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Limit Theorems for randomly weighted sums of random variables

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Abstract. Let (X_n) be a sequence of non-negative valued iid random variables with a common distribution function F, belonging to the domain of attraction of a positive stable law, and (W_n) be a sequence of bounded non-negative valued random variables. Assuming that the sequence (W_n) is iid and that (X_n) and (W_n) are independent, we obtain the limit distribution of $\left(T_n = \sum\limits_{j=1}^{n} W_j X_j\right)$, properly normalized. We extend the result to the sums of random number of random variables and obtain the limit law as geometric stable. Relaxing the condition on the weights and assuming (W_n) to be independent random variables, we obtain the limit distribution of (T_n), but under the setup that F belongs to the domain of normal attraction of a positive stable law. Also, for triangular array $(W_{n,1}, W_{n,2}, \ldots, W_{n,n})$ of weights, with non-negative bounded iid random variables as components, $n \geq 1$, we obtain the limit distribution of $(\sum\limits_{j=1}^{n} W_{n,j} X_j)$, when F belongs to the domain of normal attraction of a positive stable law.

1. Introduction

Let (X_n) be a sequence of identically distributed random variables (r.v.) with a common distribution function (d.f.) F. Assuming that $X_n, n \geq 1$, are mutually independent and that F belongs to the domain of attraction of a stable law with exponent $\alpha, 0 < \alpha < 2$, $\alpha \neq 1$, Beuerman (1975) established that the sequence $\left(\sum\limits_{k=1}^{n} f(\frac{k}{n}) X_k\right)$, properly normalized, also converges to a stable law with the same exponent, but with scale change, where $f(\cdot)$ is a non negative continuous function over $[0, 1]$. As a special case, he deduced the limit distribution of the sequence $\left(\sum\limits_{k=1}^{n} \frac{A_{n-k}^{(r)}}{A_n^{(r)}} X_k\right)$ of Cesaro sums, where $A_n^{(r)} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)\Gamma(r+1)}$, $n \geq 1$, $r \geq 1$, by taking $f(x) = \frac{A_{n-k}^{(r)}}{A_n^{(r)}}$ at $x = \frac{k}{n}, k = 1, 2 \ldots n$, and piece-wise continuous at other points. Wiber (2006) established central limit theorem for the sequence $\left(\sum\limits_{k=1}^{n} a_k X_k\right)$, when X_n's are square integrable and a_n's are real constants. Kim and Kim (2005) considered triangular arrays $\{a_{n,j}, 1 \leq j \leq n\}$ of real constants as weights and obtained central limit theorem for $(\sum\limits_{j=1}^{n} a_{n,j} X_j)$, when (X_n) is a dependent sequence.

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In the study of sums with random weights, let the weights be denoted by the sequence (W_n) of non-negative r.v.s and let the sequence of weighted sums be denoted by $\left(T_n = \sum_{j=1}^{n} W_j X_j\right)$. Tang and Tsitsiashvili (2003) have discussed the behaviour of (T_n) for large n, assuming that (X_n) is a sequence of subexponential r.v.s, (W_n) is a sequence of dependent bounded r.v.s with values over $[a, b]$ (where a and b are some positive constants) and that (X_n) and (W_n) are independent. Tang and Zhongyi (2014) have extended the results, by relaxing the condition on (W_n). To be precise, they have considered $W_n, n \geq 1$, taking values over $(0, b]$, for some constant b. They have also discussed extension to the whole of positive axis, but under some additional conditions. Some applications in the areas of portfolio management, risk investment, capital allocations etc. have been discussed. Gao and Wang (2010) and Hazra and Maulik(2012) have studied the tail probability of randomly weighted sums, when the underlying r.v.s have heavy tailed distributions. The same has been extended to the bivariate set up, by Li.J (2018). For similar work, one can see Hashorva et al.(2010). In the study of multiplicative cascades, Mendalbrot (1974) has discussed the behaviour of sums with random weights. Asimit et.al.(2017), discuss the asymptotic structure of the tail probabilities of the weighted sum of order statistics of dependent r.v.s, when the underlying d.f.s belong to the max-domain of attraction of Frechet law and Gumbel law and the weights are random. When the r.v.s $X_n, n \geq 1$, are negatively dependent Shen and Lin (2008) have discussed the large deviation results for (T_n) and Dong and Wang(2017) have studied the moderate deviation results. Rosalsky and Sreehari (1998) consider triangular arrays $(X_{n,1}, X_{n,2}, \ldots X_{n,n})$ of r.v.s $n \geq 1$, and $(W_{n,1}, W_{n,2}, \ldots W_{n,n})$ of weights, $n \geq 1$, and discuss the convergence in probability and almost sure convergence of the sequence $(\sum_{j=1}^{n} W_{n,j} X_{n,j})$. Our interest is in establishing the convergence in distribution of the sequence (T_n), assuming that the r.v. X_1 belongs to the domain of attraction of a positive stable law, which is closer in spirit to Beuerman (1975). We also investigate the limit distribution of the weighted sums of random number of random variables, on the lines of Klebanov et al. (1985).

Klebanov et al.(1985) have introduced the geometric stable laws associated with the sums of geometric number of i.i.d. random variables, as the solutions of the equation $\left(X = p^{1/\alpha} \sum_{j=1}^{N} X_j\right)$, where $X, X_1, X_2, \ldots X_N$ are i.i.d. random variables and N is a geometric r.v. with values over the set of positive integers, with parameter p, $o < p < 1$, under the additional assumption that (X_n) and N independent. It is interesting to note that, a non-negative geometric stable r.v. is given by the Laplace transform $(1 - s^{\alpha})^{-1}$, $0 < \alpha < 1$. Mohan et al.(1993) have obtained geometric stable laws as the limit laws of sums of random number of r.v.s. Set $\left(U_n = \sum_{j=1}^{N_n} X_j\right)$, where N_n is a geometric r.v. with values over the set of positive integers and with the parameter p_n asymptotically equal to 1/n. When F belongs to the domain of attraction of a strictly stable law with characteristic function $g(t)$, assuming that (X_n) and (N_n) are independent, they have established that (U_n), properly normalised, converges to a geometric stable law with the characteristic function $(1 + log\ g(t))^{-1}$.

In this paper, we establish limit theorems for sums with random weights, when the underlying sequence (X_n) of r.v.s is iid non-negative valued with a common d.f.F that belongs to the domain of attraction of a positive stable law. In the next section, we discuss the limiting behaviour of $\left(T_n = \sum_{j=1}^{n} W_j X_j\right)$, when (W_n) is a sequence of non-negative valued, bounded iid random variables. We also obtain the limit distribution of $\left(U_n = \sum_{j=1}^{N_n} W_j X_j\right)$, where $N_n, n \geq 1$, are mutually independent geometric r.v.s with values over the set of positive integers. We give an example to show that the above results need not necessarily hold when W_1 is unbounded. In Section 3, we obtain the limit distribution of (T_n), properly normalized, assuming that (W_n) is a sequence of independent non-negative valued bounded r.v.s, but under the more stringent set up that F belongs to the domain of normal attraction of a positive stable law. In the last section, we consider triangular arrays $\{W_{n,j}, j = 1, 2, \dots n\}$ of weights and obtain the limit distribution of the sequence $\left(R_n = \sum_{j=1}^{n} W_{n,j} X_j\right)$, assuming that $W_{n,1}, W_{n,2} \dots W_{n,n}$ are non-negative bounded iid r.v.s, $n \geq 1$, and that F belongs to the domain of normal attraction of a positive stable law.

Preliminaries. We now present some results on stable laws and domain of attraction, so that the reading of the paper is made easier. For a positive stable law with exponent α, $0 < \alpha < 1$, the Levy spectral function is given by $\nu(u) = -c_1 u^{-\alpha}, u > 0$, and the scale parameter c is given by $c = -c_1 M(\alpha) \cos \frac{\pi\alpha}{2}$, where

$$M(\alpha) = \int_0^{\infty} (e^{-v} - 1) \frac{dv}{v^{1+\alpha}}. \tag{1}$$

Let $S_n = \sum_{j=1}^{n} X_j$. If there exists a sequence (B_n) with $B_n \to \infty$ as $n \to \infty$, such that $(B_n^{-1} S_n)$ converges to a positive stable law with exponent $\alpha, 0 < \alpha < 1$, then the d.f. F is said to belong to the domain of attraction of a positive stable law with exponent α and the same is denoted by $F \in DA(\alpha)$. Further, the tail of F is regularly varying with $\overline{F}(x) = 1 - F(x) = \dfrac{L(x)}{x^{\alpha}}$, where $L(\cdot)$ is a slowly varying (s.v) function (for definition and properties of regularly varying functions, one can see for example, Feller(1966) or Embrechts et.al (1997)) . In particular, if $\overline{F}(x) \sim c_1 x^{-\alpha}$, as $x \to \infty$ ('$\sim$' means asymptotically equal), for some $c_1 > 0$, then F is said to belong to the domain of normal attraction of a positive stable law with exponent α and the same is denoted by $F \in DNA(\alpha)$. Here B_n takes the form $B_n = a n^{1/\alpha}$ for some $a > 0$. For more details on stable laws and domains of attraction see, Gnedenko and Kolmogorov (1954) and Feller (1966). Throughout the paper we assume that $c_1 = 1$, so that the scale parameter is $c = -M(\alpha) \cos \frac{\pi\alpha}{2}$.

2. Identically distributed weights

In this section, we obtain the limit distributions of (T_n) and (U_n), assuming that $F \in DA(\alpha)$ and that (W_n) is a sequence of non negative, bounded i.i.d. random variables. We also discuss through an example, that the results need not necessarily hold when W_1 is unbounded.

Theorem 2.1. *Suppose that there exists a sequence $(B_n, B_n \to \infty$ as $n \to \infty)$ of constants such that the sequence $(B_n^{-1} S_n)$ converges to a positive stable law with exponent α and scale c. Then $(B_n^{-1} T_n)$ converges weakly to a positive stable law with exponent α and scale $c\, EW_1^{\alpha}$.*

Proof. Let $G(\cdot)$ denote the d.f. of W_1. With no loss of generality, assume that W_1 takes values in $(0, 1)$. Define $Z_n = W_n X_n$, $n \geq 1$, so that $T_n = \sum_{j=1}^{n} W_j X_j = \sum_{j=1}^{n} Z_j$. Observe that $Z_1, Z_2 \dots$ is an iid sequence and denote by $J(\cdot)$ the d.f. of Z_1. For any $x > 0$, let $\overline{J}(x) = 1 - J(x)$. Note that

$$\overline{J}(x) = \int_0^1 \overline{F}\left(\frac{x}{y}\right) dG(y). \tag{2}$$

Then for any $k > 0$ and $x > 0$, we have

$$\begin{aligned} \frac{\overline{J}(kx)}{\overline{J}(x)} &= \frac{\int_0^1 \overline{F}\left(\frac{kx}{y}\right) dG(y)}{\int_0^1 \overline{F}\left(\frac{x}{y}\right) dG(y)} \\ &= \frac{\int_0^1 \frac{\overline{F}\left(\frac{kx}{y}\right)}{\overline{F}\left(\frac{x}{y}\right)} \overline{F}\left(\frac{x}{y}\right) dG(y)}{\int_0^1 \overline{F}\left(\frac{x}{y}\right) dG(y)}. \end{aligned}$$

Given that $F \in DA(\alpha)$, by Theorem 2, § 35, Gnedenko and Kolmogorov (1954) note that

$$\lim_{x\to\infty} \frac{\overline{F}(kx)}{\overline{F}(x)} = k^{-\alpha}.$$

Consequently, for any given $\epsilon_1 > 0$, one can find a $x_1 > 0$ such that for all $x \geq x_1$,

$$(1-\epsilon_1)k^{-\alpha} < \frac{\overline{F}(kx)}{\overline{F}(x)} < (1+\epsilon_1)k^{-\alpha}$$

Since $y \in (0,1)$, note that for any $x > 0$, $\dfrac{x}{y} > x$. In turn, whenever $x \geq x_1$,

$$(1-\epsilon_1)k^{-\alpha} < \frac{\overline{F}\left(\frac{kx}{y}\right)}{\overline{F}\left(\frac{x}{y}\right)} < (1+\epsilon_1)k^{-\alpha},$$

which implies that for $x \geq x_1$,

$$(1-\epsilon_1)k^{-\alpha} < \frac{\overline{J}(kx)}{\overline{J}(x)} < (1+\epsilon_1)k^{-\alpha}.$$

Taking limit as $x \to \infty$ and then as $\epsilon_1 \to 0$, we get

$$\lim_{x\to\infty} \frac{\overline{J}(kx)}{\overline{J}(x)} = k^{-\alpha}.$$

Appealing to Theorem 2, § 35, Gnedenko and Kolmogorov (1954), we observe that $J \in DA(\alpha)$.

Given that $(B_n^{-1}S_n)$ converges to a stable with exponent α and scale $c = -M(\alpha)\cos\frac{\pi\alpha}{2}$, for any $x > 0$, we note that $n\overline{F}(xB_n) \to x^{-\alpha}$ as $n \to \infty$. For $y \in (0,1)$ and $x > 0$, $\dfrac{x}{y} > x$ implies that $n\overline{F}\left(\frac{x}{y}B_n\right) \leq n\overline{F}(xB_n)$. Hence for any given $\epsilon > 0$ one can find a $n_2 > 0$ such that, $n\overline{F}\left(\frac{xB_n}{y}\right) \leq (x^{-\alpha}+\epsilon)$ for all $y \in (0,1)$ and for all $n \geq n_2$. By applying dominated convergence theorem, we see that

$$\begin{aligned}\lim_{n\to\infty} n\overline{J}(xB_n) &= \lim_{n\to\infty} \int_0^1 n\overline{F}\left(\frac{xB_n}{y}\right) dG(y) \\ &= \int_0^1 \lim_{n\to\infty} n\overline{F}\left(\frac{xB_n}{y}\right) dG(y) = \left(\frac{EW_1^{\alpha}}{x^{\alpha}}\right).\end{aligned} \tag{3}$$

We have hence established that $(B_n^{-1}T_n)$ converges to a stable law with exponent α and scale $c\,EW_1^{\alpha}$. The proof is complete. □

For the next theorem, we assume that (N_n) is a sequence of geometric r.v.s with support over the set of positive integers and with parameter p_n asymptotic to $\frac{1}{n}$. We further assume that (N_n) is independent of (X_n) and (W_n).

Theorem 2.2. *Let $(B_n^{-1}S_n)$ converge to a positive stable law with exponent α and scale c. Then,*
(A) the sequence $(B_n^{-1}S_{N_n})$ converges to a positive valued geometric stable law with exponent α and scale c
(B) the sequence $(B_n^{-1}U_n)$ converges to a positive valued geometric stable law with exponent α and scale $c\,EW_1^{\alpha}$.

Proof. Given that $(B_n^{-1}S_n)$ converges to a positive stable law with exponent α and scale c, from Mohan et al.(1993), part (A) of the theorem immediately follows. Putting $Z_j = W_jX_j,\ j = 1, 2, \ldots, n$, one can see that $\left(T_n = \sum\limits_{j=1}^{n} Z_j\right)$, where (Z_n) is a sequence of i.i.d. random variables. From the above theorem, note that $(B_n^{-1}T_n)$ converges to a stable law with exponent α and scale $c\ EW_1^{\alpha}$. Again appealing to Mohan et al.(1993), part (B) of the theorem follows. □

Remark 2.3. *In the above theorems, weights are assumed to be non negative, bounded r.v.s. On the other hand, if they are allowed to be unbounded, then the result need not necessarily hold, as demonstrated by the following example.*

Example 2.4. *Let (X_n) be iid Pareto with d.f. $\overline{F}(x) = x^{-\alpha}$ if $x \geq 1$, $= 1$ if $x < 1$ and (W_n) be iid Pareto with d.f. $\overline{G}(x) = x^{-\alpha_1}$ if $x \geq 1$, $= 1$ if $x < 1$. Suppose that $0 < \alpha, \alpha_1 < 1$ and that $\alpha_1 \neq \alpha$. Note that $F \in DA(\alpha)$. For any $x > 1$ we have,*

$$\overline{J}(x) = \int_0^{\infty} \overline{F}\left(\frac{x}{y}\right) dG(y) \qquad = \alpha_1 \int_1^{x} \frac{y^{\alpha}}{x^{\alpha}} \frac{dy}{y^{1+\alpha_1}} + \alpha_1 \int_x^{\infty} \frac{dy}{y^{1+\alpha_1}} \tag{4}$$

$$= \begin{cases} \dfrac{\alpha}{\alpha - \alpha_1} \dfrac{1}{x^{\alpha_1}} - \dfrac{\alpha_1}{\alpha - \alpha_1} \dfrac{1}{x^{\alpha}} & \text{if } \alpha_1 < \alpha \\ \dfrac{\alpha_1}{\alpha_1 - \alpha} \dfrac{1}{x^{\alpha}} - \dfrac{\alpha}{\alpha_1 - \alpha} \dfrac{1}{x^{\alpha_1}} & \text{if } \alpha_1 > \alpha. \end{cases} \tag{5}$$

Hence $\overline{J}(x) \sim \dfrac{\alpha}{\alpha - \alpha_1} \dfrac{1}{x^{\alpha_1}}$, as $x \to \infty$, when $\alpha_1 < \alpha$ and $\overline{J}(x) \sim \dfrac{\alpha_1}{\alpha_1 - \alpha} \dfrac{1}{x^{\alpha}}$, as $x \to \infty$, when $\alpha_1 > \alpha$. Consequently, $J(\cdot) \in DA(\alpha)$ when $\alpha_1 > \alpha$, and $J(\cdot) \in DA(\alpha_1)$ when $\alpha > \alpha_1$.

When $\alpha_1 > \alpha$, both $\left(n^{-1/\alpha}S_n\right)$ and $\left(n^{-1/\alpha}T_n\right)$ converge to stable laws with exponent α and respectively, with scales $c = -M(\alpha)\cos\frac{\pi\alpha}{2}$ and $c_1 = -\dfrac{\alpha_1}{\alpha_1 - \alpha} M(\alpha)\cos\frac{\pi\alpha}{2}$. When $\alpha_1 < \alpha$, $\left(n^{-1/\alpha}S_n\right)$ converges to a stable law with exponent α but, $\left(n^{-1/\alpha}T_n\right)$ fails to converge to any law. On the other hand $\left(n^{-1/\alpha_1}T_n\right)$ converges to a stable law with exponent α_1. Also, when $\alpha_1 > \alpha$, from Mohan et al.(1993), we note that both $(n^{-1/\alpha}S_{N_n})$ and $(n^{-1/\alpha}U_n)$ converge to geometric stable laws with exponent α and scales c and c_1 respectively.

3. Mutually independent weights.

In this section, we assume that $F \in DNA(\alpha)$, $0 < \alpha < 1$, and that (W_n) is a sequence of independent r.v.s with values over $(0,\ a)$, for some $a > 0$. We also assume that (X_n) and (W_n) are mutually independent. Recalling that $S_n = \sum\limits_{j=1}^{n} X_j$ and $T_n = \sum\limits_{j=1}^{n} W_jX_j,\ n \geq 1$, we establish the following.

Theorem 3.1. *Let $EW_n^{\alpha} \to \theta_{\alpha}$ as $n \to \infty$, for some $\theta_{\alpha} > 0$. Given that $(n^{-1/\alpha}S_n)$ converges to a positive stable law with exponent α, $0 < \alpha < 1$, and scale c, the sequence $(n^{-1/\alpha}T_n)$ converges to a positive stable law with the same exponent α but with scale $c\ \theta_{\alpha}$.*

Proof. Assume, with no loss of generality, that $a = 1$ so that $W_n,\ n \geq 1$, take values over $(0,\ 1)$. Denote by $G_n(\cdot)$, the d.f of W_n and by $H_n(\cdot)$, that of $W_nX_n,\ n \geq 1$. By Theorem 4, § 25, Gnedenko and Kolmogorov (1954), the theorem is established, once we show that

(i) for any $x > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} (1 - H_j(xn^{1/\alpha})) = \theta_{\alpha} x^{-\alpha} \tag{6}$$

(ii)

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} n^{-2/\alpha}\sum_{j=1}^{n}\left\{\int_0^{\epsilon n^{1/\alpha}} x^2 dH_j(x) - \left(\int_0^{\epsilon n^{1/\alpha}} x dH_j(x)\right)^2\right\} = 0, \tag{7}$$

and

(iii) for any $\tau > 0$,

$$\lim_{n\to\infty} n^{-1/\alpha}\sum_{j=1}^{n}\int_0^{\tau n^{1\alpha}} x dH_j(x) = r(\tau), \tag{8}$$

where $r(\tau)$ is a real valued function.

Define $\overline{F}(x) = 1 - F(x)$ and $\overline{H}_n(x) = 1 - H_n(x),\ x > 0,\ n \geq 1$. Then for any $x > 0$ we have the relation,

$$\overline{H}_n(x) = P(W_n X_n > x) = \int_0^1 \overline{F}(\tfrac{x}{y}) dG_n(y). \tag{9}$$

From the fact that $F \in DNA(\alpha)$, we know that $\overline{F}(x) \sim c_1 x^{-\alpha}$ as $x \to \infty$, for some $c_1 > 0$. Assume that $c_1 = 1$, with no loss of generality. For any given $\epsilon_1 > 0$, one can find a $x_1 > 0$ such that for all $x \geq x_1$,

$$(1-\epsilon_1)x^{-\alpha} < \overline{F}(x) < (1+\epsilon_1)x^{-\alpha}. \tag{10}$$

For any $x > 0$, choose an integer n_1 such that $x n_1^{1/\alpha} > x_1$. Then from (10), for all $n \geq n_1$,

$$(1-\epsilon_1)n^{-1}x^{-\alpha} < \overline{F}(xn^{1/\alpha}) < (1+\epsilon_1)n^{-1}x^{-\alpha}.$$

Since $y \in (0,\ 1)$, for any $x > 0$, $\dfrac{x}{y} > x$ and hence for all $n \geq n_1$,

$$(1-\epsilon_1)n^{-1}y^{\alpha}x^{-\alpha} \leq \overline{F}\left(\frac{xn^{1/\alpha}}{y}\right) \leq (1+\epsilon_1)n^{-1}y^{\alpha}x^{-\alpha}.$$

From (9), for all $n \geq n_1$ one gets,

$$(1-\epsilon_1)n^{-1}x^{-\alpha}EW_j^{\alpha} \leq \overline{H}_j(xn^{1/\alpha}) \leq (1+\epsilon_1)n^{-1}x^{-\alpha}EW_j^{\alpha},$$

and in turn,

$$\frac{(1-\epsilon_1)}{x^{\alpha}}\sum_{j=1}^{n}\frac{EW_j^{\alpha}}{n} \leq \sum_{j=1}^{n}\overline{H}_j(xn^{1/\alpha}) \leq \frac{(1+\epsilon_1)}{x^{\alpha}}\sum_{j=1}^{n}\frac{EW_j^{\alpha}}{n} \tag{11}$$

Given that $EW_n^{\alpha} \to \theta_{\alpha}$ as $n \to \infty$, we note that the Cesaro sequence, $\dfrac{1}{n}\sum\limits_{j=1}^{n} EW_j^{\alpha} \to \theta_{\alpha}$ as $n \to \infty$. In (11), taking limit as $n \to \infty$ and then as $\epsilon_1 \to 0$ one gets,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\Big(1 - H_j(xn^{1/\alpha})\Big) = \theta_{\alpha}x^{-\alpha}.$$

i.e. (6) is established. (7) is proved, once we show that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} n^{-2/\alpha}\sum_{j=1}^{n}\int_0^{\epsilon n^{1/\alpha}} x^2 dH_j(x) = 0. \tag{12}$$

Since $y \in (0,\, 1)$, for any given $\epsilon_1 > 0$, one can find a $x_1 > 0$ such that for any $x \geq x_1$,

$$\overline{F}\left(\frac{x}{y}\right) \leq (1+\epsilon_1) y^{\alpha} x^{-\alpha}$$

In turn, for all $x \geq x_1$ and $n \geq 1$, one can show that

$$\overline{H}_n(x) \leq (1+\epsilon_1) x^{-\alpha} E W_n^{\alpha} \tag{13}$$

For n large such that $\epsilon n^{1/\alpha} \geq x_1$, define $T_{1,n} = n^{-2/\alpha} \sum\limits_{j=1}^{n} \int\limits_{0}^{x_1} x^2 dH_j(x)$ and $T_{2,n} = n^{-2/\alpha} \sum\limits_{j=1}^{n} \int\limits_{x_1}^{\epsilon n^{1/\alpha}} x^2 dH_j(x)$. Note that (12) holds, once we show that $T_{k,n} \to 0$ as $n \to \infty$, $k = 1, 2$. Observe that

$$0 < T_{1,n} < n^{\frac{-2}{\alpha}} \sum_{j=1}^{n} x_1^2 H_j(x) \leq n^{(1-\frac{2}{\alpha})} x_1^2,$$

implies $T_{1,n} \to 0$ as $n \to \infty$.

By product formula,

$$\begin{aligned}
\int\limits_{x_1}^{\epsilon n^{\frac{1}{\alpha}}} x^2 dH_j(x) &= -\int\limits_{x_1}^{\epsilon n^{\frac{1}{\alpha}}} x^2 d\overline{H}_j(x) \\
&= \int\limits_{x_1}^{\epsilon n^{\frac{1}{\alpha}}} \overline{H}_j(x) dx^2 - \epsilon^2 n^{\frac{2}{\alpha}} \overline{H}_j(\epsilon n^{\frac{1}{\alpha}}) + x_1^2 \overline{H}_j(x_1).
\end{aligned}$$

Consequently,

$$\begin{aligned}
T_{2,n} &= n^{\frac{-2}{\alpha}} \sum_{j=1}^{n} \int_{x_1}^{\epsilon n^{\frac{1}{\alpha}}} \overline{H}_j(x) 2x dx - \epsilon^2 \sum_{j=1}^{n} \overline{H}_j(\epsilon n^{\frac{1}{\alpha}}) + n^{\frac{-2}{\alpha}} x_1^2 \sum_{j=1}^{n} \overline{H}_j(x_1) \\
&= V_{1,n} - V_{2,n} + V_{3,n},\ say
\end{aligned} \tag{14}$$

From (13), we have

$$\begin{aligned}
V_{1,n} &\leq 2(1+\epsilon_1) n^{-2/\alpha} \sum_{j=1}^{n} E W_j^{\alpha} \int_{x_0}^{\epsilon n^{1/\alpha}} x^{1-\alpha} dx \\
&\leq 2(1+\epsilon_1) n^{-2/\alpha} \sum_{j=1}^{n} E W_j^{\alpha} \frac{\epsilon^{(2-\alpha)} n^{\frac{2-\alpha}{\alpha}}}{(2-\alpha)} \\
&= \frac{2(1+\epsilon_1)\epsilon^{2-\alpha}}{(2-\alpha)} \sum_{j=1}^{n} \frac{E W_j^{\alpha}}{n}.
\end{aligned}$$

Given that $E W_n^{\alpha} \to \theta_\alpha$ as $n \to \infty$, we note that the Cesaro sequence, $\sum\limits_{j=1}^{n} \frac{E W_j^{\alpha}}{n} \to \theta_\alpha$ as $n \to \infty$. Consequently,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} V_{1,n} = 0. \tag{15}$$

For $\epsilon n^{1/\alpha} > x_1$, from (13) we have

$$V_{2,n} \leq \epsilon^{2-\alpha}(1+\epsilon_1) \sum_{j=1}^{n} \frac{E W_j^{\alpha}}{n}.$$

Again from the fact that $\sum\limits_{j=1}^{n} \frac{EW_j^\alpha}{n} \to \theta_\alpha$ as $n \to \infty$, we get

$$\lim_{\epsilon \to 0} \limsup_{n\to\infty} V_{2,n} = 0. \tag{16}$$

Also, $V_{3,n} \le n^{(1-2/\alpha)} x_1^2$ implies that

$$\lim_{n\to\infty} V_{3,n} = 0. \tag{17}$$

From (14) to (17), we note that $\lim\limits_{\epsilon\to 0} \lim\limits_{n\to\infty} T_{2,n} = 0$, which in turn implies (12). Hence (ii) is established. We now proceed to prove (iii). By applying product formula again, for $\tau n^{1/\alpha} > x_1$ one gets,

$$\begin{aligned}
\int_0^{\tau n^{\frac{1}{\alpha}}} x dH_j(x) &= -\int_0^{\tau n^{\frac{1}{\alpha}}} x d\overline{H}_j(x) \\
&= \int_0^{\tau n^{\frac{1}{\alpha}}} \overline{H}_j(x)dx - \tau n^{\frac{1}{\alpha}} \overline{H}_j(\tau n^{\frac{1}{\alpha}}) \\
&= \int_0^{x_1} \overline{H}_j(x)dx + \int_{x_1}^{\tau n^{\frac{1}{\alpha}}} \overline{H}_j(x)dx - \tau n^{\frac{1}{\alpha}} \overline{H}_j(\tau n^{\frac{1}{\alpha}}).
\end{aligned} \tag{18}$$

Hence for $\tau n^{\frac{1}{\alpha}} > x_1$,

$$\begin{aligned}
n^{\frac{-1}{\alpha}} \sum_{j=1}^{n} \int_0^{\tau n^{\frac{1}{\alpha}}} x dH_j(x) &= n^{\frac{-1}{\alpha}} \sum_{j=1}^{n} \int_0^{x_1} \overline{H}_j(x)dx + n^{\frac{-1}{\alpha}} \sum_{j=1}^{n} \int_{x_1}^{\tau n^{\frac{1}{\alpha}}} \overline{H}_j(x) \\
&\quad -\tau n^{\frac{-1}{\alpha}} \sum_{j=1}^{n} \overline{H}_j(\tau n^{\frac{1}{\alpha}})
\end{aligned} \tag{19}$$

It can be trivially seen that

$$\lim_{n\to\infty} n^{-1/\alpha} \sum_{j=1}^{n} \int_0^{x_1} \overline{H}_j(x)dx \le \lim_{n\to\infty} x_1 n^{-(\frac{1}{\alpha}-1)} = 0 \tag{20}$$

Since $y \in (0,1)$, for all $x \ge x_1$, from (10) we get,

$$(1-\epsilon_1) y^\alpha x^{-\alpha} \le \overline{F}(\tfrac{x}{y}) \le (1+\epsilon_1) y^\alpha x^{-\alpha},$$

which implies that for $x > x_1$ and $n \ge 1$,

$$(1-\epsilon_1) x^{-\alpha} EW_n^\alpha \le \overline{H}_n(x) \le (1+\epsilon_1) x^{-\alpha} EW_n^\alpha. \tag{21}$$

In (21), since ϵ_1 is arbitrary,taking limit as $n \to \infty$ and then as $\epsilon_1 \to 0$, one can show that,

$$\lim_{n\to\infty} n^{-1/\alpha} \sum_{j=1}^{n} \int_{x_1}^{\tau n^{1/\alpha}} \overline{H}_j(x)dx = \frac{\tau^{(1-\alpha)}\theta_\alpha}{(1-\alpha)} \tag{22}$$

and

$$\lim_{n\to\infty} \tau n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \overline{H}_j(\tau n^{1/\alpha}) = \tau^{(1-\alpha)} \theta_\alpha. \tag{23}$$

From (20), (22) and (23) we get

$$\lim_{n\to\infty} n^{-1/\alpha}\sum_{j=1}^{n}\int_{0}^{\tau n^{1/\alpha}} x dH_j(x) = \frac{\alpha\tau^{(1-\alpha)}\theta_\alpha}{(1-\alpha)}.$$

Hence (iii) is established with $r(\tau) = \dfrac{\alpha\tau^{(1-\alpha)}\theta_\alpha}{(1-\alpha)}$.

From (i), (ii) and (iii) we note that $(n^{-1/\alpha}T_n)$ converges weakly to a positive stable law with exponent α and scale $c\,\theta_\alpha$. □

Remark 3.2. *From the proof, one may notice that for the theorem to hold, it is enough to assume that the Cesaro sequence,* $\left(\dfrac{EW_1^\alpha + EW_2^\alpha + \ldots + EW_n^\alpha}{n}\right)$ *converges to* θ_α*, instead of the stronger condition,* $EW_n^\alpha \to \theta_\alpha$ *as* $n\to\infty$*. Suppose that* $P\left(W_n = \frac{1}{4}\right) = P\left(W_n = \frac{1}{2}\right) = \frac{1}{2}$ *for n odd and* $P\left(W_n = \frac{1}{3}\right) = P\left(W_n = \frac{1}{2}\right) = \frac{1}{2}$ *for n even. Then* $EW_n^\alpha = \frac{1}{2}\left(\frac{1}{4^\alpha}+\frac{1}{2^\alpha}\right)$ *for n odd and* $= \frac{1}{2}\left(\frac{1}{3^\alpha}+\frac{1}{2^\alpha}\right)$ *for n even. One can see that* (EW_n^α) *fails to converge but*

$$\left(\frac{EW_1^\alpha + EW_2^\alpha + \ldots + EW_n^\alpha}{n}\right) \to \frac{1}{4}\left(\frac{2}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}\right) = \theta_\alpha, \tag{24}$$

say. Then $\left(n^{-1/\alpha}T_n\right)$ *converges to a stable law with exponent* α *and scale* $c\,\theta_\alpha$*.*

Taking $P(W_n = a_n) = 1$*, where* a_n *is a positive constant,* $n \geq 1$*, we have the following corollary.*

Corollary 3.3. *Let* $\left(n^{-1/\alpha}S_n\right)$ *converge to a stable law with exponent* α *and scale c and let* (a_n) *be the sequence of positive constants such that* $\dfrac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n} \to \theta_\alpha(>0)$ *as* $n\to\infty$*. Then* $\left(n^{-1/\alpha}\sum\limits_{j=1}^{n} a_j X_j\right)$ *converges to a stable law with exponent* α *and scale* $c\,\theta_\alpha$*.*

4. Triangular arrays of weights

Let $W_{n,1}, W_{n,2}, \ldots, W_{n,n}$ be iid nonnegative, bounded r.v's with common d.f. $G_n(\cdot)$, $n \geq 1$ and let $F \in DNA(\alpha)$, $0<\alpha<1$. Define $R_n = \sum\limits_{j=1}^{n} W_{n,j}X_j$, $n \geq 1$. Assuming that $\{W_{n,1}, W_{n,2}, \ldots, W_{n,n}\}$ and $\{X_n\}$ are independent, we obtain the limit distribution of $(n^{-1/\alpha}R_n)$.

Theorem 4.1. *Suppose that* $(n^{-1/\alpha}S_n)$ *converges to a positive stable law with exponent* α *and scale c and that* $\lim\limits_{n\to\infty} EW_{n,1}^\alpha = \theta_\alpha(>0)$*. Then* $(n^{-1/\alpha}R_n)$ *converges to a positive stable law with exponent* α *and scale* $c\,\theta_\alpha$*.*

Proof. Define $Z_{n,j} = W_{n,j}X_j$, $j = 1, 2, \ldots, n$, $n \geq 1$, and observe that $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ are iid. With no loss of generality, assume that the r.v's $W_{n,j}$, $j = 1, 2, \ldots, n$, $n \geq 1$, take values in $(0,1)$. Denote by $G_n(\cdot)$, the d.f. of $W_{n,1}$ and by $K_n(\cdot)$ that of $Z_{n,1}$, $n \geq 1$. Note that for any $x > 0$,

$$\overline{K}_n(x) = \int_0^1 \overline{F}\left(\frac{x}{y}\right) dG_n(y),$$

where $\overline{K}_n(x) = 1 - K_n(x)$.
In order to establish the theorem, we appeal to Theorem 4, § 25, Gnedenko and Kolmogorov (1954). From the fact that $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ are iid, we need show that for any $x > 0$,

$$\lim_{n\to\infty} n\left(1 - K_n(xn^{1/\alpha})\right) = \theta_\alpha x^{-\alpha},$$

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} n^{(1-\frac{2}{\alpha})}\left\{\int_0^{\epsilon n^{1/\alpha}} x^2 dK_n(x) - \left(\int_0^{\epsilon n^{1/\alpha}} x dK_n(x)\right)^2\right\} = 0$$

and that for any $\tau > 0$,

$$\lim_{n\to\infty} n^{(1-\frac{1}{\alpha})}\int_0^{\tau n^{1/\alpha}} x dK_n(x) = r(\tau),$$

where $r(\tau)$ is a real valued function of τ.

The above conditions can be established by proceeding as in the proof of Theorem 3.1, we omit the details. □

Remark 4.2. *One can see that the d.f.s belonging to the domain of attraction of positive stable law also belong to the max-domain of attraction of Frechet law. As such, the results of Asimit et al.(2017) hold under our set up. In other wards, the tail probability of* $\sum_{j=1}^{n} W_j X_{j,n}$ *can be obtained, where* $X_{1,n} \geq X_{2,n} \cdots \geq X_{n,n}$ *is the order statistics of* $X_1, X_2, \ldots X_n$*. It can be trivially seen that* $\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} X_{j,n}$*. However,* $\sum_{j=1}^{n} W_j X_j$ *and* $\sum_{j=1}^{n} W_j X_{j,n}$ *are not same. It will be of interest to investigate the limiting distribution of* $\sum_{j=1}^{n} W_j X_{j,n}$*, if it exist, in the set up of Asimit et al.(2017).*

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