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# An explicit expression for moments of order statistics for four parameter generalized gamma distribution

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**Abstract.** Nadarajah and Pal (2008) found the explicit closed form expressions for the moments of order statistics from the two parameter gamma and three parameter generalized gamma distributions. Those expressions are shown to be very useful in terms of computing the moments of order statistics. This paper extends the same idea to obtain the explicit expressions for the exact moments of order statistics with the appropriate consideration for change of origin and scale. The said distribution represents a class of probability distributions (more than 50) as limiting forms.

## 1. Introduction

Let $X_1, X_2, ...X_n$ be a random sample of size n from a population having probability density function (pdf) f(x) and distribution function (DF) F(x), while $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the corresponding order statistics. The pdf of $X_{r:n}$, the rth order statistics is given by Arnold et al. (2008)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x). \tag{1}$$

A random variable X having a four parameter generalized Gamma distribution (Amoroso distribution) if its pdf is given by

$$f(x) = \frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1}\exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\} \tag{2}$$

for $x$, $a$, $\theta$, $\alpha$, $\beta$ in R, $\alpha > 0$; support $x \geq$ a if $\theta > 0$, $x \leq$ a if $\theta < 0$. The corresponding DF of X is given by

$$\begin{aligned} F(x) &= 1 - \frac{\gamma\left(\alpha, \left(\frac{x-a}{\theta}\right)^{\beta}\right)}{\Gamma(\alpha)} \quad \text{for } \theta > 0,\ \beta \geq 0 \\ &= \frac{\gamma\left(\alpha, \left(\frac{x-a}{\theta}\right)^{\beta}\right)}{\Gamma(\alpha)} \quad \text{for } \theta < 0,\ \beta \geq 0 \end{aligned} \tag{3}$$

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where $\gamma(.,.)$ denotes the incomplete gamma function defined by

$$\gamma(\alpha, y) = \int_0^y t^{\alpha-1} exp(-t)dt.$$

The said distribution (Crooks, 2010) is a continuous, univariate, unimodal probability density which represents a class of probability distributions (more than 50) as limiting forms. As a result, this single distribution summarizes and regularizes a wide number of interesting and common probability distributions.

An extensive numbers of articles on moments of order statistics are available in the literature. Tarter (1966) derived the exact expression for product moments of order statistics from truncated logistic distribution using Euler transformation. Saleh et al. (1975) derived exact expression for the first and second order moments of order statistics from the truncated exponential distribution. Margolin and Winkour (1967) obtained the exact expressions of first two moments of order statistics from the geometric distribution. Nadarajah and Pal (2008) established the explicit closed form expressions for the moments of order statistics from the two parameter gamma and three parameter generalized gamma distributions. They used these expressions in two quality control datasets and illustrate that the computational time is consistently smaller as compared to the integration formula for moments of order statistics. This paper extends the same idea to obtain explicit expressions for the exact moments of order statistics from four parameter generalized gamma distribution with the appropriate consideration for change of origin and scale; that are finite sums of special type of functions known as Lauricella function of type A and B defined by Lauricella (1893)

$$F_A^{(n)}(a, b_1, ...., b_n; c_1, ..., c_n; x_1, ..., x_n) = \sum_{m_1=0}^{\infty} ... \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n} (b_1)_{m_1} ... (b_n)_{m_n} x_1^{m_1} ... x_n^{m_n}}{(c_1)_{m_1} ... (c_n)_{m_n} m_1! ... m_n!}, \tag{4}$$

$$F_B^{(n)}(a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} ... \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_1} ... (b_n)_{m_n} x_1^{m_1} ... x_n^{m_n}}{(c)_{m_1+\cdots+m_n} m_1! ... m_n!}, \tag{5}$$

respectively, where $(f)_k = f(f+1)...(f+k-1)$ denotes the ascending factorial where k = 1, 2,....

Rest of the paper is organized as follows. Section 2 provides the preliminaries. The explicit expressions for $E\left(\frac{X_{r:n} - a}{\theta}\right)^k$ when $X_1, X_2, ..., X_n$ is a random sample from (2) are derived in Section 3. The extension of this result to non-identically distributed random variables is considered in Section 4. Finally, we conclude the paper in Section 5.

## 2. Preliminaries

The following results will be used in the remainder of the article implicitly.

(i) $$\gamma(\alpha, x) = x^{\alpha} \sum_{m=0}^{\infty} \frac{(-x)^m}{(\alpha+m)m!}, \text{ for all positive } \alpha, x \tag{6}$$

(ii) $$(f)_s = \frac{\Gamma(f+s)}{\Gamma(f)}, \text{ for } s \geqslant 0 \tag{7}$$

(iii) $$1 - \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = x^{\alpha-1} \exp(-x) \sum_{m=0}^{\infty} \frac{x^{-m}}{\Gamma(\alpha - m)} \tag{8}$$

## 3. I.I.D case

In this section we shall obtain the explicit expression of the kth moment of $X_{r:n}$ (with consideration of change of origin and scale) when $X_1, X_2, ..., X_n$ is a random sample from the p.d.f given by equation (2).

**Theorem 3.1.** *For the four parameter generalized gamma distribution as given in* (2) *for* $\theta > 0$ *and* $\beta > 0$,

$$E\left(\frac{X_{r:n}-a}{\theta}\right)^k = \frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{r-1}(-1)^l\binom{r-1}{l}(\Gamma(\alpha))^{r-1-l-n}\,\Gamma\left(\frac{k}{\beta}+\alpha(n-r+l+1)\right)\alpha^{r-n-l}$$
$$\times F_A^{(n-r+l)}\left(\frac{k}{\beta}+\alpha(n-r+l+1),\alpha,\dots,\alpha;\alpha+1,\dots,\alpha+1;-1,\dots,-1\right)$$

*Proof.* The kth moment of $X_{r:n}$ with the change of origin and scale can be expressed as

$$E\left(\frac{X_{r:n}-a}{\theta}\right)^k = \frac{n!}{(r-1)!(n-r)!}\beta\sum_{l=0}^{r-1}(-1)^l\binom{r-1}{l}(\Gamma(\alpha))^{r-1-l-n}\,I(l) \tag{9}$$

where

$$I(l) = \int_0^\infty z^{\alpha\beta+k-1}\left(\gamma\left(\alpha,z^\beta\right)\right)^{n-r+l}\exp(-z^\beta)dz. \tag{10}$$

Using the series expansion of equation (6), the integral in equation (10) can be expressed as

$$\begin{aligned}
I(l) &= \int_0^\infty z^{\alpha\beta+k-1}\left(z^{\alpha\beta}\sum_{m=0}^\infty\frac{(-z^\beta)^m}{(\alpha+m)m!}\right)^{n-r+l}\exp(-z^\beta)dz\\
&= \int_0^\infty\sum_{m_1=0}^\infty\cdots\sum_{m_{n-r+l}=0}^\infty(-1)^{m_1+\ldots+m_{n-r+l}}\exp\left(-z^\beta\right)\\
&\times\frac{\left(z^\beta\right)^{\frac{k-1}{\beta}+\alpha(n-r+l+1)+m_1+\cdots+m_{n-r+l}}}{(\alpha+m_1)\ldots(\alpha+m_{n-r+l})\,m_1!\ldots m_{n-r+l}!}dz\\
&= \frac{1}{\beta}\sum_{m_1=0}^\infty\cdots\sum_{m_{n-r+l}=0}^\infty(-1)^{m_1+\ldots+m_{n-r+l}}\\
&\times\frac{\Gamma\left(\frac{k}{\beta}+\alpha(n-r+l+1)+m_1+\ldots+m_{n-r+l}\right)}{(\alpha+m_1)\ldots(\alpha+m_{n-r+l})\,m_1!\ldots m_{n-r+l}!}.
\end{aligned} \tag{11}$$

After further simplification and using equation (7) in equation (11), we have

$$\begin{aligned}
I(l) &= \frac{1}{\beta}\alpha^{r-n-l}\Gamma\left(\frac{k}{\beta}+\alpha(n-r+l+1)\right)\\
&\times\sum_{m_1=0}^\infty\cdots\sum_{m_{n-r+l}=0}^\infty(-1)^{m_1}\ldots(-1)^{m_{n-r+l}}(\alpha)_{m_1}\ldots(\alpha)_{m_{n-r+l}}\\
&\times\frac{\left(\frac{k}{\beta}+\alpha(n-r+l+1)\right)_{m_1+\cdots+m_{n-r+l}}}{(\alpha+1)_{m_1}\ldots(\alpha+1)_{m_{n-r+l}}\,m_1!\ldots m_{n-r+l}!}.
\end{aligned} \tag{12}$$

Following the definition in equation (4), the equation (12) can be written as follows:

$$I(l) = \frac{1}{\beta}\alpha^{r-n-l}\Gamma\left(\frac{k}{\beta}+\alpha(n-r+l+1)\right) \times F_A^{(n-r+l)}\left(\frac{k}{\beta}+\alpha(n-r+l+1), \alpha, \ldots, \alpha; \alpha+1, \ldots, \alpha+1; -1, \ldots, -1\right). \tag{13}$$

Combining equation (9) and equation (13) we get the desired expression. □

**Theorem 3.2.** *For the four parameter generalized gamma distribution as given in eq. (2) for $\theta < 0$ and $\beta > 0$,*

$$E\left(\frac{X_{r:n}-a}{\theta}\right)^k = \frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}(\Gamma(\alpha))^{-r-l}\Gamma\left(\frac{k}{\beta}+\alpha(l+r)\right)\alpha^{1-l-r} \times F_A^{(l+r-1)}\left(\frac{k}{\beta}+\alpha(l+r), \alpha, ..., \alpha; \alpha+1, ..., \alpha+1; -1, ..., -1\right)$$

*Proof.*

$$\begin{aligned} E\left(\frac{X_{r:n}-a}{\theta}\right)^k &= \frac{n!}{(r-1)!(n-r)!(\Gamma(\alpha))^n}\left|\frac{\beta}{\theta}\right|\int_{-\infty}^{a}\left(\frac{x-a}{\theta}\right)^{\alpha\beta+k-1}\left\{\gamma\left(\alpha, \left(\frac{x-a}{\theta}\right)^{\beta}\right)\right\}^{r-1} \\ &\times\left\{\Gamma(\alpha)-\gamma\left(\alpha, \left(\frac{x-a}{\theta}\right)^{\beta}\right)\right\}^{n-r}\exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\}dx \\ &= \frac{n!}{(r-1)!(n-r)!(\Gamma(\alpha))^n}\beta\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l} \\ &\times(\Gamma(\alpha))^{n-r-l}\int_0^{\infty} z^{\alpha\beta+k-1}\left(\gamma\left(\alpha, z^{\beta}\right)\right)^{l+r-1}e^{-z^{\beta}}dz \\ &= \frac{n!}{(r-1)!(n-r)!}\beta\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}(\Gamma(\alpha))^{-r-l}I(l), \end{aligned} \tag{14}$$

where

$$I(l) = \int_0^{\infty} z^{\alpha\beta+k-1}\left(\gamma\left(\alpha, z^{\beta}\right)\right)^{l+r-1}\exp(-z^{\beta})dz.$$

Proceeding in a similar manner as Theorem 3.1, from equation (14), we get the final expression. □

## 4. N.I.D case

Let $X_1, X_2, ...X_n$ be independent gamma random variables with the pdfs given by

$$f_i(x) = \frac{1}{\Gamma(\alpha_i)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha_i\beta-1}\exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\} \tag{15}$$

for $x$, $a$, $\theta$, $\alpha_i$, $\beta$ in R, $\alpha_i > 0$; support $x \geq$ a if $\theta > 0$, $x \leq$ a if $\theta < 0$; i = 1, 2,...,n, while $X_{1:n} < X_{2:n} ... < X_{n:n}$ denote the corresponding order statistics. To obtain $E\left(\frac{X_{r:n}-a}{\theta}\right)^k$, we use the following result of Barakat and Abdelkader (2004):

$$E(X_{r:n})^k = \sum_{j=n-r+1}^{n}(-1)^{j-n+r-1}\binom{j-1}{n-r}I_j(k) \tag{16}$$

where for $x \geq a$,

$$I_j(k) = k \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \int_a^\infty x^{k-1} \prod_{t=1}^{j} (1 - F_{i_t}(x))\, dx,$$

$F_{i_t}(.)$ is the DF of $X_{i_t}$ given by $F_{i_t}(x) = \dfrac{\gamma(\alpha_{i_t}, x)}{\Gamma(\alpha_{i_t})}$. The results are derived in the following theorems.

**Theorem 4.1.** For the four parameter generalized gamma distribution as given in equation (15) for $\theta > 0$ and $\beta > 0$,

$$E\left(\frac{X_{r:n} - a}{\theta}\right)^k = \frac{\theta}{\beta} k \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \frac{\Gamma(\delta)}{j^\delta \Gamma(\alpha_{i_1} + 1) \ldots \Gamma(\alpha_{i_j} + 1)}$$
$$\times F_A^{(j)}\left(\delta, 1, \ldots, 1; \alpha_{i_1} + 1, \ldots, \alpha_{i_j} + 1; \frac{1}{j}, \ldots, \frac{1}{j}\right)$$

*where* $\delta = \dfrac{k}{\beta} + \alpha_{i_1} + \cdots + \alpha_{i_j}$.

*Proof.* Using equation (16), the kth moment of $X_{r:n}$ with the change of origin and scale can be expressed as

$$E\left(\frac{X_{r:n} - a}{\theta}\right)^k = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} I_j(k) \tag{17}$$

where

$$I_j(k) = k \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \int_a^\infty \left(\frac{x-a}{\theta}\right)^{k-1} \prod_{t=1}^{j} \{1 - F_{i_t}(x)\}\, dx. \tag{18}$$

Using equation (3) in equation (18), we obtain

$$I_j(k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{1}{\Gamma(\alpha_{i_1} + m_1 + 1) \ldots \Gamma(\alpha_{i_j} + m_j + 1)}$$
$$\times k \int_a^\infty \left\{\left(\frac{x-a}{\theta}\right)^\beta\right\}^{\frac{k-1}{\beta} + \alpha_{i_1} + \cdots + \alpha_{i_j} + m_1 + \cdots + m_j} e^{-j\left(\frac{x-a}{\theta}\right)^\beta} dx$$
$$= \frac{\theta}{\beta} k \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{1}{j^{\delta + m_1 + \cdots + m_j}}$$
$$\times \frac{\Gamma(\delta + m_1 + \cdots + m_j)}{\Gamma(\alpha_{i_1} + m_1 + 1) \ldots \Gamma(\alpha_{i_j} + m_j + 1)}. \tag{19}$$

After further simplification using the definition from equation (4) in equation (19), we get

$$I_j(k) = \frac{\theta}{\beta} k \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \frac{\Gamma(\delta)}{\Gamma(\alpha_{i_1} + 1) \ldots \Gamma(\alpha_{i_j} + 1)\, j^\delta}$$
$$\times F_A^{(j)}\left(\delta, 1, \ldots, 1; \alpha_{i_1} + 1, \ldots, \alpha_{i_j} + 1; \frac{1}{j}, \ldots, \frac{1}{j}\right). \tag{20}$$

Putting equation (20) in equation (17), we obtain the required result. □

**Theorem 4.2.** For the four parameter generalized gamma distribution as given in equation (15) for $\theta < 0$ and $\beta > 0$,

$$E\left(\frac{X_{r:n}-a}{\theta}\right)^k = \frac{\theta}{\beta}k \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1}\binom{j-1}{n-r} \times \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma(\lambda)}{\Gamma(\alpha_{i_1})\ldots\Gamma(\alpha_{i_j})\, j^{\lambda}} \times F_B^{(j)}\left(1-\alpha_{i_1},\ldots,1-\alpha_{i_j},1,...,1;1-\lambda;j,\ldots,j\right)$$

*where* $\lambda = \frac{k}{\beta} + (\alpha_{i_1}-1)+\cdots+\left(\alpha_{i_j}-1\right)$.

*Proof.* The same expression as given in equation (17) holds with

$$I_j(k) = k \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \int_{-\infty}^{a}\left(\frac{x-a}{\theta}\right)^{k-1}\prod_{t=1}^{j}\left\{1-\frac{\gamma\left(\alpha_{i_t},\left(\frac{x-a}{\theta}\right)^{\beta}\right)}{\Gamma(\alpha)}\right\}dx. \quad (21)$$

By putting equation (8), $I_j(k)$ in equation (21) can be written as

$$I_j(k) = k \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \sum_{m_1=0}^{\infty}\cdots\sum_{m_j=0}^{\infty}\frac{1}{\Gamma(\alpha_{i_1}-m_1)\ldots\Gamma\left(\alpha_{i_j}-m_j\right)} \times \int_{-\infty}^{a}\left(\left(\frac{x-a}{\theta}\right)^{\beta}\right)^{\left(\frac{k-1}{\beta}\right)+\left(\alpha_{i_1}-1+\cdots+\alpha_{i_j}-1\right)+\left(-m_1-\cdots-m_j\right)} \times e^{-j\left(\frac{x-a}{\theta}\right)^{\beta}}dx$$
$$= \frac{\theta}{\beta}k \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \sum_{m_1=0}^{\infty}\cdots\sum_{m_j=0}^{\infty} j^{m_1+\cdots+m_j-\lambda} \times \frac{\Gamma(\lambda-m_1-\cdots-m_j)}{\Gamma(\alpha_{i_1}-m_1)\ldots\Gamma\left(\alpha_{i_j}-m_j\right)}. \quad (22)$$

Substituting

$$\Gamma(\alpha_{i_t}-m_t) = \frac{(-1)^{m_t}\Gamma(\alpha_{i_t})}{(1-\alpha_{i_t})_{m_t}},$$

$$\Gamma(\lambda-m_1-\cdots-m_j) = \frac{(-1)^{m_1+\cdots+m_j}\Gamma(\lambda)}{(1-\lambda)_{m_1+\cdots+m_j}},$$

and following the definition in equation (5), we can represent equation (22) as

$$I_j(k) = \frac{\theta}{\beta}k \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma(\lambda)}{\Gamma(\alpha_{i_1})\ldots\Gamma\left(\alpha_{i_j}\right)j^{\lambda}} \times F_B^{(j)}\left(1-\alpha_{i_1},\ldots,1-\alpha_{i_j},1,\ldots,1;1-\lambda;j,\ldots,j\right). \quad (23)$$

Replacing equation (23) in equation (17), we get the desired expression. □

## 5. Conclusions

Nadarajah and Pal (2008) obtained the evident expressions for the moments of two parameter gamma and three parameter generalized gamma order statistics as finite sums of Lauricella functions. Those expressions were shown to take less computational time in order to calculate the moments of order statistics as compared to the integration formula. This paper obtains the explicit general expressions for the moments of order statistics from four parameter generalized (which represents more than 50 probability distributions as limiting forms) as finite sum of the same special functions.

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## References

[1] Arnold, B.C.,Balakrishnan, N. and Nagaraja, J. H. (2008). A first course in order statistics, *SIAM Publishers*, Philadelphia.
[2] Barakat, R. and Abdelkader (2004).Computing the moments of order statistics from nonidentical random variables , *Statistics Methods and Applications*, **13**, 15–26.
[3] Crooks, G. E. (2010). The Amoroso distribution, *Technical Note, Berkley National Laboratory.*
[4] Lauricella, G. (1893). Sulle funzioni ipergeometriche a più variabili, *Rend. Circ. Mat. Palermo*, **7**, 111–158.
[5] Margolin, B. H. and Winkour, H. B. (1967). Exact moments of the order statistics of the Geometric distribution and their relation to inverse sampling and reliability of reduntant systems, *Journal of the American Statistical Association*, **62**, 915–925.
[6] Nadarajah, S. and Pal, M. (2008). Explicit expressions for moments of gamma order statistics, *Bulletin Brazilian Mathematical Society*, **39(1)**, 45–60.
[7] Saleh, A. K., Scott. C. and Junkins, D. B. (1975). Exact first and second order moments of order statistics from the truncated exponential distribution, *Naval Research Logisitc*, **22(1)**, 65–77.
[8] Tarter, M. E. (1966). Exact moments and product moments of order statistics from the truncated logistic distribution,*Journal of the American Statistical Association*, **61**, 514–525.