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On the Performance of the Tests for Testing Exponential versus Generalized Exponential Distribution

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Abstract. Generalized Exponential (GE) distribution is a family of distribution obtained by exponentiating the cumulative distribution function (cdf) of the exponential distribution. This paper provides partial review on GE distribution. In this paper, we derive likelihood ratio, Wald and Score test for testing exponential versus GE distribution. Asymptotic null distribution of all the test is central chi-square with 1 degree of freedom. Under the alternative hypothesis, all the test statistics are distributed as non-central chi-square with 1 degree of freedom and the same non-centrality parameter. The finite (small) sample performance of the tests is compared using simulations for the censored and uncensored observations. The results indicate that Wald test and the variance of this maintains type-I error rate and has higher power compared to the other test. A real life data is also analyzed to illustrate the use of the tests.

1. Introduction

In the past, several researchers have developed methods to generalize the known distributions. The generalized family has one additional parameter and can accommodate a wide variety of situations. One such method is to exponentiate the cdf of a continuous random variable by a positive parameter. This approach dates back to Gompertz (1825), who obtained a generalized distribution by exponentiating the cdf of the extreme value distribution to graduate mortality tables. In a series of papers Verhulst (1838, 1845 and 1847) obtained new family of distribution by exponentiating the cdf of logistic distribution. When the exponentiating parameter is an integer, the family of distribution is referred as Neyman non-parametric family. For details see Al-Hussaini and Ahsanullah (2015).

GE distribution is a two parameter distribution which was introduced by Gupta and Kundu (1999). It can accommodate both increasing and decreasing failure rates and is a competitor for the two parameter Weibull distribution and Gamma distribution. Unfortunately, Gupta and Kundu (1999) do not acknowledge the original works of Gompertz (1825) and Verhulst (1838, 1845 and 1847) nor they refer to Neyman family. This paper provides partial review on GE distribution. Some technical details regarding GE distribution is presented in section 2.

Gupta and Kundu (2003a) discuss the closeness of the Weibull and the GE distribution. They observe that the behaviour of the hazard function and the tail behaviour of the Gamma distribution and the GE

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distribution are quite similar in nature. Further they observe that corresponding to the pair of the parameters of the cdf of the Gamma distribution, there exists a GE distribution whose cdf is approximately equal to the former. In another paper, Gupta and Kundu (2003b) discuss how to discriminate between Weibull and GE distribution. In some ranges of the hazard function, the hazard function of the two distributions is not distinguishable from each other. Lognormal distribution has bump shape hazard function and thus cannot be directly compared with the hazard function of the GE distribution. In some ranges of the hazard function, the behaviour of the hazard function of lognormal distribution can be compared with the behaviour of the hazard function of GE distribution; that is when the hazard function is only increasing or only decreasing. This is investigated in detail by Kundu et al. (2005). The estimation of the parameters of the GE distribution was discussed by several authors. Gupta and Kundu (2001) compare the performance of the estimators based on maximum likelihood, method of moments, method of percentiles, least square estimation, weighted least square estimation and estimators based on the linear combination of order statistics. From their simulation study, they recommend percentile estimator for small sample sizes and maximum likelihood estimator (MLE) for moderate to large sample sizes. Best linear unbiased estimators of the location and scale parameters based on order statistics of the GE distribution were derived by Raqab and Ahsanullah (2001). However the limitation is that they can be used only for sample sizes upto ten. In extension of the previous paper, Raqab (2002) has obtained best linear unbiased estimator of the three parameter GE distribution using record values. Bayes and empirical Bayes estimators for the unknown parameter of the GE distribution was derived by Jaheen (2004) when the loss function is squared error or LINEX. Raqab and Madi (2005) derive Bayes estimator for the parameters and reliability function of the GE distribution under type-II censoring. They used informative priors. Extension of this paper was undertaken by Kundu and Gupta (2008). They consider Gamma priors and compare the performance of the Bayes estimator with the MLE using non-informative priors. The estimation of the parameter of GE distribution under various censoring schemes was considered by many researchers. Asgharzadeh (2009) derived approximate MLE of the scale parameter of the GE distribution under progressive type-II censoring scheme. The results of this paper were generalized by Pradhan and Kundu (2009), where they obtain MLEs of the parameters using EM algorithm. They also provide optimal progressive censoring plans. In another paper, Kundu and Pradhan (2009a) supplement this work to the case of hybrid censoring. The results were augmented in Kundu and Pradhan (2009b), where they use Bayes procedure for the estimation of the parameters and they also provide sub-optimal censoring plans.

The previous papers confined to progressive type-II censoring and Chen and Lio (2010) obtained maximum likelihood and moment estimators of the parameters of the GE distribution under the progressive type-I interval censored scheme. Nasiri and Pazira (2010) derive the Bayes and the MLEs of the parameters of the GE distribution when there are k outliers. They compare the performance of these two estimators using simulation. The paper by Yarmohammadi and Pazira (2010) supplements the results of some of the previous papers, where they consider Bayes and non-Bayes estimation of the parameters under type-II censoring. In addition to these papers, the researchers concentrated on the estimation of the reliability parameters namely $P(Y < X)$ under stress-strength schemes. The papers in this area are due to Kundu and Gupta (2005), Baklizi (2008), where they use lower record values for the estimation and Raqab et al. (2008) for the three parameter GE distribution. The other salient papers relating to GE distribution are due to Aslam and Shahebaz (2007), Kundu and Gupta (2007), Sultan (2007), Gupta and Kundu (2007), Kundu and Gupta (2009), Gadde (2009) and Aslam et al. (2010), where they provide overview of the results concerning GE distribution till that time while Madi and Raqab (2007) use GE distribution for the analysis of rainfall data.

Concomitants of the GE distributions are discussed in several research papers. Tahmasebi and Jafari (2015) discuss the concomitant of order statistics and record values from Morgenstern type bivariate GE distribution. In an extension to this paper Tahmasebi et al. (2015) derive product moment of Morgenstern type bivariate GE distribution. The other paper in this direction is due to Barakat et al. (2016), who study the properties of ordered statistics from Huang-Kotz FGM type bivariate GE distribution.

Cox (1977) suggests that data analysis be carried out using generalized distribution and also insists the need for identifying the sub-model that fit well to the data. This would help a scientist to understand the phenomena under consideration. Whenever two or three parameter GE distribution is used for the data analysis, it is important to check whether the underlined distribution is exponential or GE distribution. This work has not been attempted in the past and we derive likelihood ratio, Wald and Score test for testing the hypothesis that the underlined distribution is exponential against the alternative that it is GE distribution. The case of complete sample and randomly censored observations are considered. The performance of the tests in terms of estimated type-I error rate and power of the test are compared using simulation. The results indicate that Wald test or its modified version performs well.

The paper unfolds in the following sections. Section 2 presents preliminaries and the test statistics, simulation experiment is presented in section 3. The results are presented for the uncensored and censored cases in section 4. Section 5 presents the use of the tests for real data sets and the paper concludes in section 6.

2. Preliminaries and the Test Statistics

2.1. Preliminaries

Gupta and Kundu (1999) obtained two parameter GE distribution by exponentiating the cdf of 1 parameter exponential distribution. The cdf of the two parameter GE distribution is given by

$$F(x;\theta,\lambda) = (1-e^{-\lambda x})^{\theta} \ ; \quad x>0,\ \theta>0,\ \lambda>0, \tag{1}$$

where θ is shape parameter and λ is the scale parameter.
Therefore the probability density function (pdf) of the two parameter GE is given by

$$f(x;\theta,\lambda) = \theta\lambda(1-e^{-\lambda x})^{\theta-1}e^{-\lambda x} \ ; \quad x>0,\ \theta>0,\ \lambda>0. \tag{2}$$

The hazard function of the distribution is

$$h(x;\theta,\lambda) = \frac{\theta\lambda(1-e^{-\lambda x})^{\theta-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\theta}}. \tag{3}$$

2.2. Maximum Likelihood Equations for the Uncensored Case

Given a random sample of size n from the GE distribution, the likelihood is given by

$$L(\theta,\lambda;x) = \prod_{i=1}^{n}(\theta\lambda(1-e^{-\lambda x_i})^{\theta-1}e^{-\lambda x_i}),$$

where $x=(x_1,x_2,\cdots,x_n)$. The Maximum Likelihood (ML) equations for the estimation of the parameters are obtained by equating the Score vector to zero. The Score vector is given below

$$\begin{pmatrix} \frac{\delta logL}{\delta\theta} \\ \frac{\delta logL}{\delta\lambda} \end{pmatrix} = \begin{pmatrix} \frac{n}{\theta}+\sum_{i=1}^{n} log(1-e^{-\lambda x_i}) \\ \frac{n}{\lambda}+(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}-\sum_{i=1}^{n}x_i \end{pmatrix}. \tag{4}$$

The components of the Hessian matrix are given by

$$\frac{\delta^2 logL}{\delta\theta^2} = -\frac{n}{\theta^2}, \tag{5}$$

$$\frac{\delta^2 logL}{\delta\lambda^2} = \frac{-n}{\lambda^2}-(\theta-1)\sum_{i=1}^{n}\frac{x_i^2e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2}, \tag{6}$$

$$\frac{\delta^2 logL}{\delta\lambda\delta\theta} = \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}. \tag{7}$$

The observed Fisher information matrix is given by

$$I(\theta,\lambda) = \begin{pmatrix} -\frac{\delta^2 logL}{\delta\theta^2} & -\frac{\delta^2 logL}{\delta\theta\delta\lambda} \\ -\frac{\delta^2 logL}{\delta\lambda\delta\theta} & -\frac{\delta^2 logL}{\delta\lambda^2} \end{pmatrix}.$$

Under regularity condition, $\sqrt{n}((\hat{\theta}-\theta),(\hat{\lambda}-\lambda))$ follows asymptotic normal distribution with mean vector zero and covariance matrix given by inverse of the Fisher information matrix. For the regularity condition, we refer to Lehmann and Casella(1998).

2.3. Maximum Likelihood Equations for the Censored Case

Let T denote the event time which follows GE distribution and C denotes the censoring time. Under random censoring, for a sample of size n, let $x_i = min(t_i,\ c_i),\ i = 1,\cdots,n$. Further, let δ denote the indicator variable and

$$\delta_i = \begin{cases} 1 & \text{if the observation is uncensored} \\ 0 & \text{if the observation is censored} \end{cases}$$

The likelihood is given by

$$L(\theta,\lambda;x,\delta) = \prod_{i=1}^{n} [\theta\lambda(1-e^{-\lambda x_i})^{\theta-1}.e^{-\lambda x_i}]^{\delta_i}[1-(1-e^{-\lambda x_i})^{\theta}]^{(1-\delta_i)},$$

where $(x,\delta) = ((x_1,\delta_1),\cdots,(x_n,\delta_n))$.
The Score vector is given by

$$\begin{pmatrix} \frac{\delta logL}{\delta\theta} \\ \frac{\delta logL}{\delta\lambda} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{i=1}^{n}\delta_i}{\theta} + \sum_{i=1}^{n}\delta_i log(1-e^{-\lambda x_i}) - \\ \sum_{i=1}^{n}\frac{(1-\delta_i)(1-e^{-\lambda x_i})^{\theta} log(1-e^{-\lambda x_i})}{(1-(1-e^{-\lambda x_i})^{\theta})} \\ \\ \frac{\sum_{i=1}^{n}\delta_i}{\lambda} + (\theta-1)\sum_{i=1}^{n}\frac{\delta_i x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})} - \sum_{i=1}^{n}\delta_i x_i - \\ \theta\sum_{i=1}^{n}\frac{(1-\delta_i)x_i e^{-\lambda x_i}(1-e^{-\lambda x_i})^{\theta-1}}{(1-(1-e^{-\lambda x_i})^{\theta})} \end{pmatrix}. \tag{8}$$

The ML equation is given by equating the Score vector to the null vector.

$$\begin{aligned} \frac{\delta^2 logL}{\delta\theta^2} &= -\frac{\sum_{i=1}^{n}\delta_i}{\theta^2} - \sum_{i=1}^{n}(1-\delta_i)[log(1-e^{-\lambda x_i})]^2(1-e^{-\lambda x_i})^{\theta} \\ &\quad \left[\frac{(1-(1-e^{-\lambda x_i})^{\theta}) + (1-e^{-\lambda x_i})^{\theta}}{(1-(1-e^{-\lambda x_i})^{\theta})^2}\right], \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\delta^2 logL}{\delta\lambda^2} &= -\frac{\sum_{i=1}^{n}\delta_i}{\lambda^2} - (\theta-1)\sum_{i=1}^{n}\frac{\delta_i x_i^2 e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2} - \theta\sum_{i=1}^{n}(1-\delta_i)x_i^2 e^{-\lambda x_i} \\ &\quad \left[\frac{(1-(1-e^{-\lambda x_i})^{\theta})(\theta-1)e^{-\lambda xi}(1-e^{-\lambda x_i})^{\theta-2}}{(1-(1-e^{-\lambda x_i})^{\theta})^2}\right. \\ &\quad \left. -\frac{(1-(1-e^{-\lambda x_i})^{\theta})(1-e-\lambda x_i)^{\theta-1} + \theta e^{\lambda x_i}((1-e^{-\lambda x_i})^{\theta-1})^2}{(1-(1-e^{-\lambda x_i})^{\theta})^2}\right], \end{aligned} \tag{10}$$

$$\frac{\delta^2 logL}{\delta\lambda\delta\theta} = \sum_{i=1}^{n}\frac{\delta_i x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})} - \sum_{i=1}^{n}(1-\delta_i)x_i e^{-\lambda x_i}(1-e^{-\lambda x_i})^{\theta-1} \Bigg[\frac{\theta log(1-e^{-\lambda x_i})(1-(1-e^{-\lambda x_i})^{\theta})}{(1-(1-e^{-\lambda x_i})^{\theta})^2} + \frac{(1-(1-e^{-\lambda x_i})^{\theta}) + \theta(1-e^{-\lambda x_i})^{\theta} log(1-e^{-\lambda x_i})}{(1-(1-e^{-\lambda x_i})^{\theta})^2}\Bigg]. \tag{11}$$

The observed Fisher information matrix corresponds to negative of the Hessian matrix.

2.4. Test Statistic

In the sequel, we present the likelihood ratio, Wald and Score test for testing $H_0 : \theta = 1 \; v/s \; H_1 : \theta \neq 1$. We do not distinguish between censored and uncensored cases and the context would make it clear.

2.4.1. Likelihood Ratio Test (LRT)

Let $\hat{\hat{\lambda}}$ denote the restricted MLE of λ at $\theta_0 = 1$ and $(\hat{\theta}, \hat{\lambda})$ denote the unrestricted MLE of (θ, λ). The LRT test statistic is given by

$$-2log\lambda_n = 2[logL(\hat{\theta}, \hat{\lambda}; x) - logL(\theta_0, \hat{\hat{\lambda}}; x)]. \tag{12}$$

Under null hypothesis, $-2log\lambda_n$ is asymptotically distributed as central chi-square with 1 degree of freedom. We reject null hypothesis if $-2log\lambda_n \geq \chi^2_\alpha(1)$, where $\chi^2_\alpha(1)$ denotes the upper α^{th} percentile value of central chi-square distribution with 1 degree of freedom.

2.4.2. Wald Test

Using the asymptotic normality of the MLEs, Wald proposed a test which is equivalent to the LRT. In the original proposition, Wald evaluated the Fisher information matrix using the unrestricted MLEs. Cox (1974) observed that the Fisher information can also be evaluated using the null hypothesis and the restricted MLE of other nuisance parameters. In this paper, we have used both the forms of the Wald test and called it as Wald test1 and Wald test2.

The test statistic for Wald test1 (W_1) is given by

$$W_1 = \frac{\hat{\theta} - 1}{SE(\hat{\theta})}, \tag{13}$$

Where $SE(\hat{\theta})$ is obtained by the inverse of the Fisher information matrix. We reject the null hypothesis if $W_1^2 > \chi^2_\alpha(1)$, where $\chi^2_\alpha(1)$ refers to the upper α^{th} percentile value of the central chi-square distribution with 1 degree of freedom. The test statistic for Wald test2 (W_2) has the similar format as W_1 and is not given. Under null hypothesis, W_2^2 is also asymptotically distributed as central chi-square with 1 degree of freedom.

2.4.3. Score Test

The Score test statistic is given by

$$W_e = \left(\frac{\delta logL(\theta_0, \hat{\hat{\lambda}})}{\delta\theta_0}, \frac{\delta logL(\theta_0, \hat{\hat{\lambda}})}{\delta\hat{\hat{\lambda}}}\right) I^{-1}(\theta_0, \hat{\hat{\lambda}}) \left(\frac{\delta logL(\theta_0, \hat{\hat{\lambda}})}{\delta\theta_0}, \frac{\delta logL(\theta_0, \hat{\hat{\lambda}})}{\delta\hat{\hat{\lambda}}}\right)'. \tag{14}$$

Under H_0, W_e is asymptotically distributed as central chi-square with 1 degree of freedom. Reject the null hypothesis if $W_e > \chi^2_\alpha(1)$. For details regarding likelihood ratio, Wald and Score tests, refer to Rao (1973) and Cox and Hinkley (1974). For the use of modified Wald test, see Vasudeva and Rao (2009), Sumathi and Rao (2010), Sumathi and Rao (2011), Sumathi and Rao (2013) and Sumathi and Rao (2014).

3. Simulation Experiment

3.1. Uncensored Case

We have compared the performance of the four tests using simulation. Observations of size n = 20, 40, 60, 80 and 100 are generated from exponential distribution with parameter $\lambda = 0.33$. This corresponds to a life time of 3 years, which is commonly used in medical follow-up studies; then we have computed the test statistics corresponding to LRT, Wald test1, Wald test2 and Score test. Using 10000 simulations, we have estimated the proportion of times the null hypothesis is rejected, which corresponds to estimated type-I error rate. The level of significance is specified as 0.05. We also noted the lower and upper $(\frac{\alpha}{2})^{th}$ percentile values of the simulated distribution of the test statistic. This is helpful to carryout the power of the test, as the power comparison is valid when estimated type-I error rates are equal for all the tests.

For computing the power of the test, we generated observations from GE distribution with different shape parameter θ and fixed scale parameter $\lambda = 0.33$, then computed the test statistics corresponding to LRT, Wald test1, Wald test2 and Score test. The power of the test is the proportion of times the null hypothesis is rejected out of 10000 simulations.

3.2. Censored Case

To generate censored observations from the GE distribution, we have assumed that the censoring variable C follows discrete uniform distribution. Although continuous uniform distribution is commonly taken as the distribution of the censoring variable, we have used discrete uniform distribution for simplicity. This corresponds to a scenario where patients are lost to follow in medical follow-up studies.

We have considered two scenarios namely 10% censoring and 20% censoring. For any sample size n, the number of uniform variable generated corresponds to $0.1 \times n$ and $0.2 \times n$ for the 10% and 20% censoring respectively. The observations from the GE distribution having these serial number are treated as censored observations. The rest of the computation is similar to the uncensored case.

4. Result and Discussion

4.1. Estimated Type-I Error Rate for the Uncensored Case

If the estimated values of type-I error rate lie within the range of $(0.045, 0.055)$, then the test maintains the level of significance 0.05, i.e., 10% error. A similar criterion was also used by several researchers. For example, refer to DCunha and Rao (2014a,b). Table 1 presents the estimated type-I error rate for the uncensored case.

Table 1: Estimated Type-I error rate for various tests when $\alpha = 0.05$ and for various sample sizes for the uncensored case.

n	LRT	Wald test1	Wald test2	Score
20	0.0566	0.0318	0.1303	0.0397
40	0.0532	0.0419	0.0986	0.0446
60	0.0524	0.0447	0.0825	0.0475
80	0.0542	0.0472	0.0739	0.0488
100	0.0497	0.0455	0.0665	0.0470

From the Table 1, we can say that in LRT, the test maintains level of significance for all sample sizes. Wald test1 maintains level of significance for the samples of size ≥ 60. Wald test2 does not maintain level of significance in all the sample sizes. Score test maintains level of significance for the samples of size ≥ 40.

To check the adequacy of the chi-square distribution to the null distribution of the test statistic, the histogram of the simulated null distribution of the test statistic along with the pdf of the chi-square distribution with 1 degree of freedom is plotted in Figures 1 - 6 for sample sizes $n = 20, 60, 100$; to save space the Figures are not presented for other sample sizes.

Figure 1: Histogram for the LRT and Wald test1 when n=20.

Figure 2: Histogram for the Wald test2 and Score when n=20.

Figure 3: Histogram for the LRT and Wald test1 when n=60.

Figure 4: Histogram for the Wald test2 and Score when n=60.

Figure 5: Histogram for the LRT and Wald test1 when n=100.

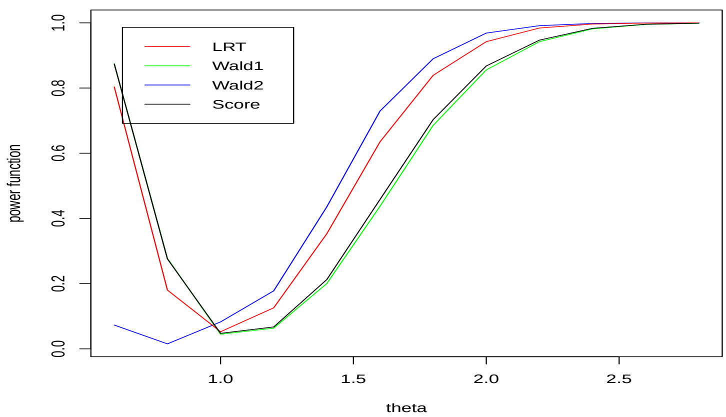
Figure 6: Histogram for the Wald test2 and Score when n=100.

For small sample sizes, the approximation is more accurate for the LRT compared to other tests. The approximation differs in the middle region and as the sample size increases the chi-square approximation is accurate to the null distribution for all the test statistics in the range of the test statistic.

4.2. Power of the Test for the Uncensored Case

The estimated power curves of all the tests is presented in Figure 7 for sample size 60.

Figure 7: Power curves for different tests when n=60 for uncensored case.



We have used the estimated upper 5^{th} percentile value of the simulated distribution of the test statistics as the critical value for computing the power.

From the Figure 7, we can say that the rate of convergence of power function to the value 1 is faster for Wald test2 compared to other tests. For the right local alternative, the power of the Wald test2 is higher than the other tests, which is followed by LRT, Score and Wald test1 respectively. For the left local alternative, the power of the Wald test1 is higher compared to other tests. However the difference is marginal when compared to the power of the Score test. The power of the Wald test2 is drastically different from other tests. The power of the test could not be computed for smaller values of θ as difficulty was encountered in the computation.

4.3. Estimated Type-I Error Rate for the Censored Case

For the Monte Carlo comparison, we have restricted to 10% and 20% random censoring, which corresponds to mild and moderate censoring. Censoring distribution is discrete uniform. This simulation corresponds to the case of persons lost to follow-up in medical studies.

Estimated type-I error rate is presented in Table 2 across the sample sizes.

Table 2: Estimated Type-I error rate for various tests when $\alpha = 0.05$. (10% and 20%censoring).

n	10%				20%			
	LRT	Wald test1	Wald test2	Score	LRT	Wald test1	Wald test2	Score
20	0.0614	0.0288	0.1439	0.0395	0.0898	0.0315	0.1477	0.0796
40	0.0728	0.0417	0.1021	0.0722	0.1260	0.0352	0.1085	0.1541
60	0.0779	0.0446	0.0871	0.0901	0.2022	0.0415	0.0879	0.2193
80	0.0862	0.0484	0.0806	0.1020	0.3522	0.0468	0.0873	0.2915
100	0.0867	0.0446	0.0742	0.1183	0.6880	0.0456	0.0803	0.3450

From the Table 2, we say that for 10% censoring, LRT, Wald test2 and Score test do not maintain level of significance for all the sample sizes, while Wald test1 maintain level of significance when the sample size

≥ 60. For 20% censoring, the conclusion is the same except that Wald test1 maintain level of significance when the sample size ≥ 80.

Histogram of estimated null distribution of the statistic indicates that the chi-square approximation to the null distribution is not accurate for all the sample sizes for the LRT, Wald test2 and Score test for 10% as well as 20% censoring. To save the space, figures are not shown here.

4.4. Power of the Test for the Censored Case

Figure 8: Power curves for different tests when n=60 for 10% censored case.

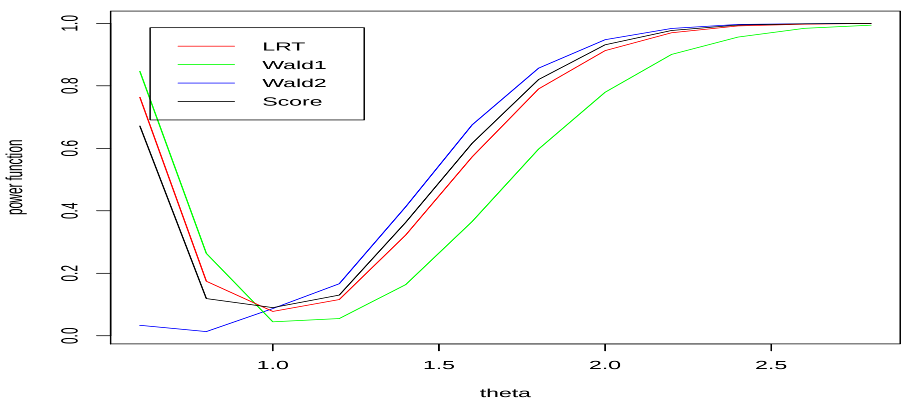
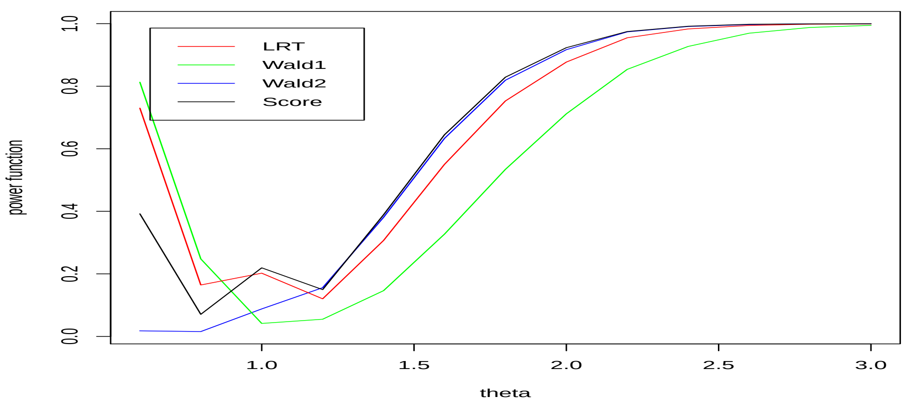


Figure 9: Power curves for different tests when n=60 for 20% censored case.



Figures 8 and 9 presents the estimated power curves for the four tests for the sample size $n = 60$. As in the uncensored case, the estimated critical values are used for the power computation. As in the uncensored case, Wald test1 emerges as a best test for left local alternative for 10% as well as 20% censoring. For the right local alternative, the Wald test2 emerges as the best test for 10% censoring as in the uncensored case. While Score test emerges as the best test for 20% censoring. For left alternatives, the power of each of the test decreases as the degree of censoring increases. For the right alternatives, the similar conclusion emerges for the LRT, Wald test1 and Wald test2, while for the Score test, the power of the test increases as the degree of censoring increases. Generally, it is expected that there is loss in the power of the test as the degree of censoring increases. The reasoning for the behaviour of the power function for the Score test is not clear at this juncture.

5. Example (Leukemia Free Survival Times)

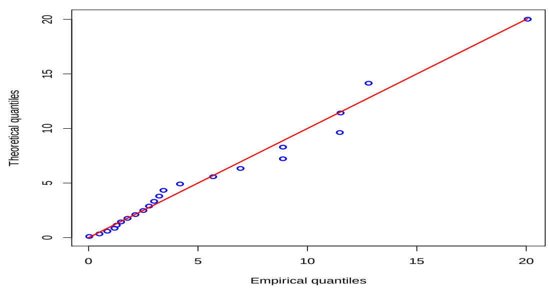
To illustrate the use of the various tests developed in previous sections, we have considered a data set on Leukemia free survival times (in months), for the 50 allogeneic transplant patients available in the text book authored by Klein and Moeschberger (2003).

Table 3: Leukemia free survival times.

The leukemia-free survival times for the 50 allo transplant patients were
0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224,
3.421, 4.178, 4.441^+, 5.691, 5.855^+, 6.941^+, 6.941, 7.993^+, 8.882, 8.882, 9.145^+,
11.480, 11.513, 12.105^+, 12.796, 12.993^+, 13.849^+, 16.612^+, 17.138^+, 20.066, 20.329^+,
22.368^+, 26.776^+, 28.717^+, 28.717^+, 32.928^+, 33.783^+, 34.211^+, 34.770^+, 39.539^+,
41.118^+, 45.033^+, 46.053^+, 46.941^+, 48.289^+, 57.401^+, 58.322^+, 60.625^+.

There are 28 censored patients which corresponds to 56% censoring. The Q-Q plot for uncensored observation is shown in Figure 10, which shows that the data is well described by the GE distribution.

Figure 10: Q-Q plot of GE distribution for Leukemia free survival times for uncensored observations.



Exponential distribution is a member of GE family of distribution, and the null hypothesis of interest is whether the data is a sample from the exponential distribution. i.e., $H_0 : \theta = 1$.

The estimates of the restricted and unrestricted MLEs for the data, including the censored observation, is given in the Table 4

Table 4: Maximum Likelihood Estimates.

Types of data	Unrestricted		Restricted
	θ	λ	λ
Uncensored	0.960825	0.1871362	0.192024
Uncensored+censored	0.4557904	0.00683785	0.05390481

The data is analyzed in two stages. In first stage, only the uncensored observations are taken, while in the second stage, all the observations are considered for carrying out the test.

The values of the test statistics along with the p-value are reported in Table 5.

Table 5: The values of the test statistics along with the p-value.

Tests	Uncensored observation		uncensored + censored observations	
	value of the statistic	p-value	value of the statistic	p-value
LRT	0.02110712	0.8844875	34.7124	3.821894×10^{-09}
Wald test1	0.02180514	0.8826067	28.22367	1.080748×10^{-07}
Wald test2	0.01977727	0.8881609	6.567576	0.01038533
Score test	0.02181101	0.882591	0.02307798	0.8792545

For the uncensored case, the LRT maintains type-I error rate for all the sample sizes, and therefore the conclusion is to accept the null hypothesis at 5% level of significance. Even though other tests do not maintain level of significance, the same conclusion is arrived at, when we use these tests. When we considered the uncensored as well as censored cases, the p-values for all the tests are less than 0.05 except for the Score test. Therefore, we have to reject the null hypothesis based on the evidence provided by the majority of the tests. This is an example where heavy censoring leads to a different conclusion than obtained by using uncensored observations.

6. Conclusion

In this paper, we have investigated the finite sample performance of the likelihood ratio, Wald and Score tests. The Monte Carlo results indicate that, LRT maintains level of significance even for a small sample of size $n = 20$ in uncensored case, but it does not maintain level of significance in censored case. Wald test1 maintains level of significance for sample size $n \geq 60$ in uncensored case as well as in 10% censored case, while it maintains level of significance for sample size $n \geq 80$ for the case of 20% censoring. Wald test2 does not maintain level of significance for all sample sizes in uncensored as well as in censored case, but the rate of convergence of the power function to value 1 is faster compared to other tests. Score test maintains level of significance for sample size $n \geq 40$ in uncensored case and it does not maintain level of significance for all the sample sizes in censored case. Barring Wald test2, LRT and Score test have better power property compared to the Wald test1.

Looking at the above conclusion, we recommend LRT for the uncensored case. For the censored case, we recommend Wald Test1, but minimum sample size should be ≥ 80. Finite sample performances of the likelihood ratio, Wald and Score test were considered by various researchers in the past in different context (see Nairy and Rao (2003), Bhat and Rao (2007), Guddattu and Rao (2009, 2010), Sumathi and Rao (2009, 2010, 2011, 2013, 2014), Aruna and Rao (2014) and the references cited in these papers). No uniform pattern is emerging from this investigation. A promising approach is to use Bayesian test, even though they are computationally tedious. Frequentist comparison of likelihood based classical tests and Bayes tests in GE can be a topic for future research.

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