ProbStat Forum, Volume 10, October 2017, Pages 85–93
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

On the Characteristic Function of Extrema with Some Applications

Aparna Aravindakshan M[a], Chandran C[a]

[a]*Department of Statistics, University of Calicut, Kerala - 673 635, India.*

Abstract. The characteristic function of sums of n independent and identically distributed random variables can be expressed as the n^{th} power of the characteristic function of single random variable. However, the characteristic function of maximum of n independent and identically distributed random variables is not available in the literature in closed form. This paper identifies the conditions under which the characteristic function of maxima and minima is expressible in closed form. Some applications of these results are illustrated.

1. Introduction

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d) non-degenerate random variables (r.v) with common distribution function (d.f) $F_X(x)$ and characteristic function (c.f) $\phi_X(t)$. Let $S_n = \sum_{i=1}^{n} X_i$, then $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of $\{X_n\}_{n=1}^{\infty}$. The c.f of S_n is given by,

$$\phi_{S_n}(t) = (\phi_X(t))^n, n \geq 1.$$

When the r.v are independent but not identically distributed,

$$\phi_{S_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t), n \geq 1.$$

It is these explicit forms of $\phi_{S_n}(t)$ that helps to derive stability properties of the sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d or independent non-identically distributed r.v in terms of the partial sum sequence $\{S_n\}_{n=1}^{\infty}$. For example, central limit theorem, laws of large numbers, etc., of sequence $\{X_n\}_{n=1}^{\infty}$ (see Billingsley (1995) and Laha and Rohatgi (1979)). However, in general the d.f of the partial sum S_n does not have an explicit form for either the i.i.d case or in the independent non-identically distributed case. In such situations generally the asymptotic distributions of S_n is used in the literature. For example, the central limit theorem says that, if there exists sequence of constants $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, with $b_n > 0$, such that the normalized sequence of partial sums $\{\frac{S_n - a_n}{b_n}\}_{n=1}^{\infty}$ converges in law to some non-degenerate random variable Z whose d.f is G, then G is an α-stable distribution for some $\alpha > 0$. The normal distribution arises as a special case of this when $\alpha = 2$. This approximation is extensively used in the statistics literature when the exact distribution

2010 *Mathematics Subject Classification.*
Keywords. Distribution function, Characteristic function, Maxima, Minima, Distributions closed under extrema.
Received: 22 March 2017; Revised: 11 April 2017, Re-revised 12 June 2017, 25 August 2017, Accepted: 04 September 2017

Email addresses: aparna.aravind@gmail.com (Aparna Aravindakshan M), c_cheruvalath@rediffmail.com (Chandran C)

of S_n cannot be computed exactly. For example, if the second moment of X_n exists for a sequence of i.i.d r.v $\{X_n\}_{n=1}^{\infty}$, S_n has an asymptotic normal distribution.

Two other statistics in the literature which are equally important as S_n are the partial maxima and partial minima given by,

$$\begin{aligned} M_n &= \max(X_1, X_2, \ldots, X_n) \quad (1)\\ m_n &= \min(X_1, X_2, \ldots, X_n). \quad (2) \end{aligned}$$

These statistics are also extensively used in the statistics literature. For example, these statistics or their functions are the minimal sufficient statistics in the non-regular family of distributions. Similarly the maximum likelihood estimators and the uniformly most powerful test functions for non-regular family of distributions are functions of the above statistics. In reliability, these are the statistics used to study the parallel and series systems. Unlike the d.f of S_n, the d.f of M_n and m_n are in explicit form given by,

$$\begin{aligned} F_{M_n}(x) &= [F_X(x)]^n \quad (3)\\ F_{m_n}(x) &= 1 - [1 - F_X(x)]^n. \quad (4) \end{aligned}$$

For a detailed account of the distribution of M_n, m_n and all other order statistics and their functions see David (1970). However, these exact distributions cannot be used for modeling purposes in many situations. The famous Fisher-Tippet theorem identifies the limit distribution of M_n and m_n whenever they exist. The limit distributions of the linearly normalized maxima can be identified as one among the three extreme value distributions namely Frechet, Weibull and Gumbel distributions, which are the max-stable distributions on $\Re$. These three extreme value distributions can be put into a single family called the Generalized Extreme Value Distributions. For details, see Embrechts et.al. (1997), Leadbetter et.al. (1983) and Resnick (1987). Theorem 2.1 of Pancheva (2010) gives three equivalent conditions which characterize the generalized max stable distributions on $\Re$ under a more general normalization, for details see Pancheva (2010).

As discussed above, does there exist any situation in which we can express the c.f of maxima or minima or both in an explicit form? This natural curiosity is the motivation behind this piece of work. The c.f of M_n and m_n can be expressed as:

$$\begin{aligned} \phi_{M_n}(t) &= E(e^{itM_n}), \quad t\epsilon\Re \\ &= \int_{\Re} e^{itx} dF_{M_n}(x) \\ &= \begin{cases} \int_{\Re} e^{itx} f_{M_n}(x)dx, & \text{if F is continuous} \\ \sum_x e^{itx} P(M_n = x), & \text{if F is discrete} \end{cases} \\ &= \begin{cases} \int_{\Re} e^{itx} n[F_X(x)]^{(n-1)} f_X(x)dx, & \text{if F is continuous} \\ \sum_x e^{itx} [F_{M_n}(x) - F_{M_n}(x-)], & \text{if F is discrete.} \end{cases} \quad (5) \end{aligned}$$

$$\begin{aligned} \phi_{m_n}(t) &= E(e^{itm_n}), \quad t\epsilon\Re \\ &= \int_{\Re} e^{itx} dF_{m_n}(x) \\ &= \begin{cases} \int_{\Re} e^{itx} f_{m_n}(x)dx, & \text{if F is continuous} \\ \sum_x e^{itx} P(m_n = x), & \text{if F is discrete} \end{cases} \\ &= \begin{cases} \int_{\Re} e^{itx} n[1 - F_X(x)]^{(n-1)} f_X(x)dx, & \text{if F is continuous} \\ \sum_x e^{itx} [F_{m_n}(x) - F_{m_n}(x-)], & \text{if F is discrete.} \end{cases} \quad (6) \end{aligned}$$

No further simplifications to the integrals or sums in equations (5) and (6) are possible unless or until we impose some restrictions over F. Let us try to evaluate these through some specific situations.

Example 1.1. *Suppose X follows exponential distribution with d.f $F_X(x) = 1 - e^{-\theta x}$, $x > 0$, $\theta > 0$. Now, $F_{M_n}(x) = (1 - e^{-\theta x})^n$ and $F_{m_n}(x) = 1 - e^{-n\theta x}$. It is to be noted that minima of exponentially distributed random variable with parameter θ again has an exponential distribution with parameter $n\theta$. Let $\phi_{X;\theta}(t)$ be the c.f of X. Then, $\phi_{X;\theta}(t) = \frac{\theta}{\theta - it}$. Now, due to the one to one correspondence between d.f and c.f, the c.f of minima, $\phi_{m_n;\theta}(t) = \frac{n\theta}{n\theta - it} = \phi_{X;n\theta}(t)$.*

What will be the form of c.f of maxima in Example 1.1.? We narrow down our study to distributions whose minima (maxima) belongs to the same family and try to get a form for c.f of maxima (minima) in this paper. Rest of this paper is arranged as follows: Some basic concepts required to prove the main results are given in Section 2. In Section 3, a relation between d.f of maxima and minima is derived and its special significance when the distribution has closure property under maxima or minima is given. Section 4 gives the c.f of maxima and minima when X is closed under either minima or maxima. Section 5 discusses an application of the results derived in Section 4.

2. Basic Concepts

In this section we discuss some basic concepts required to prove the main results in the foregoing sections. These concepts are illustrated through suitable examples.

Definition 2.1. *Let X be a random variable with d.f $F_{X;\alpha}$ involving a parameter α and $(X_1, X_2, \ldots, X_n)$ be a random sample of independent observations of size n from $F_{X;\alpha}$. Then $F_{X;\alpha}$ is said to be closed under minima, m_n, with respect to the parameter α if for every $n \geq 1$, the d.f of m_n is $F_{X;g_n(\alpha)}$, for some $g_n(\alpha)$ which is a function of the parameter α.*

Example 2.2. *Suppose X has exponential distribution with d.f $F_{X;\theta}(x) = 1 - e^{-\theta x}$, $x > 0$, $\theta > 0$, then $F_{m_n;\theta}(x) = 1 - e^{-n\theta x} = F_{X;g_n(\theta)}(x)$, where $g_n(\theta) = n\theta$. Hence, exponential distribution is closed under minima.*

Example 2.3. *Suppose, X follows Pareto distribution with d.f $F_{X;\alpha}(x) = 1 - (\frac{p}{x})^{\alpha}$, $x > p$, $p > 0$, $\alpha > 0$ then $F_{m_n;\alpha}(x) = 1 - (\frac{p}{x})^{n\alpha} = F_{X;g_n(\alpha)}(x)$. Here, $g_n(\alpha) = n\alpha$. i.e., Pareto distribution is closed under minima.*

Example 2.4. *If X has Weibull distribution with d.f given by, $F_{X;\theta}(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\alpha}}$, $x > 0$, $\theta > 0$, $\alpha > 0$ then, $F_{m_n;\theta}(x) = 1 - e^{-n\left(\frac{x}{\theta}\right)^{\alpha}} = F_{X;g_n(\theta)}(x)$. In this case, $g_n(\theta) = \theta/n^{1/\alpha}$. If we define X to have Weibull distribution with d.f $F_{X;\theta}(x) = 1 - e^{-\frac{x^{\alpha}}{\theta}}$, $x > 0$, $\theta > 0$, $\alpha > 0$, then $F_{m_n;\theta}(x) = 1 - e^{-\frac{nx^{\alpha}}{\theta}} = F_{X;g_n(\theta)}(x)$, and $g_n(\theta) = \theta/n$. Hence, Weibull distribution is closed under minima.*

Example 2.5. *Let X has Geometric distribution with parameter $0 < p < 1$, then,*

$$F_{X;p}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1-p)^{[x]+1}, & x \geq 1 \end{cases}$$

and hence,

$$F_{m_n;p}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1-p)^{n([x]+1)}, & x \geq 1. \end{cases}$$

So, m_n has Geometric distribution with parameter $0 < 1-(1-p)^n < 1$. Here $g_n(p) = 1-(1-p)^n$. Therefore, geometric distribution is closed under minima.

Definition 2.6. *Let X be a random variable with d.f $F_{X;\alpha}$ involving a parameter α and $(X_1, X_2, \ldots, X_n)$ be a random sample of independent observations of size n from $F_{X;\alpha}$. Then $F_{X;\alpha}$ is said to be closed under maxima, M_n, with respect to the parameter α, if for every $n \geq 1$, the d.f of M_n is $F_{X;h_n(\alpha)}$, for some $h_n(\alpha)$ which is a function of the parameter α.*

Example 2.7. *Suppose X has power distribution with d.f $F_{X;\alpha}(x) = \left(\frac{x}{\theta}\right)^{\alpha}$, $0 < x < \theta$, $\theta > 0$, $\alpha > 0$ then, $F_{M_n;\alpha}(x) = \left(\frac{x}{\theta}\right)^{n\alpha} = F_{X;h_n(\alpha)}(x)$. Here, $h_n(\alpha) = n\alpha$. i.e., power distribution is closed under maxima.*

Example 2.8. *If X has an inverse Weibull distribution with $F_{X;\theta}(x) = e^{-\left(\frac{\theta}{x}\right)^{\alpha}}, x > 0, \theta > 0, \alpha > 0$ then, $F_{M_n;\theta}(x) = e^{-n\left(\frac{\theta}{x}\right)^{\alpha}} = F_{X;h_n(\theta)}(x)$. In this case $h_n(\theta) = n^{1/\alpha}\theta$. Hence, inverse Weibull distribution is closed under maxima.*

Example 2.9. *If X has Bernoulli distribution with d.f*

$$F_{X;p}(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \le x < 1, \quad 0 < p < 1 \\ 1, & x \ge 1. \end{cases}$$

Then,

$$F_{m_n;p}(x) = \begin{cases} 0, & x < 0 \\ 1-p^n, & 0 \le x < 1, \quad 0 < p < 1 \\ 1, & x \ge 1. \end{cases} = F_{X;g_n(p)}(x)$$

and

$$F_{M_n;p}(x) = \begin{cases} 0, & x < 0 \\ (1-p)^n, & 0 \le x < 1, \quad 0 < p < 1 \\ 1, & x \ge 1. \end{cases} = F_{X;h_n(p)}(x).$$

Here, $g_n(p) = p^n$ and $h_n(p) = 1-(1-p)^n$. Hence, Bernoulli distribution is closed under maxima and minima.

Remark 2.10. *Every two point distribution is closed under both maxima and minima.*

The above examples are of standard distributions. Given an arbitrary F one can construct distribution G having closure property.

Example 2.11. *If X has an arbitrary distribution $F_X(x)$, then the family of distributions $G(x) = [F_X(x)]^{\lambda}, \lambda > 0$, which is well known in the literature as exponentiated family of distributions, is closed under maxima with $h_n(\lambda) = n\lambda$ and the family $H(x) = 1 - [1 - F_X(x)]^{\lambda}, \lambda > 0$ is closed under minima with $g_n(\lambda) = n\lambda$.*

Theorem 2.12. *If X is a positive valued random variable with continuous distribution $F_{X;\alpha}(x)$ which is closed under minima (maxima), then the d.f of $\frac{1}{X}$ (inverse of X) given by, $G_{\frac{1}{X};\alpha}(x) = 1 - F_{X;\alpha}(1/x)$, is closed under maxima (minima).*

Proof. Since $X > 0$, $G_{\frac{1}{X};\alpha}(x) = 1 - F_{X;\alpha}(1/x)$ is a d.f. Suppose $F_{X;\alpha}(x)$ is closed under minima with respect to the parameter α, then,

$$F_{m_n;\alpha}(x) = 1 - (1 - F_{X;\alpha}(x))^n = F_{X;g_n(\alpha)}(x).$$

Now,

$$\begin{aligned} G_{M_n;\alpha}(x) &= [G_{\frac{1}{X};\alpha}(x)]^n \\ &= [1 - F_{X;\alpha}(1/x)]^n \\ &= 1 - F_{X;g_n(\alpha)}(1/x) \\ &= G_{\frac{1}{X};g_n(\alpha)}(x). \end{aligned}$$

Similarly we can prove the case when X is closed under maxima. □

Remark 2.13. *Every max-stable (min-stable) distribution is closed under maxima (minima).*

Remark 2.14. *Exponential distribution discussed in Example 2.1. is closed under minima according to Definition 2.1. but is not closed under maxima in the sense of Definition 2.2., since $F_{M_n;\theta}(x) = (1-e^{-\theta x})^n \neq F_{X;g_n(\theta)}(x)$. However, this distribution is max-stable and min-stable in the sense of Pancheva (2010) with $\mathbb{L}_n(x) = -\frac{1}{\theta} ln[1-(1-e^{-\theta x})^n]$ and $\mathbb{L}_n(x) = nx$ respectively, where $\mathbb{L}_n(x)$ is the function discussed in Pancheva (2010).*

3. Distribution Function of Maxima and Minima

In this section we derive a relation between d.f of maxima and minima. The d.f of maxima in terms of d.f of minima and d.f of minima in terms of d.f of maxima are derived. This has special significance when the distribution have closure property under maxima or minima. Probability density function (probability mass function) also has a similar representation.

Lemma 3.1. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with d.f $F_X(x)$. Then, the d.f of partial maxima M_n and partial minima m_n are given respectively by,*

$$F_{M_n}(x) = \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} F_{m_k}(x) \tag{7}$$

and

$$F_{m_n}(x) = \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} F_{M_k}(x). \tag{8}$$

Proof. Replacing $[F_X(x)]^n$ by $[1-(1-F_X(x))]^n$ in (3) and expanding and simplifying using binomial theorem we get,

$$\begin{aligned} F_{M_n}(x) &= \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} [1 - (1 - F_X(x))^k] \\ &= \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} F_{m_k}(x). \end{aligned} \tag{9}$$

Similarly using binomial expansion in (4) and simplifying we get,

$$\begin{aligned} F_{m_n}(x) &= \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} [F_X(x)]^k \\ &= \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} F_{M_k}(x). \end{aligned} \tag{10}$$

□

The following two corollaries give the corresponding representations for the probability density function and probability mass function respectively.

Corollary 3.2. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with continuous d.f $F_X(x)$ and density $f_X(x)$. Then, the probability density function of partial maxima M_n and partial minima m_n are given respectively by,*

$$f_{M_n}(x) = \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} f_{m_k}(x) \tag{11}$$

and

$$f_{m_n}(x) = \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} f_{M_k}(x). \tag{12}$$

Corollary 3.3. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with discrete d.f $F_X(x)$ and probability mass function $P(X = x)$. Then, the probability mass function of partial maxima M_n and partial minima m_n are given respectively by,*

$$P(M_n = x) = \sum_{k=1}^{n} (-1)^{(k-1)} \binom{n}{k} P(m_k = x) \tag{13}$$

and

$$P(m_n = x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}P(M_k = x). \tag{14}$$

The following theorem gives a new representation for the d.f of maxima (minima) in terms of the d.f of X, when X is closed under minima (maxima).

Theorem 3.4. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with d.f $F_{X;\alpha}(x)$. If X is closed under minima with respect to the parameter α,*

$$F_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;g_k(\alpha)}(x) \tag{15}$$

and if X is closed under maxima with respect to the parameter α,

$$F_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;h_k(\alpha)}(x). \tag{16}$$

Proof. Since X is closed under minima with respect to the parameter α, by replacing $F_{m_k;\alpha}(x)$ by $F_{X;g_k(\alpha)}(x)$ in (7) we get,

$$F_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;g_k(\alpha)}(x)$$

and similarly if X is closed under maxima, from (8) we have,

$$F_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}F_{X;h_k(\alpha)}(x).$$

□

When $F_{X;\alpha}$ admits a density $f_{X;\alpha}$, we have the following corollary.

Corollary 3.5. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with continuous d.f $F_{X;\alpha}(x)$. If X is closed under minima,*

$$f_{M_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}f_{X;g_k(\alpha)}(x) \tag{17}$$

and if X is closed under maxima with respect to the parameter α,

$$f_{m_n;\alpha}(x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}f_{X;h_k(\alpha)}(x). \tag{18}$$

Now, when F is discrete, we have the following corollary.

Corollary 3.6. *Let $X_1, X_2, \dots, X_n$ be i.i.d r.v with discrete d.f $F_{X;\alpha}(x)$. If X is closed under minima with respect to the parameter α,*

$$P(M_n;\alpha = x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}P(X;g_k(\alpha) = x) \tag{19}$$

and if X is closed under maxima with respect to the parameter α,

$$P(m_n;\alpha = x) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}P(X;h_k(\alpha) = x). \tag{20}$$

4. Characteristic Function of Maxima and Minima

In this section we derive the c.f of minima when X is closed under maxima and the c.f of maxima when X is closed under minima. We have seen from equations (5) and (6) that the c.f of maxima and minima does not have a compact form. However, when X is closed under maxima (minima) the c.f of minima (maxima) have a representation similar to that of d.f in Theorem 3.4.

Theorem 4.1. *Let X be a random variable with d.f $F_{X;\alpha}$ which is closed under minima with respect to the parameter α. Let the c.f of X be $\phi_{X;\alpha}(t)$, then the c.f of m_n is given by,*

$$\phi_{m_n;\alpha}(t) = \phi_{X;g_n(\alpha)}(t) \tag{21}$$

and that of M_n by,

$$\begin{aligned}\phi_{M_n;\alpha}(t) &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{m_k;\alpha}(t)\\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{X;g_k(\alpha)}(t).\end{aligned} \tag{22}$$

Proof. We have, $X \sim F_{X;\alpha}$ and $m_n \sim F_{X;g_n(\alpha)}$. Hence, by the one to one correspondence between d.f and c.f,

$$\phi_{m_n;\alpha}(t) = \phi_{X;g_n(\alpha)}(t).$$

Now if X is continuous,

$$\phi_{M_n;\alpha}(t) = \int_{\Re} e^{itx} f_{M_n;\alpha}(x)dx$$

and from (17) on simplification we get,

$$\phi_{M_n;\alpha}(t) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}\phi_{X;g_k(\alpha)}(t).$$

Similarly if X is discrete from (18) we get,

$$\phi_{M_n;\alpha}(t) = \sum_{k=1}^{n}(-1)^{(k-1)}\binom{n}{k}\phi_{X;g_k(\alpha)}(t).$$

□

Example 4.2. *In the case of exponential distribution with d.f $F_{X;\theta}(x) = 1 - e^{-\theta x}$, $\phi_{X;\theta}(t) = \frac{\theta}{\theta - it}$. Then c.f of minima, $\phi_{m_n;\theta}(t) = \frac{n\theta}{n\theta - it}$ and hence c.f of maxima is $\phi_{M_n;\theta}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{k\theta}{k\theta - it}$.*

Example 4.3. *Suppose X has Weibull distribution with d.f $F_{X;\theta}(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\alpha}}$, $x > 0,\ \theta > 0,\ \alpha > 0$ then, $\phi_{X;\theta}(t) = \sum_{j=0}^{\infty}\frac{(it)^j\theta^j}{j!}\Gamma(1+\frac{j}{\alpha})$. Now, $\phi_{m_n;\theta}(t) = \phi_{X;\theta/n^{1/\alpha}}(t)$ and hence $\phi_{M_n;\theta}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{X;\theta/n^{1/\alpha}}(t)$.*

Next we prove the result for the c.f of minima when the original distribution is closed under maxima.

Theorem 4.4. *Let X be a random variable with d.f $F_{X;\alpha}$ which is closed under maxima with respect to the parameter α. Let the c.f of X be $\phi_{X;\alpha}(t)$, then the c.f of M_n is given by,*

$$\phi_{M_n;\alpha}(t) = \phi_{X;h_n(\alpha)}(t) \tag{23}$$

and that of m_n by,

$$\begin{aligned}\phi_{m_n;\alpha}(t) &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{M_k;\alpha}(t)\\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\phi_{X;h_k(\alpha)}(t). \qquad (24)\end{aligned}$$

Proof. Proof is similar to that of Theorem 4.1. □

5. Some Applications of the Results

In this section we discuss some applications of the results proved in the previous section to evaluate the moments of extrema. The k^{th} moment of extema of a distribution exists iff the k^{th} moment of the base random variable exists and if the k^{th} moment of the random variable X exists only if the c.f of X is differentiable k times. Let us denote the k^{th} derivative of the c.f of X by $\phi^{(k)}_{X;\alpha}(t)$.

Theorem 5.1. *Let the d.f $F_{X;\alpha}$ be closed under minima with respect to the parameter α. Then,*

$$E[(m_n;\alpha)^k] = E[(X;g_n(\alpha))^k] \qquad (25)$$

and

$$E[(M_n;\alpha)^k] = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E[(X;g_k(\alpha))^k]. \qquad (26)$$

Proof. Differentiating (21) and (22) k times, dividing by i^k and letting $t=0$ we get the results. □

Corollary 5.2. *Let X be a r.v having d.f $F_{X;\alpha}$ which is closed under minima with respect to the parameter α, then the expectation of the partial minima m_n is given by,*

$$E[m_n;\alpha] = E[X;g_n(\alpha)] \qquad (27)$$

and the expectation of partial maxima M_n is given by,

$$E[M_n;\alpha] = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E[X;g_k(\alpha)]. \qquad (28)$$

We have a similar result when the d.f F is closed under maxima.

Theorem 5.3. *If the d.f $F_{X;\alpha}$ is closed under maxima with respect to the parameter α, then,*

$$E[(M_n;\alpha)^k] = E[(X;h_n(\alpha))^k] \qquad (29)$$

and

$$E[(m_n;\alpha)^k] = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E[(X;h_k(\alpha))^k]. \qquad (30)$$

Proof. Proof follows as that of Theorem 5.1. □

Corollary 5.4. *Let X be a r.v having d.f $F_{X;\alpha}$ which is closed under maxima with respect to the parameter α, then the expectation of the partial maxima M_n is given by,*

$$E[M_n;\alpha] = E[X;h_n(\alpha)] \qquad (31)$$

and the expectation of partial minima m_n is given by

$$E[m_n;\alpha] = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}E[X;h_k(\alpha)]. \qquad (32)$$

This result can be used to find the expected lifetime of a series or parallel system. An illustration of the results in Theorem 5.1. and Theorem 5.3. is given in the following example.

Example 5.5. *Consider a parallel system consisting of 5 components each having life distribution* $F_{X;\theta}(x) = 1 - e^{-\theta x}$, $x > 0$, $\theta > 0$. *What will be the expected life time of the system?*

A parallel system fails if all of its components fail. So the lifetime of the system is given by, $F_{M_5;\theta}(x) = P(M_5 \leq x) = [P(X \leq x)]^5$. Since exponential distribution is closed under minima, $F_{m_k;\theta}(x) = F_{X;g_k(\theta)} = F_{X;k\theta} = 1 - e^{-k\theta x}$, $k = 1, 2, ..., 5$ and $E[m_k; \theta] = E[X; g_k(\theta)] = \frac{1}{k\theta}$. Therefore,

$$\begin{aligned} E[M_5;\theta] &= \sum_{k=1}^{5}(-1)^{k-1}\binom{5}{k}E[X;g_k(\theta)] \\ &= \sum_{k=1}^{5}(-1)^{k-1}\binom{5}{k}\frac{1}{k\theta} \\ &= \frac{137}{60\theta}. \end{aligned}$$

Acknowledgement

The 1st author thanks the Indian Council of Medical Research (**ICMR**), for the financial support which led to this research work. The authors thank the referees for their valuable suggestions which has improved the original version of this paper. They also thank one of the referees in particular who pointed out to clarify the connection of Definitions 2.1 and 2.2 with the definition of max-stability in Pancheva (2010).

References

[1] Billingsley, P. (1995) *Probability and Measure*, **3**rd edition, John Wiley & sons.
[2] David, H.A. (1970). *Order Statistics*, **1**st edition, John Wiley & Sons, New York.
[3] Embrechts, P., Kluppelberg, C. and Mikosch, T. (1997) *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.
[4] Laha, R.G. and Rohatgi,V.K. (1979) *Probability Theory*, John Wiley.
[5] Leadbetter, M.R., Lindgren, G., and Rootzen, H. (1983) *Extremes and Related Properties of Random Sequences and Processes*. Springer-Verlag, New York.
[6] Pancheva, E. (2010) Max-Semistability: A Survey. *ProbStat* Forum, Vol 03, 11-24.
[7] Resnick, S.I. (1987) *Extreme Values, Regular Variation and Point Processes*, Springer, New York.