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Interval estimation for lifetime distribution of k-unit parallel system

S.S.Godase[a], **D.T.Shirke**[b], **D.N.Kashid**[b]

[a]*Department of Statistics, SGM College, Karad, Satara,(MS), India 415 124.*
[b]*Department of Statistics, Shivaji University, Kolhapur,(MS), India 416 004.*

Abstract. In this paper we consider estimation of scale parameter of the half logistic distribution when life time distribution of a unit in k-unit parallel system belongs to the half logistic distribution. We construct confidence intervals (CIs) and tolerance intervals (TIs) using generalized pivotal quantity (GPQ).We derive the Modified Maximum Likelihood Estimator (MMLE) using the approach of Tiku and Suresh as the likelihood equations are intractable. Simulation studies are carried out to evaluate the performance of these CIs and TIs and compared them with existing ones. These intervals are illustrated through real life data due to Lawless (1982).

1. Introduction

In statistical point estimation problem, by taking sample size larger and larger one can obtain an estimator which will be quite close to the true value of the parameter in some sense, with a very high probability. Whatever be the sample size, the point estimator may not be equal to the true parameter value. This is so as there is a margin of uncertainty which can be expressed by the mean square error of estimator. The point estimator itself cannot tell how much error of uncertainty exists. To overcome this difficulty, one can use another method of estimation of parameter called the interval estimation. In this method we provide random subset of the parameter space which contains the true parameter value with certain probability. Some of the methods to obtain confidence sets are pivotal quantity method, method based on a suitable test and method based on large sample theory.

The concept of generalized variable has recently become popular in small sample inferences for complex problems such as Behrens-Fisher problem. These techniques have been shown to be efficient in specific distributions by using maximum likelihood estimators(MLEs). The generalized variable method was motivated by the fact that the small sample optimal confidence intervals in statistical problems involving nuisance parameters may not be available. The method of generalized confidence intervals (GCI) will be used whenever standard pivotal quantities either do not exist or are difficult to obtain. Weerahandi (1993) introduced the concept of generalized confidence interval. For some problems where the classical procedures are not optimal, generalized confidence interval performed well. Krishnamoorthy and Mathew (2003) developed exact confidence interval and tests for single lognormal mean using ideas of generalized p-values and generalized confidence intervals. Guo and Krishnamoorthy (2005) explained a problem of interval estimation and

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Email addresses: suwarna_godase@rediffmail.com (S.S.Godase), dtshirke@gmail.com (D.T.Shirke), dnk_stats@unishivaji.ac.in (D.N.Kashid)

testing for the difference between the quantiles of two populations using generalized variable approach.The literature survey reveals that during last ten years number of researchers have reported inference for the well known models using generalized variable approach, which motivated us to study the problem of confidence interval for the parameter of half logistic distribution under consideration.

A parallel system is one which keeps functioning until at least one of its components is functioning. A k-out-of-n system is one which functions as long as at least k of its n components are functioning. Kumbhar and Shirke (2004) gave tolerance limits for lifetime distribution of k-unit parallel system. Potdar and Shirke (2014) explained inference for the scale parameter of lifetime distribution of k-unit parallel system based on progressively censored data.

Tiku (1967) obtained modified maximum likelihood equations which have explicit solutions by replacing the intractable terms by linear approximations. Kantam et al. (2013) obtained estimation and testing in Type I generalized half logistic distribution by using MMLE. In this article we use MMLE to construct CIs and TIs.

In Section 2, the half logistic distribution in lifetime distribution of a k-unit parallel system is considered and discuss maximum likelihood estimator and modified maximum likelihood estimator. Section 3, provides CIs using large sample theory and using GPQ. Section 4, provides TIs using large sample theory and using GPQ. In section 5, the performance of the CIs and TIs by using MLE and MMLE for small samples is investigated using simulations. Results of the simulation study have been reported in same sections. One real data set has been analyzed for illustrative purpose in section 6.

2. Model and estimation of the scale parameter

Consider a k unit parallel system with independent and identically distributed lifetimes of components. Let $X_1, X_2, ..., X_k$ be the lifetimes, (where X_i is the lifetime of ith component) having half logistic distribution with scale parameter θ. Let lifetime of the system is $X = max(X_1, X_2, ..., X_k)$. The cdf of X is

$$F(x;\theta) = \left[\frac{1-e^{-x/\theta}}{1+e^{-x/\theta}}\right]^k; \qquad x \geq 0, \theta > 0 \tag{1}$$

The pdf of X is given by

$$f(x;\theta) = \frac{k}{\theta}\frac{2e^{-x/\theta}}{(1+e^{-x/\theta})^2}\left[\frac{1-e^{-x/\theta}}{1+e^{-x/\theta}}\right]^{(k-1)}; \qquad x \geq 0, \theta > 0 \tag{2}$$

where θ is scale parameter. In this paper interval estimation (confidence interval and tolerance interval) based on MLE and MMLE of scale parameter of above model is studied.

2.1. Maximum Likelihood Estimation

Log likelihood function is

$$L = nlog(k) - nlog(\theta) + \sum_{i=1}^{n}\log\left[\frac{2e^{-x_i/\theta}}{(1+e^{-x_i/\theta})^2}\right] + (k-1)\sum_{i=1}^{n}\log\left[\frac{1-e^{-x_i/\theta}}{1+e^{-x_i/\theta}}\right] \tag{3}$$

The MLE of θ can be obtained by solving $\frac{dL}{d\theta} = 0$, where

$$\frac{dL}{d\theta} = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{x_i e^{-x_i/\theta}}{1-e^{-x_i/\theta}} - \frac{2(k-1)}{\theta^2}\sum_{i=1}^{n}\frac{x_i e^{-x_i/\theta}}{1+e^{-x_i/\theta}} \tag{4}$$

The solution can be obtained by Newton-Raphson Method by taking initial solution $\hat{\theta}_o = \bar{X}$
Then the Fisher Information is given by

$$I(\theta) = \frac{-n}{\theta^2} + \frac{2}{\theta^4}\sum_{i=1}^{n}E\left[\frac{X_i^2 e^{-X/\theta}}{(1+e^{-X_i/\theta})^2}\right] + \frac{2}{\theta^3}\sum_{i=1}^{n}E\left[\frac{X_i(1-e^{-X_i/\theta})}{1+e^{-X_i/\theta}}\right] \tag{5}$$

$$+\frac{2(k-1)}{\theta^4}\sum_{i=1}^{n}E\left[\frac{X_i^2e^{-X_i/\theta}(1+e^{-2X_i/\theta})}{(1-e^{-X_i/\theta})^2(1+e^{-X_i/\theta})^2}\right]+\frac{4(k-1)}{\theta^3}\sum_{i=1}^{n}E\left[\frac{X_ie^{-X_i/\theta}}{(1-e^{-X_i/\theta})(1+e^{-X_i/\theta})}\right] \quad (6)$$

2.2. Modified Maximum Likelihood Estimation

In many situations, the maximum likelihood equations have no explicit solutions. This is due to the fact that a few terms in the maximum likelihood equations are intractable; see, for example, Tiku (1967, 1968). In this paper we use modified maximum likelihood estimation method and obtain MMLE for a scale parameter θ of half logistic distribution under consideration.
Let $x_1 < x_2 < ... < x_n$ be an ordered sample of size n from (2) (because the theory of order statistics is required in the estimation, an ordered sample is itself first considered). Put $\frac{x_i}{\theta} = z_i$ in equation (4) and it becomes

$$\frac{dL}{d\theta} = -n + \sum_{i=1}^{n} z_i - (k-1)\sum_{i=1}^{n}\frac{z_ie^{-z_i}}{1-e^{-z_i}} - (k+1)\sum_{i=1}^{n}\frac{z_ie^{-z_i}}{1+e^{-z_i}} \quad (7)$$

The maximum likelihood equation (7) does not have explicit solution for θ. This is due to the fact that the terms $G(z_i) = \frac{z_ie^{-z_i}}{1-e^{-z_i}}$ and $K(z_i) = \frac{z_ie^{-z_i}}{1+e^{-z_i}}$ are intractable. Therefore, we use the modified maximum likelihood approach to derive approximate MLE for θ with reference to Kantam et al. (2013) and then we obtain CIs and TIs for above model. In order to obtain an analytical expression for θ, the expression in equation (7) is approximated by some linear function in the respective population quantiles. The linearization is done in such a way that the derived MMLE retains all the desirable asymptotic properties of the MLEs. Let,

$$G(z_i) = \frac{z_ie^{-z_i}}{1-e^{-z_i}} \approx a_i + b_iz_i \quad (8)$$

$$K(z_i) = \frac{z_ie^{-z_i}}{1+e^{-z_i}} \approx c_i + d_iz_i \quad (9)$$

where a_i, b_i, c_i, d_i are to be suitably found. After using the approximation in (7) the solution for θ is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i - (k-1)\sum_{i=1}^{n} b_ix_i - (k+1)\sum_{i=1}^{n} d_ix_i}{n + (k-1)\sum_{i=1}^{n} a_i + (k+1)\sum_{i=1}^{n} c_i} \quad (10)$$

This estimator is named the MMLE of θ, which is a linear estimator in $x_i^{'}s$. To obtain a_i, b_i, c_i, d_i, let $p_i = \frac{i}{n+1}, i = 1, 2, ..., n$ and let t_i, t_i^* be the solutions of following equations, for example

$$F(t_i) = p_i - \sqrt{\frac{p_iq_i}{n}} = p_i^* \quad (11)$$

$$F(t_i^*) = p_i + \sqrt{\frac{p_iq_i}{n}} = p_i^{**} \quad (12)$$

with $q_i = 1 - p_i$ and where F(.) is cdf of (2). The intercepts a_i, c_i and slope b_i, d_i of linear approximations (8),(9) are respectively given by

$$b_i = \frac{G(t_i^*) - G(t_i)}{t_i^* - t_i} \quad (13)$$

$$d_i = \frac{k(t_i^*) - k(t_i)}{t_i^* - t_i} \quad (14)$$

$$a_i = G(t_i^*) - b_it_i^* \quad (15)$$

$$c_i = K(t_i^*) - d_it_i^* \quad (16)$$

Using the cdf of (2), the expressions for t_i, t_i^* are given by

$$t_i = log\left[\frac{1+p_i^{*1/k}}{1-p_i^{*1/k}}\right] \tag{17}$$

$$t_i^* = log\left[\frac{1+p_i^{**1/k}}{1-p_i^{**1/k}}\right] \tag{18}$$

In the following, we shall see two approximate methods of finding confidence intervals for scale parameter θ using MLE and MMLE.

Lemma 2.1

Distribution of $(\frac{\hat{\theta}_n}{\theta})$ and $(\frac{\hat{\theta}}{\theta})$, both are free from θ where $\hat{\theta}_n$ is MLE and $\hat{\theta}$ is MMLE.
Proof: The proof is similar to the one given by Gulati and Mi (2006). This lemma can be used to find GPQ.

3. Confidence Intervals

3.1. Using LS approach

By using asymptotic normal distribution of MLE, we construct confidence interval for θ. Let $\hat{\theta}_n$ is the MLE of θ. Therefore by Cramer (1946) $\hat{\theta}_n \to AN(\theta, \sigma^2(\hat{\theta}_n))$, where $\sigma^2(\hat{\theta}_n) = \frac{1}{I(\hat{\theta}_n)}$ be the asymptotic variance.
Therefore, $100(1-\alpha)\%$ asymptotic confidence interval for θ is given by

$$\left(\hat{\theta}_n - \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\theta}_n)}, \hat{\theta}_n + \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\theta}_n)}\right) \tag{19}$$

where $\tau_{\alpha/2}$ is the upper $100(\alpha/2)th$ percentile of standard normal distribution.
For more details please refer to Potdar and Shirke (2014).

According to Tiku and Suresh (1992) the derived MMLEs retain all the desirable asymptotic properties of the MLEs. Hence simply by replacing MLEs with MMLEs we can obtain CI using LS approach based on MMLE.

3.2. Using GV approach

The concept of a generalized confidence interval is due to Weerahandi (1993). One may refer to Weerahandi (1995) for a detailed discussion along with numerous examples. Consider a random variable X (scalar or vector) whose distribution $f(x, \theta, \delta)$ depends on a scalar parameter of interest θ and a nuisance parameter (parameter that is not of direct inferential interest)δ, where δ could be a vector. Suppose we are interested in computing a confidence interval for scale parameter θ. Let, x denotes the observed value of X. To construct a GCI for θ, we first define a GPQ, $T(X; x, \theta, \delta)$ which is a function of random variable X, its observed data x, the parameters θ and δ. A quantity $T(X; x, \theta, \delta)$is required to satisfy the following two conditions.
i) For a fixed x, the probability distribution of $T(X; x, \theta, \delta)$ is free of unknown parameters θ and δ;
ii) The observed value of $T(X; x, \theta, \delta)$, namely $T(x; x, \theta, \delta)$ is simply θ.
The percentiles of $T(X; x, \theta, \delta)$ can then be used to obtain confidence intervals for θ. Such confidence intervals are referred to as generalized confidence intervals. For example, if $T_{1-\tau}$ denotes the $100_{1-\tau}$ th percentile of $T(X; x, \theta, \delta)$, then $T_{1-\tau}$ is a generalized upper confidence limit for θ. Therefore $100(1-\tau)\%$ two-sided GCI for parameter θ is given by

$$(T_{\tau/2}, T_{1-\tau/2}) \tag{20}$$

Here we define GPQ as

$$T_1(X; x, \theta) = \frac{\hat{\theta}_o}{\frac{\hat{\theta}}{\theta}} \tag{21}$$

where $\hat{\theta}_o$ is the MLE obtained using observed data. We note the following:
i) Distribution of $T_1(X; x, \theta)$ is free from θ, which follows from Lemma (2.1) and
ii) $T_1(X; x, \theta) = \theta$, since for observed data, $\hat{\theta} = \hat{\theta}_o$. A GCI based on $T_1(X; x, \theta)$ is obtained by using the following algorithm.

I. Algorithm to obtain GCI for θ using GPQ

1. Input N, n, k, θ, τ.
2. Generate independently and identically distributed observations $(U_1, U_2, ..., U_n)$ from U (0,1).
3. For the given value of the parameter θ, set

$$x_i = -\theta log\left[\frac{1 - U_i^{1/k}}{1 + U_i^{1/k}}\right] \qquad for \quad i = 1, 2, ..., n. \tag{22}$$

Then $(x_1, x_2, ..., x_n)$ is the required sample from the distribution of a k-unit parallel system with half logistic distribution as the component life distribution.
4. Based on observations in step 3, obtain MLE of θ (say $\hat{\theta}_o$), using Newton-Raphson method.
5. Generate random sample of size n from F (.) with parameter θ =1.
6. Based on observations in step 5, obtain MLE of θ (say $\hat{\theta}$), using Newton-Raphson method.
7. Compute GPQ, $T_1 = \frac{\hat{\theta}_o}{\hat{\theta}}$
8. Repeat steps (5) to (7) N times, so as to get $T_{11}, T_{12}, ..., T_{1N}$.
9. Arrange $T_{1i}^{'}s$ in an ascending order. Denote them by $T_{(11)}, T_{(12)}, ..., T_{(1N)}$.
10. Compute a $100(1-\alpha)\%$ GCI for θ as$(T_{(1,([(\tau 2)N]))}, T_{(1,([(1-\tau 2)N]))}$.
Extending above algorithm one can estimate coverage probability of the proposed GCI.
In the above algorithm, we can replace MLE by MMLE and obtain GCI, based on MMLE.

4. Tolerance Intervals

4.1. Using LS approach (Large Sample Tolerance Intervals)

There are two types of tolerance intervals namely β-expectation tolerance interval and β-content-(1-γ) coverage tolerance interval.

4.1.1. β-expectation TI for the distribution function F (.; θ)

Let $X_\beta(\theta)$ be the lower quantile of order β of the distribution function F (.;θ). Then, we have

$$X_\beta(\theta) = -\theta log\left[\frac{1 - \beta^{1/k}}{1 + \beta^{1/k}}\right] \tag{23}$$

Since θ is unknown, we replace it by its MLE. Hence maximum likelihood estimate of $X_\beta(\theta)$

$$X_\beta(\hat{\theta}) = -\hat{\theta} log\left[\frac{1 - \beta^{1/k}}{1 + \beta^{1/k}}\right] \tag{24}$$

having an approximate upper β-expectation TI for F (.; θ) as

$$I_1(X) = (0, X_\beta(\hat{\theta})). \tag{25}$$

We approximate$E[F(X_\beta(\theta);\theta)]$ using Atwood (1984) and is given as

$$E[F(X_\beta(\hat{\theta});\theta)] \approx \beta - 0.5F_{02}Var(\hat{\theta}) + \frac{F_{01}Var(\hat{\theta})F_{11}}{F_{10}} \tag{26}$$

where $F_{10} = \frac{\partial F(x;\theta)}{\partial x}$,$F_{01} = \frac{\partial F(x;\theta)}{\partial \theta}$,$F_{11} = \frac{\partial^2 F(x;\theta)}{\partial x \partial \theta}$,$F_{02} = \frac{\partial^2 F(x;\theta)}{\partial \theta^2}$ with $x = X_\beta(\theta)$and all the derivatives are evaluated at X_β and θ. We can replace MLE by MMLE and obtain β-expectation TI for F (.; θ) based on MMLE. Simulated and approximate values of expected coverage of $I_1(X)$ using MLE and MMLE have been reported in section 5 for different values of n, β and k.

4.1.2. β-content-$(1-\gamma)$ coverage Tolerance Interval

Let $I_2(X) = (0, D\hat{\theta})$ be an upper β-content-(1-γ) coverage TI for the distribution having distribution function (5). The constant $D(>0)$ for $\beta(0,1)$, $\gamma(0,1)$ is to be determined such that

$$P\{F(D\hat{\theta};\theta) \leq \beta\} = 1-\gamma \tag{27}$$

That is

$$P\left\{\hat{\theta} \leq \theta \frac{-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]}{D}\right\} = 1-\gamma \tag{28}$$

Using asymptotic normality of $\hat{\theta}$, (28) can be equivalently written as

$$P\left\{Z \leq \frac{\theta}{var(\theta)} \frac{-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]}{D} - 1\right\} = 1-\gamma, \tag{29}$$

where $Z \to N(0,1)$. This gives

$$D = \frac{-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]}{1 + \frac{var(\theta)}{\theta} Z_{1-\gamma}} \tag{30}$$

Hence, an upper tolerance limit of $\beta - content - (1-\gamma)$ coverage tolerance interval $(I_2(X))$ is given by

$$U(X) = \hat{\theta}\left\{\frac{-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]}{1 + \frac{var(\theta)}{\theta} Z_{1-\gamma}}\right\} \tag{31}$$

4.2. Using GV approach (Generalized Tolerance Intervals)

The problem of computing a one-sided tolerance limit reduces to that of computing a one-sided confidence limit for the percentile of the relevant probability distribution. That is a $(\beta,(1-\gamma))$ upper tolerance limit is simply an (1-γ)th upper confidence limit for the (100β)th percentile of the population. It is easily seen that a $(\beta,(1-\gamma))$ upper tolerance limit for $F(.;\theta)$ is simply a $100(1-\gamma)\%$ upper confidence limit for $-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]$. We use the GV approach for obtaining the aforementioned upper confidence limit. Let $\hat{\theta}_o$ is the MLE obtained using observed data. The GPQ for constructing a confidence interval for θ is given by $T_1(X;x,\theta) = \frac{\hat{\theta}_o}{\hat{\theta}_i/\theta}$,i=1,2,...,N. The GPQ for $-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]$ is given by $T_2 = \frac{\hat{\theta}_o}{\hat{\theta}_i/\theta}\left\{-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]\right\}$, i= 1, 2,..., N. The $(1-\gamma)$th quantile of T_2 is a $(1-\gamma)$th generalized upper confidence bound for $-log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]$. Hence $(\beta,(1-\gamma))$ upper tolerance limit for $G(.;\theta)$ is $(0, T_{2,1-\gamma})$.

A generalized tolerance interval based on$T_2(X;x,\theta)$is obtained by using the following algorithm.

4.3. Algorithm to obtain Generalized Tolerance Interval for F (. ;) using GPQ (GV approach)

1. Input n, N, k, θ, β, γ.
2. Input random sample of size n from F (.) with an unknown parameter θ.
3. Based on observations in step 2, obtain MLE of θ (say $\hat{\theta}_o$), using Newton-Raphson method.
4. Generate random sample of size n from F (.) with parameter θ=1.
5. Based on observations in step 4, obtain MLE of θ (say$\hat{\theta}$), using Newton-Raphson method.
6. Compute GPQ, $T_2 = \frac{\hat{\theta}_o}{\hat{\theta}_i/\theta}\left\{ - log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]\right\}$, i= 1, 2,..., N.
7. Repeat steps (4) to (6) N times, so as to get $T_{21}, T_{22}, ..., T_{2N}$.
8. Arrange $T_{2i}^{'}s$ in an ascending order. Denote them by $T_{(21)}, T_{(22)}, ..., T_{(2N)}$.
9. Compute an upper tolerance limit of generalized TI $I_2(X) = (0, T_{2,1-\gamma})$.

Extending above algorithm one can estimate coverage probability of the proposed generalized TI.
In the above algorithm, we can replace MLE by MMLE and obtain generalized TI, based on MMLE.

5. Simulation study

We conduct extensive simulation experiments to evaluate performance of CIs (LS and GV) based on MLE and MMLE. Using, interval (19) large sample CI is computed. GCI is computed using algorithm given in section (3). In the simulation study, we generate n observations on $X = Max\{X_i, i = 1, 2, ..., k\}$, where $X_i, i = 1, 2, ..., k$ are iid half logistic distribution with scale parameter θ. Using Newton-Raphson method, we obtain the MLE and using Kantam's approach we obtain MMLE; based on the generated n observations. Repeating the process 10,000 times we estimate coverage of both intervals for n=3,4,5,6,7,8,9,10,15,30,50 and k=2,3.

Figures in the first row are based on MLE, while figures in the second row are based on MMLE. From tables 1-4, we observe that simulated coverage of GCI does not differ significantly whether it can be computed from MLE as well as MMLE. However, large sample approach underestimates the coverage probabilities for most of the scenarios, especially when the sample size is small and (or) the parameter θ is large. Also the performance of the proposed GCI does not depend on θ. However, as the sample size is large, the two estimators (MLE, MMLE) are equally efficient.

We investigate coverage (numerical and simulation) of β-expectation TI for F (.,θ) with k = 3 and β= 0.90, 0.95, 0.975, 0.99 by using MLE and MMLE. Figures in the 1st row are based on MLE, while figures in the 2nd row are based on MMLE. An upper β-expectation tolerance limit is given in equation (24). Results of the simulation study for the β-expectation tolerance interval, which is tabulated in table 5, indicate that, the estimated expectation and simulation mean for small sample size are marginally lower than the nominal value. As the sample size increases, the performance of tolerance intervals improves.

A simulation study of an upper β-content- (1-γ) coverage TI, having an upper limit (31) is also conducted, for k=2 and for known values of n,β, θ and γ. In this simulation study 5000 samples from F (.;θ) were generated and for each of the samples U(X) was computed, for different combinations of β, θ , γ. The proportion of samples for which -$\theta log\left[\frac{1-\beta^{1/k}}{1+\beta^{1/k}}\right]$ exceeded U(X) was computed 100 times and the mean of these 100 proportions is taken as simulated value of γ. The simulation study for the generalized TI was carried out as algorithm (4.1). Table 7 gives the simulated values of confidence level γ when k=2, θ =2 respectively.

The proposed confidence interval and tolerance interval performs satisfactory for small to moderate sample sizes. These intervals are superior to the large sample confidence intervals.

6. Real Data Example

Lawless (1982) provided real data, which represents the number of million revolutions before failure for each of 23 ball bearings in a life test: 17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.8,68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4. Half logistic distribution satisfactory fit to this data. We consider this data as outcome for lifetime for k-unit parallel system. Results obtained from this data are given in Table 9.

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Table 1: Mean coverage of Confidence Intervals (using MLE and MMLE) for half logistic distribution (k-unit parallel system): I_1) Large Sample procedure I_2) Generalized variable approach when θ=1, k=2

coverage	0.90		0.95		0.97		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
3	0.8587	0.9094	0.8906	0.9509	0.9066	0.9785	0.9421	0.9955
	0.7559	0.9012	0.7712	0.9458	0.8341	0.9668	0.8661	0.9883
4	0.8698	0.9046	0.9114	0.9557	0.9118	0.9765	0.9592	0.9946
	0.8312	0.9024	0.8821	0.9584	0.9045	0.9664	0.9245	0.9898
5	0.8857	0.9058	0.9127	0.9568	0.9233	0.9768	0.9534	0.9921
	0.8346	0.9035	0.8885	0.9532	0.9242	0.9731	0.9427	0.9934
6	0.8819	0.9030	0.9214	0.9522	0.9381	0.9874	0.9662	0.9935
	0.8647	0.9065	0.9276	0.9511	0.9270	0.9787	0.9513	0.9950
7	0.8837	0.9065	0.9012	0.9528	0.9357	0.9798	0.9582	0.9936
	0.8883	0.9047	0.9381	0.9520	0.9319	0.9724	0.9674	0.9931
8	0.8704	0.9098	0.9241	0.9509	0.9245	0.9778	0.9754	0.9906
	0.8914	0.9048	0.9350	0.9537	0.9517	0.9721	0.9659	0.9937
9	0.8854	0.9004	0.9151	0.9568	0.9421	0.9718	0.9668	0.9947
	0.8940	0.9024	0.9382	0.9553	0.9484	0.9701	0.9721	0.9964
10	0.8942	0.9017	0.9231	0.9549	0.9487	0.9780	0.9634	0.9965
	0.8955	0.9035	0.9374	0.9565	0.9541	0.9760	0.9772	0.9935
15	0.8921	0.9039	0.9264	0.9566	0.9462	0.9708	0.9721	0.9911
	0.9069	0.9066	0.9493	0.9531	0.9562	0.9734	0.9732	0.9975
30	0.8927	0.9097	0.9547	0.9509	0.9619	0.9767	0.9814	0.9997
	0.8999	0.9024	0.9530	0.9520	0.9686	0.9786	0.9883	0.9948
50	0.8901	0.9164	0.9532	0.9507	0.9752	0.9744	0.9921	0.9906
	0.9099	0.9035	0.9554	0.9543	0.9710	0.9779	0.991	0.9936

Table 2: Mean coverage of Confidence Intervals (using MLE and MMLE) for half logistic distribution (k-unit parallel system): I_1) Large Sample procedure I_2) Generalized variable approach when θ=1, k=3

coverage	0.90		0.95		0.97		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
3	0.8506 0.7962	0.9055 0.9050	0.887 0.8548	0.9515 0.9449	0.9162 0.8738	0.9489 0.9667	0.9395 0.9132	0.9984 0.9899
4	0.8321 0.8631	0.9087 0.8996	0.9144 0.9082	0.9534 0.9505	0.9133 0.9242	0.9764 0.9669	0.9519 0.9514	0.9968 0.9904
5	0.8685 0.8991	0.9094 0.9034	0.9223 0.9371	0.9501 0.9534	0.9254 0.9883	0.9748 0.9702	0.9583 0.9472	0.9975 0.9968
6	0.8651 0.8954	0.9064 0.9068	0.9244 0.9421	0.9582 0.9556	0.9306 0.9581	0.9701 0.9768	0.9621 0.9731	0.9945 0.9947
7	0.8593 0.8951	0.9015 0.9025	0.9113 0.9451	0.9557 0.9584	0.9373 0.9524	0.9764 0.9724	0.9745 0.9874	0.9962 0.9910
8	0.8721 0.8955	0.9034 0.9033	0.9267 0.9428	0.9536 0.9560	0.9270 0.9597	0.9781 0.9758	0.9762 0.9798	0.9932 0.9911
9	0.8564 0.9064	0.9022 0.9004	0.9187 0.9439	0.9559 0.9504	0.9417 0.9601	0.9732 0.9734	0.9735 0.9793	0.9948 0.9963
10	0.8522 0.9012	0.9007 0.9071	0.9209 0.9445	0.9501 0.9584	0.9407 0.9581	0.9735 0.9787	0.9564 0.9884	0.9954 0.9984
15	0.8511 0.8996	0.9002 0.9069	0.9212 0.9430	0.9542 0.9588	0.9416 0.9668	0.9704 0.9786	0.9731 0.9898	0.9987 0.9934
30	0.8724 0.9004	0.9082 0.9020	0.9574 0.9560	0.9518 0.9564	0.9669 0.9678	0.9789 0.9780	0.9837 0.9980	0.9914 0.9903
50	0.8985 0.9032	0.9188 0.9099	0.9554 0.9504	0.9529 0.9567	0.9765 0.9721	0.9777 0.9788	0.9994 0.9904	0.9910 0.9908

Table 3: Mean coverage of Confidence Intervals (using MLE and MMLE) for half logistic distribution (k-unit parallel system): I_1) Large Sample procedure I_2) Generalized variable approach when θ=5, k=2

coverage	0.90		0.95		0.97		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
3	0.8546 0.7652	0.9065 0.9050	0.8751 0.8341	0.9524 0.9448	0.9125 0.8364	0.9725 0.9669	0.9347 0.8741	0.9953 0.9878
4	0.8684 0.8344	0.9025 0.9051	0.9123 0.8962	0.9536 0.9585	0.9109 0.8762	0.9748 0.9664	0.9494 0.9193	0.9975 0.9932
5	0.8725 0.8669	0.9022 0.9084	0.9264 0.8937	0.9558 0.9564	0.9242 0.9223	0.9767 0.9784	0.9539 0.9365	0.9930 0.9965
6	0.8575 0.8671	0.9035 0.9032	0.9273 0.9141	0.9517 0.9528	0.9252 0.9321	0.9884 0.9721	0.9584 0.9541	0.9919 0.9942
7	0.8481 0.8954	0.9098 0.9028	0.9126 0.9142	0.9504 0.9534	0.9311 0.9356	0.9734 0.9724	0.9530 0.9780	0.9941 0.9920
8	0.8493 0.8932	0.9004 0.9021	0.9288 0.9365	0.9588 0.9512	0.9474 0.9463	0.9709 0.9734	0.9593 0.9612	0.9961 0.9951
9	0.8613 0.8957	0.9051 0.9054	0.9195 0.9387	0.9557 0.9584	0.9536 0.9521	0.9774 0.9780	0.9612 0.9753	0.9938 0.9982
10	0.8742 0.9061	0.9039 0.9087	0.9215 0.9463	0.9506 0.9521	0.9489 0.9597	0.9791 0.9737	0.9669 0.9791	0.9909 0.9938
15	0.8870 0.9010	0.9094 0.9090	0.9337 0.9445	0.9538 0.9550	0.9420 0.9669	0.9764 0.9781	0.9741 0.9889	0.9934 0.9955
30	0.8967 0.9065	0.9037 0.9021	0.9542 0.9535	0.9519 0.9521	0.9652 0.9780	0.9795 0.9770	0.9771 0.9878	0.9909 0.9980
50	0.9009 0.9036	0.9019 0.9087	0.9503 0.9564	0.9507 0.9595	0.9710 0.9784	0.9784 0.9781	0.9840 0.9902	0.9997 0.9973

Table 4: Mean coverage of Confidence Intervals (using MLE and MMLE) for half logistic distribution (k-unit parallel system): I_1) Large Sample procedure I_2) Generalized variable approach when θ=5, k=3

coverage	0.90		0.95		0.97		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
3	0.8346	0.9018	0.8794	0.9615	0.9134	0.9706	0.9342	0.9945
	0.7881	0.8949	0.8652	0.9586	0.8612	0.9668	0.9094	0.9889
4	0.8464	0.9036	0.9155	0.9533	0.9116	0.9745	0.9431	0.9965
	0.8593	0.9065	0.8947	0.9534	0.8994	0.9787	0.9531	0.9902
5	0.8737	0.9003	0.9235	0.9509	0.9258	0.9785	0.9505	0.9926
	0.8962	0.9031	0.9283	0.9563	0.9421	0.9754	0.9714	0.9901
6	0.8551	0.9099	0.9243	0.9564	0.9263	0.9859	0.9546	0.9919
	0.8753	0.9068	0.9351	0.9514	0.9497	0.9762	0.9773	0.9954
7	0.8448	0.9064	0.9126	0.9551	0.9327	0.9764	0.9583	0.9909
	0.8854	0.9014	0.9381	0.9584	0.9541	0.9732	0.9784	0.9975
8	0.8419	0.9058	0.9244	0.9536	0.9412	0.9799	0.9517	0.9944
	0.8962	0.9011	0.9453	0.9536	0.9673	0.9764	0.9782	0.9984
9	0.8639	0.9020	0.9136	0.9590	0.9546	0.9706	0.9634	0.9959
	0.8971	0.9064	0.9487	0.9531	0.9618	0.9721	0.9823	0.9951
10	0.8711	0.9064	0.9244	0.9564	0.9490	0.9788	0.9637	0.9967
	0.9013	0.9033	0.9498	0.9564	0.9635	0.9708	0.9843	0.9962
15	0.8804	0.9055	0.9337	0.9542	0.9447	0.9725	0.9765	0.9921
	0.9061	0.9062	0.9558	0.9501	0.9728	0.9780	0.9960	0.9937
30	0.8955	0.9082	0.9502	0.9554	0.9641	0.9744	0.9734	0.9904
	0.9035	0.9024	0.9562	0.9584	0.9734	0.9789	0.9924	0.9957
50	0.9016	0.9109	0.9554	0.9565	0.9770	0.9714	0.9807	0.9908
	0.9012	0.9034	0.9564	0.9532	0.9741	0.9788	0.9950	0.9924

Table 5: Simulated mean and estimated expectation of the coverage of approximate β-expectation TI using MLE and MMLE for half logistic distribution(k unit parallel system)

	$\alpha = 3$							
	$\beta(\theta = 1.0)$				$\beta(\theta$=2.0)			
n	0.90	0.95	0.97	0.99	0.90	0.95	0.97	0.99
2	0.8232	0.8623	0.9025	0.9425	0.8425	0.8841	0.9024	0.9623
	(0.8351)	(0.8526)	(0.8914)	(0.9514)	(0.8521)	(0.8741)	(0.9142)	(0.9667)
	0.8140	0.0.8714	0.9120	0.9428	0.8389	0.8630	0.9150	0.9520
	(0.7924)	(0.8832)	(0.9047)	(0.9417)	(0.8340)	(0.8605)	(0.9240)	(0.9567)
3	0.8578	0.8902	0.9394	0.9531	0.8415	0.9124	0.9433	0.9640
	(0.8435)	(0.8864)	(0.9376)	(0.9442)	(0.8427)	(0.9075)	(0.9302)	(0.9641)
	0.8471	0.9171	0.9341	0.9507	0.8434	0.8964	0.9349	0.9518
	(0.8514)	(0.9034)	(0.9244)	(0.9514)	(0.8594)	(0.9094)	(0.9434)	(0.9546)
4	0.8711	0.9133	0.9407	0.9746	0.8360	0.9032	0.9344	0.9711
	(0.8704)	(0.9079)	(0.9546)	(0.9737)	(0.8317)	(0.9143)	(0.9494)	(0.9802)
	0.0.8574	0.8931	0.9475	0.9644	0.8508	0.9166	0.9464	0.9675
	(0.8424)	(0.8934)	(0.9442)	(0.9746)	(0.8433)	(0.9094)	(0.9423)	(0.9755)
5	0.8646	0.9045	0.9435	0.9780	0.8620	0.9230	0.9591	0.9745
	(0.8784)	(0.9147)	(0.9341)	(0.9774)	(0.8742)	(0.9324)	(0.9646)	(0.9782)
	0.8664	0.9284	0.9515	0.9647	0.8734	0.9042	0.9418	0.9635
	(0.8779)	(0.9121)	(0.9427)	(0.9720)	(0.8794)	(0.9184)	(0.9536)	(0.9741)
6	0.8784	0.9338	0.9535	0.9841	0.8718	0.9150	0.9594	0.9782
	(0.8772)	(0.9360)	(0.9566)	(0.9822)	(0.8842)	(0.9222)	(0.9564)	(0.9749)
	0.8640	0.9174	0.9574	0.9727	0.8709	0.9108	0.9488	0.9870
	(0.8724)	(0.9044)	(0.9511)	(0.9819)	(0.8717)	(0.9149)	(0.9560)	(0.9895)
7	0.8633	0.9435	0.9623	0.9837	0.8843	0.9327	0.9671	0.9846
	(0.8617)	(0.9474)	(0.9604)	(0.9849)	(0.8811)	(0.9443)	(0.9712)	(0.9944)
	0.0.8532	0.9219	0.9579	0.9737	0.8797	0.9281	0.9520	0.9708
	(0.8405)	(0.9147)	(0.9654)	(0.9745)	(0.8835)	(0.9250)	(0.9634)	(0.9832)
8	0.8898	0.9233	0.9635	0.9808	0.8670	0.9237	0.9665	0.9888
	(0.8712)	(0.9208)	(0.9647)	(0.9849)	(0.8789)	(0.9469)	(0.9694)	(0.9841)
	0.0.8947	0.9347	0.9610	0.9737	0.8770	0.9339	0.9528	0.9894
	(0.8871)	(0.9349)	(0.9538)	(0.9742)	(0.8846)	(0.944)	(0.9639)	(0.9884)

Table 6: Simulated mean and estimated expectation of the coverage of approximate β-expectation TI using MLE and MMLE for for half logistic distribution(k unit parallel system). Continued

	$\alpha = 3$							
	$\beta(\sigma = 1.0)$				$\beta(\sigma=2.0)$			
n	0.90	0.95	0.97	0.99	0.90	0.95	0.97	0.99
9	0.8704	0.9221	0.9622	0.9847	0.8717	0.9342	0.9614	0.9853
	(0.8822)	(0.9204)	(0.9634)	(0.9827)	(0.8764)	(0.9432)	(0.9780)	(0.9874)
	0.0.8987	0.9318	0.9649	0.9838	0.8771	0.9211	0.9526	0.9860
	(0.8935)	(0.9244)	(0.9647)	(0.9873)	(0.8704)	(0.9394)	(0.9643)	(0.9897)
10	0.8874	0.9235	0.9524	0.9846	0.8834	0.9414	0.9601	0.9813
	(0.8864)	(0.9314)	(0.9590)	(0.9809)	(0.8918)	(0.9432)	(0.9744)	(0.9881)
	0.0.8897	0.9347	0.9628	0.9857	0.8743	0.9309	0.9630	0.9840
	(0.8808)	(0.9411)	(0.9643)	(0.9849)	(0.8810)	(0.9447)	(0.9714)	(0.9844)
15	0.8934	0.9364	0.9641	0.9933	0.8797	0.9333	0.9644	0.9832
	(0.8970)	(0.9341)	(0.9737)	(0.9847)	(0.8742)	(0.9338)	(0.973)	(0.9849)
	0.0.8905	0.9489	0.9749	0.9805	0.8817	0.9364	0.9615	0.9875
	(0.8933)	(0.9434)	(0.9724)	(0.9919)	(0.8935)	(0.9447)	(0.9708)	(0.9844)
30	0.8891	0.9404	0.9770	0.9930	0.8987	0.9528	0.9649	0.9945
	(0.9032)	(0.9497)	(0.9618)	(0.9924)	(0.9012)	(0.9540)	(0.9628)	(0.9827)
	0.0.9040	0.9414	0.9731	0.9897	0.8827	0.9415	0.9715	0.9855
	(0.9034)	(0.9347)	(0.9745)	(0.9940)	(0.8904)	(0.9546)	(0.9748)	(0.9849)
50	0.9097	0.9422	0.9624	0.9988	0.9035	0.9522	0.9790	0.9910
	(0.9055)	(0.9546)	(0.9634)	(0.9994)	(0.9144)	(0.9549)	(0.9743)	(0.9928)
	0.0.9007	0.9580	0.9737	0.9914	0.8960	0.9520	0.9720	0.9986
	(0.9132)	(0.9546)	(0.9748)	(0.9940)	(0.9083)	(0.9534)	(0.9814)	(0.9994)

Table 7: Coverage probabilities of Tolerance Intervals for half logistic distribution(k unit parallel system) I_1) Large sample procedure I_2) Generalized variable approach θ=1.0, k=2.0

	γ=0.90				γ=0.95			
coverage	β=0.90		β=0.95		β=0.90		β=0.95	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
2	0.6872	0.9034	0.6620	0.8912	0.5625	0.9415	0.5784	0.9414
	0.6751	0.9081	0.6741	0.8905	0.5502	0.9490	0.5649	0.9420
3	0.8084	0.8901	0.7741	0.8944	0.7150	0.9477	0.7512	0.9475
	0.7946	0.9083	0.7803	0.8994	0.7244	0.9410	0.7501	0.9448
4	0.8216	0.9099	0.7914	0.9054	0.8278	0.9519	0.8534	0.9433
	0.8416	0.9075	0.8025	0.8919	0.8246	0.9402	0.8428	0.9518
5	0.8520	0.9016	0.8326	0.9071	0.8789	0.9547	0.8746	0.9505
	0.8612	0.9075	0.8489	0.9052	0.8847	0.9533	0.8824	0.9520
6	0.8652	0.9029	0.8656	0.9028	0.8964	0.9510	0.9018	0.9546
	0.8688	0.9010	0.8607	0.9017	0.9011	0.9519	0.9035	0.9515
7	0.8704	0.9011	0.8748	0.9052	0.8994	0.9548	0.9064	0.9545
	0.8724	0.8947	0.8721	0.9005	0.9035	0.9547	0.9048	0.9585
8	0.8669	0.8902	0.8652	0.9044	0.9019	0.9635	0.9075	0.9549
	0.8723	0.9054	0.8699	0.9097	0.9021	0.9594	0.9147	0.9577
9	0.8649	0.9091	0.8458	0.9047	0.9046	0.9516	0.9090	0.9548
	0.8794	0.9033	0.8735	0.9019	0.9197	0.9444	0.9134	0.9533
10	0.8784	0.9018	0.8659	0.9137	0.9145	0.9474	0.9015	0.9664
	0.8733	0.9072	0.8720	0.8980	0.9249	0.9520	0.9287	0.9514
15	0.8851	0.9034	0.8619	0.8975	0.9161	0.9591	0.9233	0.9501
	0.8749	0.9050	0.8649	0.9081	0.9234	0.9540	0.9247	0.9575
30	0.8851	0.9194	0.8721	0.9022	0.9291	0.9578	0.9344	0.9545
	0.8802	0.9048	0.8798	0.9071	0.9344	0.9541	0.9348	0.9579
50	0.9061	0.9075	0.8849	0.9064	0.9438	0.9599	0.9246	0.9515
	0.9024	0.9089	0.8932	0.9087	0.9405	0.9501	0.9447	0.9522

Table 8: Coverage probabilities of Tolerance Intervals for half logistic distribution(k unit parallel system) I_1) Large sample procedure I_2) Generalized variable approach θ=2.0, k=2.0

	γ=0.90				γ=0.95			
coverage	β=0.90		β=0.95		β=0.90		β=0.95	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
2	0.6546	0.8915	0.6789	0.8930	0.5847	0.9464	0.5649	0.9512
	0.6625	0.8987	0.6714	0.8951	0.5801	0.9420	0.5642	0.9494
3	0.7984	0.8994	0.8124	0.8920	0.7724	0.9441	0.7487	0.9434
	0.7846	0.8946	0.7948	0.8924	0.7439	0.9418	0.7411	0.9420
4	0.8214	0.9048	0.8312	0.8943	0.8648	0.9574	0.8597	0.9505
	0.8370	0.9021	0.8361	0.9084	0.8502	0.9582	0.8510	0.9404
5	0.8315	0.9098	0.8541	0.9077	0.8812	0.9491	0.9031	0.9581
	0.8524	0.9037	0.8581	0.8961	0.8945	0.9583	0.9019	0.9532
6	0.8565	0.9064	0.8491	0.9038	0.9184	0.9572	0.9088	0.9465
	0.8694	0.9019	0.8656	0.9019	0.9254	0.9484	0.9102	0.9545
7	0.8535	0.9082	0.8528	0.9025	0.8896	0.9550	0.9194	0.9575
	0.8651	0.9046	0.8664	0.9027	0.8987	0.9561	0.9136	0.9544
8	0.8742	0.9018	0.8729	0.9053	0.9091	0.9540	0.9145	0.9515
	0.8746	0.9134	0.8637	0.9044	0.9102	0.9527	0.9210	0.9574
9	0.8578	0.9015	0.8665	0.9108	0.9055	0.9549	0.8829	0.9515
	0.8798	0.9028	0.8735	0.9067	0.9132	0.9518	0.8938	0.9566
10	0.8414	0.9139	0.8506	0.9009	0.8965	0.9506	0.9135	0.9514
	0.8784	0.9014	0.8760	0.9147	0.9294	0.9589	0.9216	0.9542
15	0.8675	0.9004	0.8625	0.8954	0.9235	0.9448	0.9142	0.9555
	0.8704	0.9084	0.8734	0.9063	0.9264	0.9546	0.9225	0.9587
30	0.8628	0.9095	0.8509	0.9054	0.9122	0.9522	0.9228	0.9689
	0.8846	0.9062	0.8846	0.9075	0.9346	0.9541	0.9380	0.9547
50	0.8915	0.9051	0.8855	0.9049	0.9247	0.9543	0.9497	0.9551
	0.8991	0.9041	0.8975	0.9034	0.9401	0.9588	0.9574	0.9587

Table 9: Confidence intervals (using LS and GV approach) for real data.

Coverage	Using Estimator	Using LS approach	Using GV approach
90%	MLE	(12.16498,23.688612) Length=11.523632	(11.745021,20.20297) Length=8.457949
	MMLE	(11.38794,25.283543) Length=13.895603	(10.402121,20.805491) Length=10.40337
95%	MLE	(16.25966,36.54188) Length=20.28222	(8.180416,16.23732) Length=8.056904
	MMLE	(15.737967,37.332039) Length=21.594072	(8.366712,14.836193) Length=6.469481
99%	MLE	(8.801689,26.805990) Length=18.004301	(6.711657,15.89151) Length=9.179853
	MMLE	(9.644007,25.425999) Length=15.781992	(7.884905,14.698532) Length=6.813627