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Exponential-Generalised Half Logistic Additive Failure Rate Model: An Inferential Study

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Abstract.
The addition of hazard functions of Exponential and Generalized half logistic model is developed. The probability model is considered and an attempt is made to present the distributional properties, estimation of parameters and testing of hypothesis about the proposed model. The findings are described.

1. Introduction

In reliability studies, combinations of components forming series, parallel, k out of n systems are quite popular. The survival probabilities of such systems are evaluated either by the system as a whole or through the survival probabilities of the components that define the system. It is well known that in a series system of a finite number of components with independent life time random variables, the system reliability is equal to the product of the component reliabilities. If f(x), F(x), h(x) respectively indicate the failure density, failure probability, failure rate of a component with life time random variable 'X', then we know that the reliability is given by

$$R(x) = 1 - F(x) = e^{-\int_0^x h(x)\mathrm{d}x}$$

If a series system has two components with independent but non-identical life patterns explained by two distinct random variables say X_1, X_2, with respective failure densities, failure probabilities, failure rates as $f_1(x)$, $f_2(x)$; $F_1(x)$, $F_2(x)$; $h_1(x)$, $h_2(x)$ then the system reliability is given by

$$R(x) = e^{-\int_0^x [h_1(x)+h_2(x)]\mathrm{d}x}. \tag{1}$$

From the above expression we get the failure density and failure rate of the series system whose reliability is given by (1) . Such models are already studied in the past with different choices of $h_1(x)$ and $h_2(x)$. One such situation is the popular linear failure rate distribution [LFRD]. In this model $h_1(x)$ is taken a constant failure rate (CFR) model. $h_2(x)$ is taken as an increasing failure rate (IFR) model with specific choices of exponential for $h_1(x)$ and Weibull with shape 2 for $h_2(x)$. The failure density, the cumulative distribution function, the reliability and the failure rate of LFRD model are given by

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$$f(x;\alpha,\beta) = (\alpha+\beta x)e^{-\alpha x-\frac{\beta}{2}x^2}. \tag{2}$$

$$F(x;\alpha,\beta) = 1-e^{-\alpha x-\frac{\beta}{2}x^2}, \qquad x>0,\alpha>0,\beta>0. \tag{3}$$

$$\bar{F}(x;\alpha,\beta) = e^{-\alpha x-\frac{\beta}{2}x^2}. \tag{4}$$

$$h(x;\alpha,\beta) = \alpha+\beta x. \tag{5}$$

A hazard rate given in (5) is in the form of a straight line equation justifying the name "Linear Failure Rate" for this distribution. A number of researchers made an extensive study on LFRD model. Some recent works in this regard are Bain(1974) (8), Balakrishnan and Malik (1986) (9), Ananda Sen and Bhattacharya(1995) (7), Mohie El-Din *et al* (1997) (14), Ghitany and Kotz(2007) (13), El-Baset A. Ahmad(2008) (4), Khedhairi(2008) (3), Sarhan and Zaindin(2009) (5), Sarhan and Kundu(2009) (6), Mahmoud and Al-Nagar(2009) (10), Mazen and Zaindin(2010) (12), Kantam and Priya(2011) (15), Srinivasa Rao *et al* (2013) (11) have worked out an additive life testing model combining CFR and DFR models. The rest of the paper is organised as follows: The Distributional properties of our proposed model are given in Section 2. The ML Estimation is discussed in Section 3. Discrimination of our model from exponential using likelihood ratio criterion is given in Section 4. Summary and Conclusions are given in Section 5.

2. Distributional Properties

We consider exponential distribution (constant failure rate model) and generalized half logistic distribution (increasing failure rate model). The probability density function, cumulative distribution function and hazard function of exponential distribution are given by

$$f_1(x) = \lambda e^{-\lambda x}; \quad x\geq 0,\lambda>0. \tag{6}$$

$$F_1(x) = 1-e^{-\lambda x}; \quad x\geq 0,\lambda>0. \tag{7}$$

$$h_1(x) = \lambda. \tag{8}$$

The probability density function, cumulative distribution function and hazard function of generalized half logistic distribution are given by

$$f_2(x) = \frac{\theta(2e^{-x})}{(1+e^{-x}))^{\theta+1}}; \quad x>0,\theta>0. \tag{9}$$

$$F_2(x) = 1-\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\theta}; \quad x>0,\theta>0. \tag{10}$$

$$h_2(x) = \frac{\theta}{1+e^{-x}}; \quad x > 0, \theta > 0. \tag{11}$$

The reliability of the series system with two components as exponential distribution and generalized half logistic distribution are using equation (1.1) is

$$R(x) = e^{-\lambda x}(e^x - 1)^{-\theta}; \quad x > 0, \theta > 0, \lambda > 0. \tag{12}$$

We consider the failure density corresponding to (2.7) as our exponential generalized half logistic additive failure rate (EGHLAFRM).

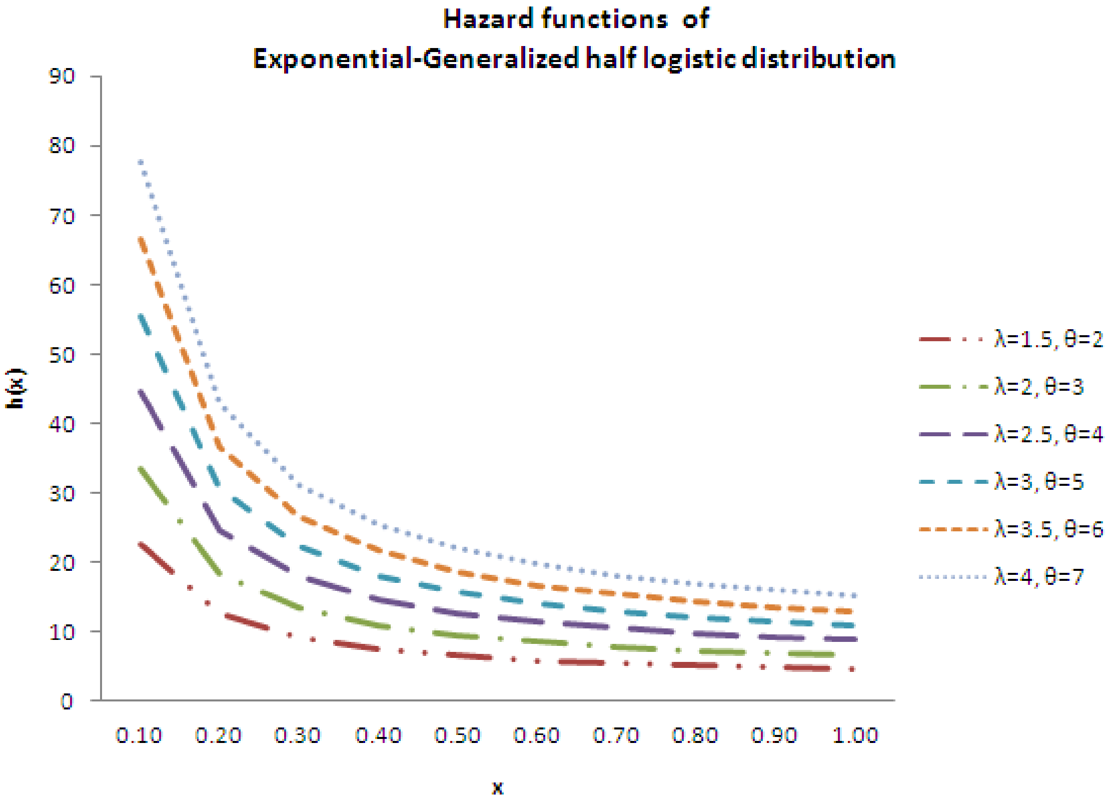
The probability density function(pdf)-g(x), the cumulative distribution function(cdf)-G(x), failure rate-h(x) of EGHLAFRM are respectively given by

$$g(x) = e^{-\lambda x}(e^x - 1)^{-\theta}\big(\theta e^x(e^x - 1)^{-1} + \lambda\big); \quad x > 0, \theta > 0, \lambda > 0. \tag{13}$$

$$G(x) = 1 - e^{-\lambda x}(e^x - 1)^{-\theta}; \quad x > 0, \theta > 0, \lambda > 0. \tag{14}$$

$$h(x) = \lambda + \theta e^x(e^x - 1)^{-1}; \quad x > 0, \theta > 0, \lambda > 0. \tag{15}$$

The shapes of the hazard curves are shown in the following graph for various values of $(\lambda, \theta) =$ (1.5,2),(2.0,3),(2.5,4),(3.0,5),(3.5,6),(4.0,7). The hazard function appears to be a decreasing function.



3. Maximum Likelihood Estimation

Let $x_1, x_2, .., x_n$ be a random sample of size 'n' drawn from EGHLAFRM with pdf $g(x; \lambda, \theta)$ then the likelihood function is given by

$$L = \prod_{i=1}^{n} g(x_i; \lambda, \theta). \tag{16}$$

$$L = \prod_{i=1}^{n} e^{-\lambda x}(e^x - 1)^{-\theta}\big(\theta e^x (e^x - 1)^{-1} + \lambda\big). \tag{17}$$

$$logL = -(\lambda - 1)\sum_{i=1}^{n} x_i + nlog\theta - \sum_{i=1}^{n} log(e^{x_i} - 1) - 2\theta\sum_{i=1}^{n} log(e^{x_i} - 1) + nlog\lambda - \lambda\sum_{i=1}^{n} x_i. \tag{18}$$

The MLEs of λ, θ can be obtained by solving the following likelihood equations

$$\frac{\partial logL}{\partial \lambda} = 0 \Rightarrow -2\sum_{i=1}^{n} x_i + \frac{n}{\lambda} = 0 \Rightarrow = \hat{\lambda} = \frac{n}{2\sum_{i=1}^{n} x_i}. \tag{19}$$

and

$$\frac{\partial logL}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - 2\sum_{i=1}^{n} log(e^{x_i} - 1) = 0 \Rightarrow \hat{\theta} = \frac{n}{2\sum_{i=1}^{n} log(e^{x_i} - 1)} \tag{20}$$

The asymptotic variance, covariance of the estimates of the parameters are obtained using the following elements of the information matrix:

$$I_{11} = -E\left(\frac{\partial^2 logL}{\partial \lambda^2}\right) = -E\left(\frac{-n}{\lambda^2}\right) = \frac{n}{\lambda^2}. \tag{21}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 logL}{\partial \lambda \partial \theta}\right) = 0. \tag{22}$$

$$I_{22} = -E\left(\frac{\partial^2 logL}{\partial \theta^2}\right) = -E\left(\frac{-n}{\theta^2}\right) = \frac{n}{\theta^2}. \tag{23}$$

The estimated asymptotic dispersion matrix of the MLEs is given by the inverse of $\begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix}$

4. Discrimination between EGHLAFRM and Exponential model

We know that the exponential distribution is having a number of preferable properties to be handled for problems of statistical inference. We therefore, are interested in assessing whether exponential distribution is an alternative to our model. In other words given a sample we are interested in studying whether the sample discriminates between our model from that of exponential. Let us designate our distribution EGHLAFRM as null population say P_0 and exponential distribution as alternative population say P_1. We propose a null hypothesis H_0 : A given sample belongs to the population P_0 against an alternative hypothesis H_1 : The sample belongs to population P_1.

Consider a sample from P_0. Let L_1, L_0 respectively stand for the likelihood functions of the sample with population P_1 and P_0. Both L_1 and L_0 contain the respective parameters of the population. The considered sample is used to get the parameters of P_1, P_0. The value of $\frac{L_1}{L_0}$ with the estimated parameters is computed. If H_0 is true, $\frac{L_1}{L_0}$ must be small, therefore for accepting H_0 with a given degree of confidence, $\frac{L_1}{L_0}$ is compared with a critical value with the help of the percentiles in the sampling distribution $\frac{L_1}{L_0}$. But the sampling distribution of $\frac{L_1}{L_0}$ is not analytical , we therefore resorted to the empirical sampling distribution through simulation. We have generated random samples of size 5(1)10,15,20,25,30 from the population P_0 with various parameter combinations ($\lambda = 1; \theta = 1, 2, 3, 4$)and got the values of L_1, L_0 along with the estimates of respective parameters for each sample. The percentiles of $\frac{L_1}{L_0}$ at various probabilities are calculated and are given in Table 4.1.

In testing of hypothesis we know that the power of a test statistic is the complementary probability of accepting a false hypothesis at a given level of significance. Let us conventionally fix 5% level of significance, so that the percentiles in Table 4.1 under the column 0.05 shall become the critical values. We generate random sample of sizes 5(1)10,15,20,25,30 from the population P_1 namely exponential. At this sample we find the estimates of the parameters of P_1 and P_0 using the respective probability models. Accordingly we got the estimates of L_1 , L_0 for the sample from P_1. Over repeated simulation runs we got the proportion of values $\frac{L_1}{L_0}$ that fall below the respective critical values of Table 4.1. These proportions would give the values of probability of type II error namely β so that $1 - \beta$ would be the power. Various power values are given in Table 4.2. We can observe that exponential can be a reasonable alternative to our model in small samples.

Table 4.1: Percentiles of $\frac{L_1}{L_0}$ for various values of λ and θ

$\lambda = 1, \theta = 1$						
n	0.99865	0.995	0.99	0.975	0.95	0.05
5	7.0417	1.9103	1.0437	0.2871	0.2865	0.2859
6	6.2854	1.8432	1.0371	0.2241	0.2234	0.2225
7	4.8680	1.9596	0.9692	0.1750	0.1744	0.1738
8	5.4677	2.4385	0.9597	0.1366	0.1362	0.1357
9	5.5794	2.0101	0.9762	0.1066	0.1063	0.1057
10	4.6914	2.0777	0.9877	0.0834	0.0830	0.0826
15	8.5481	1.1929	0.6103	0.0248	0.0243	0.0240
20	3.5754	0.7281	0.3069	0.0075	0.0073	0.0070
25	1.8872	0.5092	0.2195	0.0024	0.0022	0.0021
30	1.2026	0.4309	0.1349	0.0008	0.0007	0.0006
$\lambda = 1, \theta = 2$						
5	0.9009	0.5886	0.4805	0.2924	0.2921	0.2917
6	0.7526	0.5300	0.4122	0.2289	0.2284	0.2280
7	0.6583	0.4537	0.3396	0.1791	0.1788	0.1785
8	0.5346	0.4047	0.2869	0.1403	0.1400	0.1397
9	0.4415	0.3282	0.2492	0.1099	0.1096	0.1092
10	0.3999	0.2858	0.2131	0.0861	0.0858	0.0856
15	0.2190	0.1055	0.0808	0.0256	0.0253	0.0252
20	0.0766	0.0400	0.0297	0.0077	0.0076	0.0074
25	0.0267	0.0159	0.0115	0.0023	0.0023	0.0022
30	0.0107	0.0070	0.0045	0.0007	0.0007	0.0007
$\lambda = 1, \theta = 3$						
5	0.5198	0.4246	0.3840	0.2949	0.2947	0.2944
6	0.4245	0.3547	0.3117	0.2313	0.2310	0.2306
7	0.3494	0.2924	0.2522	0.1813	0.1811	0.1808
8	0.2841	0.2441	0.2070	0.1422	0.1419	0.1417
9	0.2288	0.1951	0.1692	0.1114	0.1113	0.1110
10	0.1858	0.1601	0.1386	0.0874	0.0873	0.0871
15	0.0766	0.0532	0.0476	0.0261	0.0259	0.0258
20	0.0246	0.0180	0.0157	0.0078	0.0077	0.0077
25	0.0076	0.0062	0.0053	0.0024	0.0023	0.0023
30	0.0027	0.0023	0.0018	0.0007	0.0007	0.0007

table contd....						
$\lambda = 1, \theta = 4$						
5	0.4178	0.3734	0.3501	0.2966	0.2964	0.2962
6	0.3360	0.3022	0.2801	0.2327	0.2325	0.2321
7	0.2715	0.2436	0.2238	0.1826	0.1824	0.1823
8	0.2169	0.1996	0.1804	0.1433	0.1432	0.1430
9	0.1746	0.1589	0.1456	0.1125	0.1123	0.1122
10	0.1387	0.1273	0.1175	0.0883	0.0882	0.0880
15	0.0504	0.0405	0.0383	0.0264	0.0263	0.0262
20	0.0156	0.0131	0.0122	0.0079	0.0079	0.0078
25	0.0048	0.0043	0.0039	0.0024	0.0024	0.0023
30	0.0016	0.0014	0.0013	0.0007	0.0007	0.0007

Table 4.2: Power of Likelihood Ratio criterion

$\lambda = 1, \theta = 1$		
n	0.025	0.05
5	0.969	0.954
6	0.967	0.934
7	0.971	0.951
8	0.987	0.964
9	0.975	0.963
10	0.981	0.976
15	0.989	0.984
20	0.988	0.985
25	0.994	0.992
30	0.997	0.996
$\lambda = 1, \theta = 2$		
5	1.000	1.000
6	1.000	1.000
7	1.000	1.000
8	1.000	1.000
9	1.000	1.000
10	1.000	1.000
15	1.000	1.000
20	1.000	1.000
25	1.000	1.000
30	1.000	1.000

table contd.... $\lambda = 1, \theta = 3$		
5	0.999	0.999
6	1.000	1.000
7	1.000	1.000
8	1.000	1.000
9	1.000	1.000
10	1.000	1.000
15	1.000	1.000
20	1.000	1.000
25	1.000	1.000
30	1.000	1.000
$\lambda = 1, \theta = 4$		
5	1.000	1.000
6	1.000	1.000
7	1.000	1.000
8	1.000	1.000
9	1.000	1.000
10	1.000	1.000
15	1.000	1.000
20	1.000	1.000
25	1.000	1.000
30	1.000	1.000

5. Summary & Conclusions

A combination of Exponential model and Generalized half logistic model is developed on lines of the well known linear failure rate model. Estimating equations by ML method are also derived. Its validity as a specified model in the presence of a simpler Exponential model as an alternative is established using a likelihood ratio criterion. The proposed model stood robust against Exponential.

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