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Characterization of One-Truncation Parameter Family of Distributions Through Expectation of Function of Order Statistics

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Abstract. For characterization of one (left or right)-truncation parameter families of distributions one needs any arbitrary non-constant function of order statistics only in place of various alternative approaches available in the literature. Path breaking different approach for characterization of general setup of one-truncation parameter family of distributions through expectation of any arbitrary non constant differentiable function of order statistics is obtained. Applications and examples are given for illustrative purpose.

1. Introduction

One-truncation parameter family of distributions with probability density function (pdf)

$$f_j(x;\theta) = \begin{cases} \begin{cases} q_1(\theta)h_1(x); & \text{for } j=1, a<\theta<x<b \\ 0, & \text{otherwise,} \end{cases} \\ \begin{cases} q_2(\theta)h_2(x); & \text{for } j=2, a<x<\theta<b \\ 0, & \text{otherwise,} \end{cases} \end{cases} \tag{1}$$

where $-\infty \leq a < b \leq \infty$ are known constant, $a<\theta<x<b$ for $j=1$, $a<x<\theta<b$ for $j=2$, $h_j;(j=1,2)$ are positive absolutely continuous functions, $q_j;(j=1,2)$ are everywhere differentiable functions is characterized.

Since $h_j(.);(j=1 \text{ or } 2)$ is positive and the range is truncated by truncation parameter θ from left or right respectively $q_1^{-1}(b)=q_2^{-1}(a)=0$. Through out the paper $q_j^{-1}(.)$ is reciprocal of $q_j(.)$.

Most powerful application of characterizations of distribution is to address a fundamental problem of identification of an appropriate model that can describe the real situation which generate the observations.

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For instant 60 observations of random phenomena observed and one group of student fit normal distribution where other group fit log-normal distribution with almost same p-value. This is one of case where characterization results provide navigation tools for correct direction of further study (research). Therefore characterizations of distribution is of general interest to mathematical community, to probabilists and statisticians as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building. Motivated by such future in this paper, identity of distribution and equality of expectation is used to, characterized (left or right)-truncation parameter family of distributions defined in (1) through expectation of any arbitrary non-constant differentiable function of order statistics which includes characterization of negative exponential distribution, Pareto distribution as special case of $f_1(x;\theta)$ where as power function distribution, uniform distribution, generalize uniform distribution as special case of $f_2(x;\theta)$.

Several characterizations of these distributions by various approaches are available in the literature. Notably for power function distribution independence of suitable function of order statistics and distributional properties of transformation of exponential variable used by Fisz (1958), Basu (1965), Govindarajulu (1966) and Dallas (1976), linear relation of conditional expectation used by Beg and Kirmani(1974), recurrence relations between expectations of function of order statistics used by Alli and Khan (1998), record valves used by Nagraja (1977), lower record statistics used by Faizan and Khan(2011), product of order statistics used by Arslan (2011) and Lorenz curve used by Moothathu (1986) are available in the literature.

Other approaches such as coefficient of correlation of order statistics of sample of size two used by Bartoszyn'ski (1980), Terreel (1983), Fernando and Rebollo (1997), maximal correlation coefficient between order statistics, of identically distributed spacings etc [used by Stapleton (1963), Arnold and Meeden (1976), Driscoll, M.F. (1978), Shimizu and Huang (1983), Abdelhamid (1985)], power contraction of order statistics by Navarro(2008), Random translation, dilation and contraction of order Statistics by Imtiyaz, Shah, Khan and Barakat(2014), moment conditions used by Lin (1988), Too and Lin (1989), moments of n-fold convolution modulo one used by Chow and Huang (1999), inequalities of chernoff-type used by Sumrita and Subir (1990) for characterization of uniform distribution.

Various approaches were used for characterization of negative exponential distribution. Amongst many other Fisz (1958), Tanis (1964), Rogers (1963) and Fergusion (1967) used properties of identical distributions, absolute continuity, constant regression of adjacentt order statistics, Fergusion (1964, 1965) and Crawford (1966), used linear regression of adjacent order statistics of random, independent and non degenerate random variables, Nagaraja (1977, 1988) used linear regression of two adjacent record values were as Khan, Mohd and Ziaul (2009) used difference of two conditional expectations, conditioned on a non-adjacent order statistics to characterized negative exponential distribution.

Economic variation in reported income and true income used by Krishnaji (1970), Nagesh (1974), independence of suitable function of order statistics used by Henrick (1970), Ahsanullah (1973, 1974), Shah (1981) and Dimaki and Evdokia (1993), linear relation of conditional expectation used by Beg and Kirmani (1974), Dallas (1976), recurrence relations between expectations of function of order statistics used by Alli and Khan (1998), exponential and related distributions used by Tavangar and Asadi(2010), for characterization of Pareto distribution.

Necessary and sufficient conditions for pdf $f(x;\theta)$ to be $f_j(x;\theta),(j = 1 or 2)$, defined in (1) is established in section 2. Section 3 is devoted for applications where as section 4 is devoted to examples for illustrative purpose.

2. Characterization

Theorem 2.1. *Let $X_1, X_2, ..., X_n$ be a random sample of size n from distribution function $F_j; j = 1, 2$. Let $X_{1:n} < X_{2:n}, ..., < X_{n:n}$ be the set of corresponding order statistics. Assume that $F_j; j = 1, 2$ continuous on the interval (a, b) where $-\infty < a < b < \infty$. Let $g(.)$ be non-constant differentiable function of t^{th} order statistic $(j = 1, t = 1$ and $j = 2, t = n)$ on the interval (a, b) where $-\infty < a < b < \infty$. Then p.d.f. $f_j(x; \theta)$ of F_j to be $f_j(x; \theta), j = 1, 2$ defined in (1) if and only if*

$$g(\theta) = E[\phi_j(X_{t:n})] = E\Big[g(X_{t:n}) + \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})}\Big]. \tag{2}$$

Proof. Given $f_j(x; \theta), j = 1, 2$ defined in (1), for necessity of (2) if $\phi_j(X_{t:n})$ is such that $g(\theta) = E[\phi_j(X_{t:n})]$ where $g(\theta)$ is differentiable function then using $f_j(x_{t:n}; \theta)$; pdf of t^{th} order statistic $j = 1, t = 1$ and $j = 2, t = n$ one gets,

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) f_1(x_{1:n}; \theta) \mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) f_2(x_{n:n}; \theta) \mathrm{d}x_{n:n}, & \text{for } j = 2, t = n \end{cases} \tag{3}$$

Based on $f_j(x : \theta)$ given in (1), substituting $f_j(x_{t:n} : \theta)$ for $j = 1, t = 1$ and $j = 2, t = n$,the (3) will be

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) n q_1^n(\theta) q_1^{-n+1}(x_{1:n}; \theta) h_1(x_{1:n}) \mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) n q_2^n(\theta) q_2^{-n+1}(x_{n:n}; \theta) h_2(x_{n:n}) \mathrm{d}x_{n:n}, & \text{for } j = 2, t = n \end{cases} \tag{4}$$

After simplification the (4) will be

$$\frac{g(\theta)}{n q_1^n(\theta)} = \int_\theta^b \phi_1(x_{1:n}) q_1^{-n+1}(x_{1:n}; \theta) h_1(x_{1:n}) \mathrm{d}x_{1:n}; \text{for } j = 1, t = 1 \tag{5}$$

and

$$\frac{g(\theta)}{n q_2^n(\theta)} = \int_a^\theta \phi_2(x_{n:n}) q_2^{-n+1}(x_{n:n}; \theta) h_2(x_{n:n}) \mathrm{d}x_{n:n}; \text{for } j = 1, t = 1 \tag{6}$$

Differentiating (5)and (6) with respect to θ on both sides and replacing $X_{1:n}$ for θ in (5) and replacing $X_{n:n}$ for θ in (6)and simplifying one gets

$$\phi_j(X_{t:n}) = g(X_{t:n}) + \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})]}, j = 1, t = 1 \text{ and } j = 2, t = n. \tag{7}$$

Note that

$$M_j(X_{t:n}) = \frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})], j = 1, t = 1 \text{ and } j = 2, t = n, \tag{8}$$

is finite function of $X_{t:n}$. Further $\phi_j(X_{t:n})$ derived in (7) reduces to (2). This establishes necessity of (2). Conversely given (2) $k_j(x_{t:n};\theta)$ be any arbitrary non constant integrable function of t^{th} order statistic, $j = 1, t = 1$ and $j = 2, t = n$ such that

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) k_1(x_{1:n};\theta) \mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) k_2(x_{n:n};\theta) \mathrm{d}x_{n:n} & \text{for } j = 2, t = n. \end{cases} \tag{9}$$

Since q_1 is increasing function with $q_1^{-1}(b) = 0$ and q_2 is decreasing function with $q_2^{-1}(a) = 0$ following identity holds.

$$g(\theta) \equiv \begin{cases} \int_\theta^b -q_1^n(\theta) \Big[\frac{d}{dx_{1:n}} g(x_{1:n}) q_1^{-n}(x_{1:n})\Big] \mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta q_2^n(\theta) \Big[\frac{d}{dx_{n:n}} g(x_{n:n}) q_2^{-n}(x_{n:n}) \mathrm{d}x_{n:n}, & \text{for } j = 2, t = n. \end{cases} \tag{10}$$

Differentiating integrand of (10) $q_j^{-n}(x_{t:n}) g(x_{t:n})$ and tacking $\frac{d}{dx_{t:n}} q_1^{-n}(x_{t:n})$ as one factor $j = 1, t = 1$ and $j = 2, t = n$ one gets (10) as

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) \Big[- q_1^{-n}(\theta) \frac{d}{dx_{1:n}} q_1^{-n}(x_{1:n})\Big] \mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) \Big[q_2^{-n}(\theta) \frac{d}{dx_{n:n}} q_2^{-n}(x_{n:n})\Big] \mathrm{d}x_{n:n} & \text{for } j = 2, t = n. \end{cases} \tag{11}$$

where $\phi_j(x_{t:n})$ is as derived in (7)($j = 1, t = 1$ and $j = 2, t = n$). From (9) and (11) one gets

$$k_j(x_{t:n};\theta) = \begin{cases} \begin{cases} -q_1^n(\theta) \frac{d}{dx_{1:n}} q_1^{-n}(x_{1:n}); & \text{for } j = 1, a < \theta < x < b \\ 0, & \text{otherwise,} \end{cases} \\ \begin{cases} q_2^n(\theta) \frac{d}{dx_{n:n}} q_2^{-n}(x_{n:n}); & \text{for } j = 2, a < x < \theta < b \\ 0, & \text{otherwise,} \end{cases} \end{cases} \tag{12}$$

Since q_1 is increasing function with $q_1^{-1}(b) = 0$ and q_2 is decreasing function with $q_2^{-1}(a) = 0$ integrating both sides of (12) on interval (a, b) for $j = 1, t = 1$ and $j = 2, t = n$ one gets

$$1 = \begin{cases} \int_\theta^b k_1(x_{1:n};\theta)\mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \\ \int_a^\theta k_2(x_{n:n};\theta)\mathrm{d}x_{n:n}, & \text{for } j = 2, t = n. \end{cases} \tag{13}$$

Using (12) and (13), $\Big[k_j(x_{t:n};\theta)\Big]_{n=1}$ reduces to $f_j(x;\theta)$ defined in (1) which establishes sufficiency of (2). □

Remark : Using $\phi(X_{t:n})$ derived in (7), the $f_j(x;\theta)$ given in (1) can be determined by

$$M_j(x_{t:n}) = \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\phi(X_{t:n}) - g(X_{t:n})}. \tag{14}$$

and pdf is given by

$$f_j(x;\theta) = \Big[(-1)^j \frac{\frac{d}{dx_{t:n}} U_j(x_{t:n})}{U_j(\theta)}\Big]_{n=1}; j = 1, 2, \tag{15}$$

where $U_j(X_{t:n})$ is decreasing function for $-\infty \le a < b \le \infty$ with $U(b) = 0$, range must be truncated by truncation parameter θ from left for $j = 1, t = 1$ and is increasing function for $-\infty \le a < b \le \infty$ with $U(a) = 0$, range must be truncated by truncation parameter θ from right for $j = 2, t = n$ such that it satisfies

$$M_j(X_{t:n}) = \frac{d}{dX_{t:n}}\Big(\log(U_j(X_{t:n}))\Big). \tag{16}$$

3. Applications

As special cases of the theorem 2.1 the following distributions are characterized.

(A) Characterization of negative exponential distribution with pdf

$$f_3(x;\theta) = \begin{cases} e^{-(x-\theta)}; & a < \theta < x < b, \\ \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{1:n}) - \Big(\frac{1}{n}\Big)\frac{d}{dX_{1:n}} g(X_{1:n})\Big], \tag{18}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 1, t = 1$ $M_1(X_{1:n})$ turns out as $-n$ and hence using (14) and (16)

$$M_1(X_{1:n}) = \tfrac{d}{dX_{1:n}} \log(U_1(X_{1:n})) = -n \Rightarrow U_1(X_{1:n}) = e^{-nX_{1:n}},$$

which is decreasing function on interval (a, b) with $U_1(b) = 0$ and range must be truncated by truncation parameter θ from left. Substituting these values in (15), $f_1(x;\theta)$ reduces to $f_3(x;\theta)$ defined in (17). Thus negative exponential distribution is characterized.

(B) Characterization of Pareto distribution with pdf

$$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{1:n}) - \Big(\frac{X_{1:n}}{cn}\Big)\frac{d}{dX_{1:n}} g(X_{1:n})\Big], \tag{20}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 1, t = 1$ $M_1(X_{1:n})$ turns out as $-\frac{cn}{X_{1:n}}$ and hence using (14) and (16)

$$M_1(X_{1:n}) = \tfrac{d}{dX_{1:n}} \log(U_1(X_{1:n})) = -\tfrac{cn}{X_{1:n}} \Rightarrow U_1(X_{1:n}) = -\tfrac{1}{cnX_{1:n}^{cn}},$$

which is decreasing function on interval (a, b) with $U_1(b) = 0$ and range must be truncated by truncation parameter θ from left. Substituting these values in (15), $f_1(x;\theta)$ reduces to $f_4(x;\theta)$ defined in (19). Thus Pareto distribution is characterized.

(C) Characterization of power function distribution with pdf

$$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, c > 0 \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{cn}\Big)\frac{d}{dX_{n:n}} g(X_{n:n})\Big], \tag{22}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{cn}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}}\log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = \left(\frac{X^c_{n:n}}{c}\right)^n,$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_5(x;\theta)$ defined in (21). Thus power function distribution is characterized.

(D) Characterization of uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{1}{\theta}; & a < x < \theta < b, c > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{23}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{n}\Big)\frac{d}{dX_{n:n}}g(X_{n:n})\Big], \tag{24}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{n}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}}\log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = X^n_{n:n},$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (23). Thus uniform distribution is characterized.

(E) Characterization of generalized uniform distribution with pdf

$$f_7(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^\alpha; & a < x < \theta < b, \alpha > -1, \\ 0, & \text{otherwise}, \end{cases} \tag{25}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dX_{n:n}}g(X_{n:n})\Big], \tag{26}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{n(\alpha+1)}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}}\log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = X^{n^{n(\alpha+1)}}_{n:n},$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_7(x;\theta)$ defined in (25). Thus uniform distribution is characterized.

4. Example

Example 4.1 Let $g_i(X_{n:n})$ be the uniformly minimum variance unbiased (UMVU) estimator;

$$g_i(X_{1:n}) = \begin{cases} X_{1:n} + 1 - \frac{1}{n}; & \text{for } i = 1, \\ \frac{c}{c-1}[1 - \frac{1}{cn}]X_{1:n}, & \text{for } i = 2, \end{cases} \tag{27}$$

$$g_i(X_{n:n}) = \begin{cases} \frac{X_{n:n}}{c+1}[c + \frac{1}{n}]; & \text{for } i = 3, \\ \frac{X_{n:n}}{2}[1 - \frac{1}{n}]; & \text{for } i = 4, \\ \frac{n\alpha+n+1}{n(\alpha+2)}X_{n:n}; & \text{for } i = 5, \end{cases} \tag{28}$$

of $\mu_1^{'}(\theta) = E(X)$; the first row moment and let the UMVU estimator of p^{th} quantile be

$$g_i(X_{1:n}) = \begin{cases} -\log(1-p) + X_{1:n} - \frac{1}{n}; & \text{for } i = 6, \\ X_{1:n}(1-p)^{-\frac{1}{c}}[1 - \frac{1}{cn}], & \text{for } i = 7, \end{cases} \tag{29}$$

$$g_i(X_{n:n}) = \begin{cases} \left(1 + \frac{1}{cn}\right)p^{-\frac{1}{c}}X_{n:n}; & \text{for } i = 3, \\ \left(1 + \frac{1}{n}\right)pX_{n:n}; & \text{for } i = 4, \\ \left(1 + \frac{1}{n(\alpha+1)}\right)p^{\frac{1}{\alpha+1}}X_{n:n}; & \text{for } i = 5, \end{cases} \tag{30}$$

and let the UMVU estimator of hazard function be

$$g_i(X_{n:n}) = \begin{cases} \left(1 - \frac{1}{n}\right)\left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 3, \\ \left(\frac{1}{X_{n:n}-t}\right)\left(1 - \frac{X_{n:n}}{n(X_{n:n}-t)}\right); & \text{for } i = 4, \\ \frac{1}{n}\left(\frac{t^\alpha}{t^\alpha - X_{n:n}^\alpha \alpha}\right)\left[\frac{t^\alpha}{t^\alpha - X_{n:n}^\alpha \alpha} - n(1+\alpha)\right]; & \text{for } i = 5, \end{cases} \tag{31}$$

Using (14) we get $M_j(X_{t:n})$, $(j = 1, t = 1$ and $j = 2, t = n)$

$$M_1(X_{1:n}) = \frac{\frac{d}{dX_{1:n}} g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} = \begin{cases} -n; & \text{for } i = 1, 6, \\ -\frac{cn}{X_{1:n}}, & \text{for } i = 2, 7, \end{cases} \tag{32}$$

which satisfies

$$M_1(X_{1:n}) = \frac{d}{dX_{1:n}} \Big[\log U_1(X_{1:n}) \Big] \Rightarrow \tag{33}$$

$$U_1(X_{1:n}) = \begin{cases} e^{-nX_{1:n}}; & \text{for } i = 1, 6, \\ \frac{1}{cnX_{1:n}^c}, & \text{for } i = 2, 7, \end{cases} \tag{34}$$

,

and

$$M_2(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} \begin{cases} \frac{cn}{X_{n:n}}; & \text{for } i = 3, 8, 11, \\ \frac{n}{X_{n:n}}; & \text{for } 4, 9, 12, \\ \frac{n(\alpha+1)}{X_{n:n}}; & \text{for } 5, 10, 13, \end{cases} \tag{35}$$

which satisfies

$$M_2(X_{n:n}) = \frac{d}{dX_{n:n}} \Big[\log U_2(X_{n:n}) \Big] \Rightarrow \tag{36}$$

$$U_2(X_{n:n}) = \begin{cases} \left(\frac{X_{n:n}}{c}\right)^c; & \text{for } i = 3, 8, 11, \\ X_{n:n}^n; & \text{for } 4, 9, 12, \\ X_{n:n}^{n(\alpha+1)}; & \text{for } 5, 10, 13, \end{cases} \tag{37}$$

Since $U_1(X_{1:n})$ decreasing function on $-\infty < a < b < \infty$ with $U_1(b) = 0$ and since $U_2(X_{n:n})$ increasing function on $-\infty < a < b < \infty$ with $U_2(a) = 0$ using method described in the remark 2.1 the pd $f_j(X; \theta)$

defined in (1) can be characterized through expectation of function of order statistics $g_i(X_{t:n})$; $t = 1$ or $t = n$ for $i = 1, 2, ..., 13$ the UMVU estimator of non constant function such as first row moment, pth quantile and hazard function by substituting $M_j(X_{t:n})$; $j = 1$ and $t = 1$ or $j = 2$ and $t = n$ defined in (14) and using $U_j(X_{t:n})$; ($j = 1, t = 1$ and $j = 2, t = n$) as appeared in (16) for (15) given below :

j	i	$M_j(X_{t:n}) = \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\phi_i(X_{t:n}) - g_i(X_{t:n})}$	$U_j(X_{t:n}) \ni M_j(X_{t:n}) = \frac{d\left(\log(U(X_{t:n}))\right)}{dX_{t:n}}$	$f_j(x,\theta) = (-1^j)\left[\frac{\frac{d}{dX_{t:n}} U_j(X_{t:n})}{U_j(\theta)}\right]_{n=1}$
1	$1, 6$	$-n$	$e^{-nX_{1:n}}$	$f_3(x;\theta) = \begin{cases} e^{-(x-\theta)}; & a < \theta < x < b, \\ 0, & \text{otherwise,} \end{cases}$
1	$2, 7$	$-\frac{cn}{X_{1:n}}$	$\frac{1}{cnX_{1:n}^{cn}}$	$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b, \\ 0, & \text{otherwise,} \end{cases}$
2	$3, 8, 11$	$\frac{cn}{X_{n:n}}$	$\left(\frac{X_{n:n}^c}{c}\right)^n$	$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, \theta = K^{-1}, \\ & K > 0, c > 0, \\ 0, & \text{otherwise,} \end{cases}$
2	$4, 9, 12$	$\frac{n}{X_{n:n}}$	$X_{n:n}^n$	$f_6(x;\theta) = \begin{cases} \frac{1}{\theta}; & a < x < \theta < b, \\ 0, & \text{otherwise,} \end{cases}$
2	$5, 10, 13$	$\frac{n(\alpha+1)}{X_{n:n}}$	$X_{n:n}^{n(\alpha+1)}$	$f_7(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}; & a << x < \theta < b, \alpha > -1, \\ 0, & \text{otherwise,} \end{cases}$

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