \tag{16}$$

Or,

$$(\alpha+k+1)\int_0^x \frac{y^k}{\Gamma(s-r+1)}[lnF(x)-lnF(y)]^{s-r}f(y)dy$$
$$= (\alpha+1)\int_0^x \frac{y^k}{\Gamma(s-r)}[lnF(x)-lnF(y)]^{s-r-1}f(y)dy. \tag{17}$$

Differentiating both sides of the equation (17) with respect to x and simplifying for $(s-r)$ times, we get,

$$(\alpha+k+1)\frac{f(x)}{F(x)}\int_0^x y^k f(y)dy = (\alpha+1)x^k f(x).$$

Thus,

$$(\alpha+k+1)\int_0^x y^k f(y)dy = (\alpha+1)x^k F(x).$$

Differentiating above equation with respect to x, we get,

$$(\alpha+k+1)x^k f(x) = (\alpha+1)x^k f(x) + (\alpha+1)kx^{k-1}F(x),$$

i.e.,

$$\frac{f(x)}{F(x)} = \left(\frac{\alpha+1}{x}\right).$$

Which proves that $f(x)$ has the form as in (6).
Hence theorem 5.1 is proved. □

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