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Characterizations of Distributions through Random Translation of Order Statistics and Record Statistics

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Abstract. Reflected exponential and extreme value type-I distributions have been characterized through random translation of order statistics, lower record statistics and upper record statistics.

1. Introduction

Chandler (1952) introduced the concept of record statistics. Suppose that $\{X_i\}, i \geq 1$ is a sequence of independent and identically distributed random variables with the distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x)$. Set $Y_n = min(max)\{X_1, X_2, X_3, ..., X_n\}$ for $n \geq 1$. We say that X_j is a lower (upper) record statistic if $Y_j < (>) Y_{j-1}, j > 1$. By definition X_1 is a lower as well as upper record statistic. The indices at which the lower record statistics occur are given by record times $\{L(n),\ n > 1\}$.That is, $X_{L(n)}$ is the n-th lower record, where $L(n) = min\{j|j > L(n-1),\ X_j < X_{L(n-1)},\ n > 1\}$ with $L(1) = 1$ denote the times of lower record statistics.
Let $X_1, X_2, X_3, ...$ be a random sample from a continuous population having the *pdf* $f(x)$ and the *df* $F(x)$. Then the *pdf* and the *df* of $X_{L(r)}$ is given as [Ahsanullah (1995) and Arnold *et al.* (1998)]

$$f_{X_{L(r)}}(x) = \frac{1}{(r-1)!}[H(x)]^{r-1} f(x) \tag{1}$$

and

$$F_{X_{L(r)}}(x) = e^{-H(x)} \sum_{j=0}^{r-1} \frac{[H(x)]^j}{j!} \tag{2}$$

where $H(x) = -lnF(x)$.

The *pdf* of $X_{U(r)}$, the r^{th} upper record statistic is [Ahsanullah (1995) and Arnold *et al.* (1998)]

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}[R(x)]^{r-1} f(x) \tag{3}$$

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and the tail of the *df* of $X_{U(r)}$ is

$$\bar{F}_{X_{U(r)}}(x) = e^{-R(x)} \sum_{j=0}^{r-1} \frac{[R(x)]^j}{j!} \tag{4}$$

where $R(x) = -ln\bar{F}(x)$ and $\bar{F}(x) = 1 - F(x)$.

Let $X_1, X_2, X_3, ..., X_n$ be n random variables. If random variables $X_1, X_2, X_3, ..., X_n$ are arranged in ascending order of magnitude such that $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq ... \leq X_{n:n}$ then $X_{r:n}$ is called the r^{th} order statistic. $X_{1:n} = min(X_1, X_2, X_3, ..., X_n)$ and $X_{n:n} = max(X_1, X_2, X_3, ..., X_n)$ are called extreme order statistics. [David and Nagaraja, (2003)].

The *pdf* and the *df* of $X_{r:n}$ is [Arnold *et al.* (1992), David and Nagaraja (2003)]

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!\ (n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \tag{5}$$

and

$$F_{X_{r:n}}(x) = \sum_{j=r}^{n} \binom{n}{r} [F(x)]^j [1-F(x)]^{n-j}. \tag{6}$$

Suppose a set of data is obtained from the exponential distribution and then has a minus sign in front of each observation. The consequent distribution will be called reversed or reflected exponential distribution. The *pdf* of reflected exponential distribution is give by

$$f(x) = \alpha e^{\alpha x}, \quad x < 0,\ \alpha > 0 \tag{7}$$

with the *df*

$$F(x) = e^{\alpha x}, \quad x < 0,\ \alpha > 0. \tag{8}$$

Shahbaz *et al.* (2016) gives the recurrence relation for single and product moment of generalized order statistics for reflected exponential distribution. Gilchrist (2000) finds the quantile function of reflected exponential distribution with the help of ordinary exponential distribution.

The *pdf* of the extreme value type I distribution is given by [Johnson *et al.* (1995)]

$$f(x) = \frac{1}{\sigma} e^{\frac{x-\mu}{\sigma}} e^{-e^{\frac{x-\mu}{\sigma}}} \quad , -\infty < x < \infty,\ \mu \geq 0, \sigma > 0.$$

At $\mu = 0$, $\sigma = 1$, the above distribution reduces to the standard extreme value distribution with the *pdf*

$$f(x) = e^x e^{-e^x} \quad , -\infty < x < \infty \tag{9}$$

and the *df* is

$$F(x) = e^{-e^x} \quad , -\infty < x < \infty. \tag{10}$$

Buetner and Kamps (2008) use the concept of random translation, random contraction and random dilation of generalized order statistics to characterize some continuous distributions. Blazquez and Mino (2009) characterized geometric distribution through random translation of record value. Martinez *et al.* (2010), Khan and Shah (2012), Khan *et al.* (2012), Shah (2013), Shah *et al.* (2015) amongst others characterized some distributions through random translation, random contraction and random dilation of order statistics, records, generalized order statistics and dual generalized order statistics.

In this paper, we characterize reflected exponential distribution through random translation of order statistics and lower record values as well as extreme value type-I distribution through random translation of upper record values.

2. Characterization Theorems

Theorem 2.1: Let $X_{L(s)}$ be the s^{th} lower record statistic from a continuous population with the $pdf\, f(x)$ and the $df\, F(x)$, then for $1 \leq r < s$

$$X_{L(s)} \underline{\underline{d}}\, X_{L(r)} + U \tag{11}$$

where $U \underline{\underline{d}}\, X_{L(s-r)}$ is the $(s-r)^{th}$ lower record statistic from reflected exponential (α) distribution and is independent of $X_{L(s)}$ if and only if X follow reflected exponential(α).

Proof: For necessary part, let mgf of $X_{L(s)}$ be $M_{X_{L(s)}}(t)$, then

$$X_{L(s)} \underline{\underline{d}}\, X_{L(r)} + U$$

implies

$$M_U(t) = \frac{M_{X_{L(s)}}(t)}{M_{X_{L(r)}}(t)}. \tag{12}$$

The mgf of $X_{L(s)}$, the s^{th} lower record statistic is given by

$$M_{X_{L(s)}}(t) = \int_{-\infty}^{\infty} e^{tx} \frac{[H(x)]^{s-1}}{(s-1)!} f(x)dx.$$

For reflected exponential (α) distribution

$$M_{X_{L(s)}}(t) = \int_{-\infty}^{0} e^{tx} \frac{[-\alpha x]^{s-1}}{(s-1)!} e^{\alpha x} dx. \tag{13}$$

Putting $y = e^x$ in equation (13), we have

$$M_{X_{L(s)}}(t) = \frac{\alpha^s}{(s-1)!} \int_0^1 \left[ln \frac{1}{y}\right]^{s-1} y^{\alpha+t} dy. \tag{14}$$

From Gradshteyn and Ryzhik (2007) (pg-551), we have

$$M_{X_{L(s)}}(t) = \left[\frac{\alpha}{t+\alpha}\right]^s. \tag{15}$$

Therefore from equation (12), we have

$$M_U(t) = \left[\frac{\alpha}{t+\alpha}\right]^{s-r} \tag{16}$$

which is the mgf of $(s-r)^{th}$ lower record statistic from reflected exponential(α) distribution.

For sufficient part, note that the pdf of $X_{L(s)}$ by the convolution method is

$$\begin{aligned} f_{X_{L(s)}}(x) &= \int_0^x f_{X_{L(r)}}(u) f_U(x-u) du \\ &= \frac{\alpha^{s-r}}{(s-r)!} \int_0^x [-(x-u)]^{s-r} e^{\alpha(x-u)} f_{X_{L(r)}}(u) du. \end{aligned} \tag{17}$$

Differentiating both sides of (17) with respect to x , we get

$$\frac{d}{dx}f_{X_{L(s)}}(x) = \frac{\alpha^{s-r}}{(s-r)!}\int_0^x \alpha\left[-(x-u)\right]^{s-r} e^{\alpha(x-u)} f_{X_{L(r)}}(u)du$$
$$-\frac{\alpha^{s-r}}{(s-r)!}\int_0^x (s-r)\left[-(x-u)\right]^{s-r-1} e^{\alpha(x-u)} f_{X_{L(r)}}(u)du. \quad (18)$$

In view of equation (17), (18) reduces to

$$\frac{d}{dx}f_{X_{L(s)}}(x) = \alpha f_{X_{L(s)}}(x) - \alpha f_{X_{L(s-1)}}(x).$$

Thus,

$$f_{X_{L(s)}}(x) = \alpha\left[F_{X_{L(s)}}(x) - F_{X_{L(s-1)}}(x)\right]. \quad (19)$$

In view of equations (1) and (2), (19) gives

$$\frac{[H(x)]^{s-1}}{(s-1)!}f(x) = \alpha F(x)\frac{[H(x)]^{s-1}}{(s-1)!}$$

which reduces to

$$\frac{f(x)}{F(x)} = \alpha$$

implies

$$F(x) = e^{\alpha x}, \quad \alpha > 0, x < 0$$

and hence the Theorem.

Corollary 2.1: Let $X_{L(s)}$ be the s^{th} lower record statistic from a continuous population with the $pdf\, f(x)$ and the $df\, F(x)$, then for $1 \leq r < s$

$$X_{L(s)} \underline{\underline{d}}\, X_{L(s-r)} + U$$

where $U \underline{\underline{d}}\, X_{L(r)}$ is the r^{th} lower record statistic from reflected exponential (α) distribution and is independent of $X_{L(s)}$ if and only if X follow reflected exponential(α).

Theorem 2.2: Let $X_{s:n}$ be the s^{th} order statistic from a sample of size n drawn from a continuous population with the $pdf\, f(x)$ and the $df\, F(x)$, then for $1 \leq r < s \leq n$

$$X_{r:n} \underline{\underline{d}}\, X_{s:r} + U \quad (20)$$

where $U \underline{\underline{d}}\, X_{r:s-1}$ is the r^{th} order statistics from a sample of size $(s-1)$ drawn from reflected exponential (α) distribution and is independent of $X_{r:n}$ if and only if X follow reflected exponential(α).

Proof: For necessary part, let mgf of $X_{r:n}$ be $M_{X_{r:n}}(t)$, then

$$X_{r:n} \underline{\underline{d}}\, X_{s:r} + U$$

implies

$$M_U(t) = \frac{M_{X_{r:n}}(t)}{M_{X_{s:n}}(t)}. \quad (21)$$

The mgf of $X_{r:n}$, the r^{th} order statistic is given by

$$M_{X_{r:n}}(t) = \int_{-\infty}^{\infty} e^{tx} \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx.$$

For reflected exponential (α) distribution

$$M_{X_{r:n}}(t) = \alpha \int_{-\infty}^{0} e^{tx} \frac{n!}{(r-1)!(n-r)!}[e^{\alpha x}]^{r-1}[1-e^{\alpha x}]^{n-r}e^{\alpha x}dx. \tag{22}$$

Putting $y = e^{\alpha x}$ in equation (22), we have

$$M_{X_{r:n}}(t) = \frac{n!}{(r-1)!(n-r)!} \int_{0}^{1} y^{r+\frac{t}{\alpha}-1}(1-y)^{n-r}dy$$

which reduces to

$$M_{X_{r:n}}(t) = \frac{\Gamma(n+1)\Gamma(r+\frac{t}{\alpha})}{\Gamma(r)\Gamma(n+\frac{t}{\alpha}+1)}.$$

Therefore,

$$M_U(t) = \frac{\Gamma(s)\Gamma(r+\frac{t}{\alpha})}{\Gamma(r)\Gamma(s+\frac{t}{\alpha})} \tag{23}$$

which is the mgf of r^{th} order statistic from a sample of size $(s-1)$ drawn from reflected exponential (α) distribution.

For sufficient part, we have from convolution formula

$$\begin{aligned} f_{X_{r:n}}(x) &= \int_{0}^{x} f_{X_{s:n}}(u) f_U(x-u) \\ &= \frac{1}{B(r,s-r)} \int_{0}^{x} \left[e^{\alpha(x-u)}\right]^{r-1} \left[1-e^{\alpha(x-u)}\right]^{s-r-1} e^{\alpha(x-u)} f_{X_{s:n}}(u)du. \end{aligned} \tag{24}$$

Differentiating (24) with respect to x, we have

$$\begin{aligned} \frac{d}{dx} f_{X_{r:n}}(x) &= \frac{\alpha r}{B(r,s-r)} \int_{0}^{x} \left[e^{\alpha(x-u)}\right]^{r} \left[1-e^{\alpha(x-u)}\right]^{s-r-1} f_{X_{s:n}}(u)du \\ &- \frac{(s-r-1)}{B(r,s-r)} \int_{0}^{x} \left[e^{\alpha(x-u)}\right]^{r+1} \left[1-e^{\alpha(x-u)}\right]^{s-r-2} f_{X_{s:n}}(u)du \end{aligned}$$

which reduces to

$$\frac{d}{dx} f_{X_{r:n}}(x) = \alpha r \left[f_{X_{r:n}}(x) - f_{X_{r+1:n}}(x)\right].$$

Thus,

$$f_{X_{r:n}}(x) = \alpha r \left[F_{X_{r:n}}(x) - F_{X_{r+1:n}}(x)\right]. \tag{25}$$

From David and Nagaraja (2003), we have

$$\left[F_{X_{r:n}}(x) - F_{X_{r+1:n}}(x)\right] = \binom{n}{r}[F(x)]^{r}[1-F(x)]^{n-r}. \tag{26}$$

In view of equations (5), (25) and (26), we have

$$f(x) = \alpha F(x)$$

implying that

$$F(x) = e^{\alpha x}$$

and hence the proof.

Theorem 2.3: Let $X_{U(r)}$ be the r^{th} upper record statistic from a continuous population with the *pdf* $f(x)$ and the *df* $F(x)$, then for $1 \leq r < s$

$$X_{U(r)} \stackrel{d}{=} X_{U(s)} + U \tag{27}$$

where $U \stackrel{d}{=} Y_{X_{r:s-1}}$ is the r^{th} order statistic from a sample of size $(s-1)$ drawn from reflected exponential (α) distribution and is independent of $X_{U(r)}$ if and only if X follow extreme value $(0, \frac{1}{\alpha})$ distribution of type I.

Proof: For necessary part, let mgf of $X_{U(r)}$ be $M_{X_{U(r)}}(t)$, then

$$X_{U(r)} \stackrel{d}{=} X_{U(s)} + U$$

implies

$$M_U(t) = \frac{M_{X_{U(r)}}(t)}{M_{X_{U(s)}}(t)}.$$

The mgf of $X_{U(r)}$, the r^{th} upper record statistic is given by

$$M_{X_{U(r)}(t)} = \int_{-\infty}^{\infty} e^{tx} \frac{[R(x)]^{r-1}}{(r-1)!} f(x) dx.$$

For extreme value $(0, \frac{1}{\alpha})$ distribution, we get

$$M_{X_{U(r)}(t)} = \frac{1}{\Gamma(r)} \int_{-\infty}^{\infty} [e^{\alpha x}]^{r+t} e^{-e^{\alpha x}} dx. \tag{28}$$

Putting $y = e^{\alpha x}$ in (28), we have

$$M_{X_{U(r)}(t)} = \frac{1}{\Gamma(r)} \int_{0}^{\infty} y^{r+\frac{t}{\alpha}-1} e^{-y} dy$$

implies

$$M_{X_{U(r)}(t)} = \frac{\Gamma(r+\frac{t}{\alpha})}{\Gamma(r)}.$$

Therefore, the mgf of U is given by

$$M_U(t) = \frac{\Gamma(s)\Gamma(r+\frac{t}{\alpha})}{\Gamma(r)\Gamma(s+\frac{t}{\alpha})} \tag{29}$$

which is the mgf of $Y_{r:s-1}$, the r^{th} order statistic from a sample of size $(s-1)$ drawn from reflected exponential (α) distribution.

For sufficient part, the *pdf* of $X_{U(r)}$ is given by convolution formula

$$\begin{aligned} f_{X_{U(r)}}(x) &= \int_0^x f_{X_{U(s)}}(u) f_{Y_{r:s-1}}(x-u)du \\ &= \frac{1}{B(r,s-r)} \int_0^x \left[e^{\alpha(x-u)}\right]^r \left[1-e^{\alpha(x-u)}\right]^{s-r-1} f_{X_{U(s)}}(u)du. \end{aligned} \tag{30}$$

Differentiating (30) with respect to x , we get

$$\begin{aligned} \frac{d}{dx} f_{X_{U(r)}}(x) &= \frac{r}{B(r,s-r)} \int_0^x \left[e^{\alpha(x-u)}\right]^r \left[1-e^{\alpha(x-u)}\right]^{s-r-1} f_{X_{U(s)}}(u)du \\ &- \frac{r}{B(r+1,s-r-1)} \int_0^x \left[e^{\alpha(x-u)}\right]^{r+1} \left[1-e^{\alpha(x-u)}\right]^{s-r-2} f_{X_{U(s)}}(u)du \end{aligned}$$

which leads to

$$\frac{d}{dx} f_{X_{U(r)}}(x) = \alpha r [f_{X_{U(r)}}(x) - f_{X_{U(r+1)}}(x)].$$

Thus, for $1 \le r < s$

$$f_{X_{U(r)}}(x) = \alpha r [F_{X_{U(r)}}(x) - F_{X_{U(r+1)}}(x)]. \tag{31}$$

Therefore, in view of (3), (4) and (31), we have

$$F(x) = 1 - e^{-e^{\alpha x}}$$

and hence the proof.

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