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Asymptotic normality of a recursive estimator of a conditional hazard function with functional stationary ergodic data

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Abstract. In this paper, we investigate a recursive kernel estimator of the conditional hazard function whenever functional stationary ergodic data are considered. Under the assumption of ergodicity, the novelty of our approach is that we do not require independence of the observations. It is shown that, under some wild conditions, the recursive kernel estimate of the three parameters (conditional density, conditional distribution and conditional hazard) are asymptotically normally distributed.

1. Introduction

Recently there has been an increasing interest in the study of functional data. For an overview of the present state on nonparametric functional data (FDA), we refer to the works of (4) and (13), and the references therein.

Conditional hazard estimation with a functional explanatory variable and a scalar response acquired considerable interest in the statistical literature. The first work was proposed by Ferraty et al. (5), where they introduce a kernel estimator and prove some asymptotic properties (with rates) in various situations including censored and/ or dependent variables. Quintela-del-Río (12) extended the results of Ferraty et al. (5) by calculating the bias and variance of these estimates, and establishing their asymptotic normality. In the case of completely observed data, another estimators have been proposed for the conditional hazard function by different approaches. In 2014, Attouch and Belabed (2) have studied the nonparametric estimator of the conditional hazard function using the k Nearest Neighbors (k-NN) estimation method and they have shown its asymptotic properties in the case of independent data. Another approach has been proposed by Massim and Mechab (11) based on the local linear method, they have established the almost complete convergence of the proposed estimator.

In the case of the functional spatial data, the works on the conditional hazard function is limited and we can refer to (9) where they studied the almost complete convergence of the kernel type estimate. These authors studied in (10) the mean squared convergence rate and proved the asymptotic normality of the

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proposed estimator.

In recent years, the statistical modeling for functional ergodic data has been an increasing interest and a great importance in various fields. The general framework of ergodic functional data has been initiated by Laïb and Louani (7),(8) who stated consistencies with rates together with the asymptotic normality of the regression function estimate. In this topic of ergodic data, several works have been published, for example, Gheriballah et al. (6) showed the almost complete convergence (with rate) of a family of robust nonparametric estimators for regression function. More recently, Ardjoun et al. (1) treated the almost complete convergence and the asymptotic normality of the estimator of conditional mode, Benziadi et al. (3) studied the almost complete rate convergence of functional recursive kernel of the conditional quantiles.

This paper is organized as follows: Section 2 introduces the estimator of the conditional hazard function. In section 3 we will define some notations and hypothesis. The asymptotic normality of the proposed estimator of the conditional hazard function is given in Section 4. Finally, the proofs of our results are given in the Appendix.

2. The recursive estimation of the conditional hazard function

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be a sequence of strictly stationary ergodic processes. Where X_i are values in semi-metric space $(\mathcal{F}, d)$ and Y_i are real-valued random variables. N_x will denote a fixed neighborhood of x. We assume that the regular version of the conditional probability of Y given X exists. Moreover, we suppose that, for all $x \in N_x$ the conditional distribution function of Y given $X = x$, the recursive kernel estimator of the conditional hazard function $h(y|x)$ such that

$$h(y|x) = \frac{f(y|x)}{1 - F(y|x)}, \text{ for } y \in \mathbb{R} \text{ and } F(y|x) < 1$$

is

$$\widehat{h}(y|x) = \frac{\widehat{f}(y|x)}{1 - \widehat{F}(y|x)}, \quad \forall y \in \mathbb{R}.$$

We define the recursive kernel estimator of the conditional distribution function by

$$\widehat{F}(y|x) = \frac{\sum_{i=1}^{n} K(a_i^{-1} d(x, X_i)) H(b_i^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(a_i^{-1} d(x, X_i))} \tag{1}$$

where K is the kernel, H is a strictly increasing distribution function and $a_i,\ b_i$ are a sequences of positive real numbers such that $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = 0$.
The recursive kernel estimator $\widehat{f}(.|x)$ of $f(.|x)$ is given by

$$\widehat{f}(y|x) = \frac{\sum_{i=1}^{n} b_i^{-1} K(a_i^{-1} d(x, X_i)) H'(b_i^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(a_i^{-1} d(x, X_i))}. \tag{2}$$

3. Notations and hypothesis

The assumptions that we will need in our study are the following:
The functional ergodic data is carried out by the following consideration: for $i = 1, ..., n$, we put $\mathcal{F}_k$ is the σ-

algebra generated by $((X_1,Y_1),...,(X_k,Y_k))$. We pose $\mathfrak{B}_k$ is the σ-algebra generated by $((X_1,Y_1),...,(X_k,Y_k),X_{k+1})$. We suppose that the strictly stationary ergodic process $(X_i,Y_i)_{i\in\mathbb{N}}$ satisfies

(H1) (i) The function $\phi(x,h) := \mathbb{P}(X \in B(x,h)) > 0, \forall h > 0$, where $B(x,h) := \{x' \in \mathcal{F}/d(x',x) < h\}$.
(ii) For all $i = 1,...,n$ there exists a deterministic function $\phi_i(x,.)$ such that almost surely $0 < \mathbb{P}(X_i \in B(x,h)|\mathcal{F}_{i-1}) \le \phi_i(x,h), \forall h > 0$ and $\phi_i(x,h) \to 0$ as $h \to 0$.
(iii) For all sequence $(h_i)_{i=1,\dots,n} > 0$, $\dfrac{\sum_{i=1}^{n} \mathbb{P}(X_i \in B(x,h_i)|\mathcal{F}_{i-1})}{\sum_{i=1}^{n} \phi(x,h_i)} \to 1.$

(H2) (i) Let $\mathcal{S}$ be a compact set of $\mathbb{R}$, the conditional distribution function $F(.|x)$ is such that, $\forall y \in \mathcal{S}$, $\exists \beta > 0$, $\inf_{y\in\mathcal{S}}(1 - F(y|x)) > \beta$, $\forall$ $(y_1,y_2)\in \mathcal{S}\times\mathcal{S}$, $\forall$ $(x_1,x_2)\in N_x \times N_x$

$$|F(y_1|x_1) - F(y_2|x_2)| \le C_1(d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2})$$

with $C_1 > 0, \beta_1 > 0, \beta_2 > 0$.

(ii) The density $f(.|x)$ is such that,$\forall y \in \mathcal{S}$, $\exists \alpha > 0$ $f(y|x) < \alpha$, $\forall (y_1,y_2)\in \mathcal{S}\times\mathcal{S}$, $\forall$ $(x_1,x_2)\in N_x \times N_x$

$$|f(y_1|x_1) - f(y_2|x_2)| \le C_2(d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2})$$

with $C_2 > 0, \beta_1 > 0, \beta_2 > 0$.

(H3) $\forall (y_1,y_2) \in \mathbb{R}^2$ $|H^{(j)}(y_1) - H^{(j)}(y_2)| \le C|y_1 - y_2|$, for $j = 0,1$
$\int |t|^{\beta_2} H^{(1)}(t)dt < \infty$ and $\int H'^{2}(t)dt < \infty$.

(H4) K is a function with support (0,1) such that $0 < C_1 < K(t) < C_2 < \infty$.

(H5) The bandwidths (a_i,b_i) satisfied : $\forall t \in [0,1]$

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} \phi(x,ta_i)}{\sum_{i=1}^{n} \phi(x,a_i)} = \beta_x(t)$$

and

$$\lim_{n\to+\infty} \frac{\sqrt{n\phi_n(x)}}{\varphi_n(x)} \left(\sum_{i=1}^{n} a_i^{\beta_1} \phi(x,a_i) + \sum_{i=1}^{n} b_i^{\beta_2} \phi(x,a_i) \right) = 0$$

where $\varphi_n(x) = n^{-1}\sum_{i=1}^{n} \phi(x,a_i)$.

Comments on hypotheses:

The condition (H1) involves the ergodic nature of the data and the small ball techniques used in this paper. The hypothesis (H1)(iii) is a direct consequence of Beck's theorem. The assumption (H2) presents the Lipschitz's condition to the conditional distribution function and conditional density function, it means that the both functions are continuous with respect to each variable and permits us to evaluate the bias term without using the differentiability. Hypothesis (H3) impose some regularity conditions upon the distribution function H used in our estimates. The condition (H4) is very standard in nonparametric function estimation. The assumptions (H5) is a technical condition.

4. Main result: Asymptotic normality

Theorem 4.1. *Under hypotheses (H1)-(H5), we have for all $x \in \mathcal{A}$*

$$\left(\frac{n\varphi_n(x)}{\sigma_h^2(x,y)}\right)^{1/2}(\widehat{h}(y|x) - h(y|x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n \to \infty \tag{3}$$

where $\mathcal{A} = \{x, \quad \sigma_h^2(x,y) \neq 0\}$ and $\sigma_h^2(x,y) = \dfrac{\alpha_2 h(y|x)}{\alpha_1^2(1-F(y|x))}$
with $\alpha_1 = K(1) - \int_0^1 K'(s)\beta_x(s)ds$ and $\alpha_2 = K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds$.

The proof of this theorem is based on the following decomposition and lemmas below:

$$\widehat{h}(y|x) - h(y|x) = \frac{1}{1-\widehat{F}(y|x)}\left[\widehat{f}(y|x) - f(y|x)\right] + \frac{h(y|x)}{1-\widehat{F}(y|x)}\left[\widehat{F}(y|x) - F(y|x)\right]. \tag{4}$$

Lemma 4.2. *Under hypothesis of the theorem 4.1, we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_F^2(x,y)}\right)^{1/2}(\widehat{F}(y|x) - F(y|x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n \to \infty \tag{5}$$

where $\sigma_F^2(x,y) = \dfrac{\alpha_2 F(y|x)\big(1-F(y|x)\big)}{\alpha_1^2}$.

Lemma 4.3. *Under hypothesis of the theorem 4.1, we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_f^2(x,y)}\right)^{1/2}(\widehat{f}(y|x) - f(y|x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n \to \infty \tag{6}$$

where $\sigma_f^2(x,y) = \dfrac{\alpha_2 f(y|x)}{\alpha_1^2}$.

The proof of lemma 4.2 is based on the following decomposition where we put, for any $x \in \mathcal{F}$, and $i = 1, ..., n; K_i = K(a_i^{-1}d(x, X_i))$ and $H_i = H(b_i^{-1}(Y_i - y))$.

We start by writing

$$\widehat{F}(y|x) - F(y|x) = \widehat{B}_{n,1}(x,y) + \frac{\widehat{R}_{n,1}(x,y)}{\widehat{F}_D(x)} + \frac{\widehat{Q}_{n,1}(x,y)}{\widehat{F}_D(x)} \tag{7}$$

where

$$\begin{aligned}
\widehat{Q}_{n,1}(x,y) &= (\widehat{F}_N(y|x) - \overline{F}_N(y|x)) - F(y|x)(\widehat{F}_D(x) - \overline{F}_D(x)),\\
\widehat{B}_{n,1}(x,y) &= \frac{\overline{F}_N(y|x)}{\overline{F}_D(x)} \text{ and } \widehat{R}_{n,1}(x,y) = -\widehat{B}_{n,1}(x,y)(\widehat{F}_D(x) - \overline{F}_D(x))
\end{aligned}$$

with

$$\begin{aligned}
\widehat{F}_N(y|x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n} K(a_i^{-1}d(x,X_i))H(b_i^{-1}(Y_i-y_i)),\\
\overline{F}_N(y|x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n} \mathbb{E}[K(a_i^{-1}d(x,X_i))H(b_i^{-1}(Y_i-y_i)|\mathcal{F}_{i-1}),\\
\widehat{F}_D(x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n} K(a_i^{-1}d(x,X_i)),\\
\overline{F}_D(x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n} \mathbb{E}[K(a_i^{-1}d(x,X_i))|\mathcal{F}_{i-1}].
\end{aligned}$$

The proof of the lemma 4.2 is a consequence of the following lemmas, whose proofs are given in the Appendix

Lemma 4.4. *Under the hypothesis of theorem 4.1*

$$\left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{1/2}\widehat{Q}_{n,1}(x,y)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1)\ \text{ as } n\to\infty.$$

Lemma 4.5. *Under the hypothesis (H1) and (H5), we have*

$$\widehat{F}_D(y|x)-1=o_p(1).$$

Lemma 4.6. *Under the hypothesis (H1), (H2) and (H3) we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{1/2}\widehat{B}_{n,1}(x,y)=o_p(1) \text{ as } n\to\infty.$$

Lemma 4.7. *Under the hypothesis (H1), (H2) and (H5) we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{1/2}\widehat{R}_{n,1}(x,y)=o_p(1) \text{ as } n\to\infty.$$

The proof is based on the following decomposition and lemmas below.

$$\widehat{f}(y|x)-f(y|x)=\widehat{B}_{n,2}(x,y)+\frac{\widehat{R}_{n,2}(x,y)}{\widehat{F}_D(x)}+\frac{\widehat{Q}_{n,2}(x,y)}{\widehat{F}_D(x)} \tag{8}$$

where

$$\begin{aligned}
\widehat{Q}_{n,2}(x,y) &= (\widehat{f}_N(y|x)-\overline{f}_N(y|x))-f(y|x)(\widehat{F}_D(x)-\overline{F}_D(x)),\\
\widehat{B}_{n,2}(x,y) &= \frac{\overline{f}_N(y|x)}{\overline{F}_D(x)}-f(y|x) \ \text{ and } \ \widehat{R}_{n,2}(x,y)=-\widehat{B}_{n,2}(x,y)\left(\widehat{f}_N(y|x)-\overline{f}_N(y|x)\right)
\end{aligned}$$

with

$$\begin{aligned}
\widehat{f}_N(y|x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n}\frac{b_i^{-1}K(a_i^{-1}d(x,X_i))H'(b_i^{-1}(Y_i-y_i))}{\mathbb{E}\left(K(a_i^{-1}d(x,X_i))\right)},\\
\overline{f}_N(y|x) &= \frac{1}{n\varphi_n}\sum_{i=1}^{n}\frac{\mathbb{E}[b_i^{-1}K(a_i^{-1}d(x,X_i))H'(b_i^{-1}(Y_i-y_i))|\mathcal{F}_{i-1}]}{\mathbb{E}\left(K(a_i^{-1}d(x,X_i))\right)}.
\end{aligned}$$

Lemma 4.8. *Under the hypothesis of theorem 4.1*

$$\left(\frac{n\varphi_n(x)}{\sigma_f^2}\right)^{1/2} \widehat{Q}_{n,2}(x,y) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \ \text{as } n \to \infty.$$

Lemma 4.9. *Under the hypothesis (H1), (H2) and (H3) we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_f^2}\right)^{1/2} \widehat{B}_{n,2}(x,y) = o_p(1) \text{ as } n \to \infty.$$

Lemma 4.10. *Under the hypothesis (H1), (H2) and (H5) we have*

$$\left(\frac{n\varphi_n(x)}{\sigma_f^2}\right)^{1/2} \widehat{R}_{n,2}(x,y) = o_p(1) \text{ as } n \to \infty.$$

5. Confidence bands

A usual application of asymptotic normality is to establish confidence bands for the estimates. Our goal in this section is the application of our asymptotic normality result (Theorem 4.1) to build the confidence intervals for the true value of $h(y|x)$ for a given curve $X = x$. In nonparametric estimation, the asymptotic variance depends on certain unknown functions. In our case, we have

$$\sigma_h^2(x,y) = \frac{\alpha_2 h(y|x)}{\alpha_1^2(1 - F(y|x))}$$

where $h(y|x), F(y|x), \alpha_1$ and α_2 are unknown a priori and have to be estimated in practice. Then one can obtain a confidence bands even if $\sigma_h^2(x,y)$, is functionally specified. Now a plug-in estimate for the asymptotic standard deviation $\sigma_h^2(x,y)$, can be easily obtained using the estimators $\widehat{h}(y|x), \widehat{F}(y|x), \widehat{\alpha}_1$ and $\widehat{\alpha}_2$ of $h(y|x), F(y|x), \alpha_1$ and α_2 respectively, that is

$$\widehat{\sigma}_h^2(x,y) = \frac{\widehat{\alpha}_2 \widehat{h}(y|x)}{\widehat{\alpha}_1^2(1 - \widehat{F}(y|x))}.$$

We estimate empirically the constants α_1 and α_2, as follows:

$$\widehat{\alpha}_1 = \frac{1}{n\phi(x,a_1)} \sum_{i=1}^{n} K(a_1^{-1} d(x,X_i)), \quad \widehat{\alpha}_2 = \frac{1}{n\phi(x,a_1)} \sum_{i=1}^{n} K^2(a_1^{-1} d(x,X_i)).$$

Remark that in the special case when $K = \mathbb{I}_{[0,1]}$, it becomes immediately that $\alpha_1 = \alpha_2 = 1$.
Now the asymptotic confidence band at asymptotic level $1 - \zeta$ for $h(y|x)$ is given by

$$\left[\widehat{h}(y|x) - u_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}_h^2(x,y)}{n\varphi_n(x)}\right)^{1/2}, \ \widehat{h}(y|x) + u_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}_h^2(x,y)}{n\varphi_n(x)}\right)^{1/2}\right]$$

where $u_{1-\frac{\zeta}{2}}$ denotes the $1 - \frac{\zeta}{2}$ quantile of the standard normal distribution.

6. Concluding remarks

This article provides a theoretical framework about recursive conditional hazard function estimator with functional stationary ergodic data. The resulting recursive conditional hazard function estimator has been shown to be consistent and asymptotically normally distributed under appropriate conditions. To prove the results, the methodology is based upon the martingale approximation used in (7). The ergodic hypothesis used in the paper avoids complicated probabilistic calculations of the mixing condition.

The hypothesis (H1) plays an important role, which involve the recursive estimate, the ergodic nature of the data and the small ball techniques. It's clear that this hypothesis is quite milder than Laib and Louani. Indeed, unlike in Laib and Louani (7), it is not necessary to write (approximately) the concentration function $\mathbb{P}(X_i \in B(x,h))$ and the conditional concentration function $\mathbb{P}(X_i \in B(x,h)|\mathcal{F}_{i-1})$ a product of two independent nonnegative functions of the center and of the radius.

The recursive estimate is very fast in practice because the smoothing parameter is linked to the observation (X_i, Y_i), which permits to update our estimator for each additional observation.

It should be noted that our work is a generalization of the results obtained by other authors in two different directions (the estimation method and the data correlation). Indeed, the classical kernel method used by Ferraty et al. (5) and Quintela-del-Río (12) can be considered as a special case of this study by considered that $a_i = h_K$ and $b_i = h_H$. On the other hand, in the independent case, condition (H1(ii)) is automatically verified and for all $i = 1, ..., n$ take $\phi_i(x,.) = \phi(x,.)$ and we obtain the same result given by Quintela-del-Río (12) when (X_i, Y_i) are independent.

7. Appendix

Proof of Lemma 4.4.
We use the same ideas in Laib and Louani (2011).
For all $i = 1, ..., n$, we define

$$\eta_{ni} = \left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{\frac{1}{2}} (H_i(y) - F(y|x)) \frac{K_i(x)}{n\varphi_n(x)}$$

and, we define $\xi_{ni} = \eta_{n_i} - \mathbb{E}\big(\eta_{n_i}|\mathcal{F}_{i-1}\big)$.
It is easily seen that

$$\left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{\frac{1}{2}} \widehat{Q}_{n,1}(x,y) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_{ni}$$

as ξ_{ni} is a triangular array of martingale differences according the σ-algebra $\mathcal{F}_{i-1}$, we are in position to apply the central limit theorem based on unconditional lindeberg condition to establish the asymptotic normality of $\widehat{Q}_{n,1}(x,y)$. This can be done if we establish the following statements:

a) $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\xi_{ni}^2|\mathcal{F}_{i-1}\right) \longrightarrow 1$ in probability ,

b) $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\xi_{ni}^2 \mathbb{I}_{\xi_{ni}^2 > \epsilon n}\right) \longrightarrow 0$ holds for any $\epsilon > 0$ (Lindeberg condition).

Proof of part (a)

$$\begin{aligned}
\mathbb{E}\left(\xi_{ni}^2|\mathcal{F}_{i-1}\right) &= \mathbb{E}\left(\big(\eta_{ni} - \mathbb{E}(\eta_{ni}|\mathcal{F}_{i-1})\big)^2\right) \\
&= \mathbb{E}\left(\eta_{ni}^2|\mathcal{F}_{i-1}\right) - \mathbb{E}^2\big(\eta_{ni}|\mathcal{F}_{i-1}\big).
\end{aligned}$$

The statement (a) follows then if we show that:

(1) $\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^2\big(\eta_{ni}|\mathcal{F}_{i-1}\big)=0$ in probability,

(2) $\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\eta_{ni}^2|\mathcal{F}_{i-1}\right)=1$ in probability.

To prove (1), we put $A=\left(\frac{n\varphi_n(x)}{\sigma_F^2}\right)^{\frac{1}{2}}$, we have

$$\begin{aligned}
\mathbb{E}\left(\eta_{ni}|\mathcal{F}_{i-1}\right) &= \frac{A}{n\varphi_n(x)}\mathbb{E}\left(\big(H_i\left(y\right)-F\left(y|x\right)\big)K_i(x)|\mathcal{F}_{i-1}\right)\\
\left|\mathbb{E}\left(\eta_{ni}|\mathcal{F}_{i-1}\right)\right| &= \frac{A}{n\varphi_n(x)}\left|\mathbb{E}\left[\left[\left(\mathbb{E}\big(H_i(y)|\mathfrak{B}_{i-1}\big)-F(y|x)\right)K_i(x)\right]|\mathcal{F}_{i-1}\right]\right|\\
&= \frac{A}{n\varphi_n(x)}\left|\mathbb{E}\left[\left[\mathbb{E}\big(H_i(y)|X_i\big)-F(y|x)K_i(x)\right]|\mathcal{F}_{i-1}\right]\right|
\end{aligned}$$

under (H1) and (H4), we have

$$C\phi_i(x,a_i)\le\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\le C'\phi_i(x,a_i)$$

Next, an integration by parts and a change of variable allow to get

$$\mathbb{E}\big(H_i(y)|X_i\big)=\int_{\mathbb{R}}H^{(1)}(t)F(y-b_it|X_i)dt.$$

Thus, we have

$$\left|\mathbb{E}\left(H_i(y)|X_i\right)-F(y|x)\right|\le C'a_i^{\beta_1}$$

combining this results, we have

$$\begin{aligned}
\left|\mathbb{E}\left(\eta_{ni}|\mathcal{F}_{i-1}\right)\right| &\le AC''\frac{\phi_i(x,a_i)}{n\varphi_n(x,a_i)}a_i^{\beta_2}\\
\frac{1}{n}\left(\mathbb{E}\left(\eta_{ni}|\mathcal{F}_{i-1}\right)\right)^2 &\le a_i^{2\beta_1}\frac{1}{\sigma_F^2}\frac{\sum_{i=1}^{n}\left(a_i^{\beta_1}\phi_i(x,a_i)\right)^2}{n\varphi_n(x,a_i)}\longrightarrow 0 \text{ (Under (H3))}.
\end{aligned}$$

Now, we have to prove (2), to do this, we observe that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\left(\eta_{ni}\right)^2|\mathcal{F}_{i-1}\right) &= \frac{A^2}{n\varphi_n^2(x,a_i)}\sum_{i=1}^{n}\mathbb{E}\left(\big(H_i(y)-F(y|x)\big)^2K_i^2(x)|\mathcal{F}_{i-1}\right)\\
&= \frac{A^2}{n\varphi_n^2(x,a_i)}\sum_{i=1}^{n}\Big[\mathbb{E}\left(K_i^2(x)H_i^2(y)|\mathcal{F}_{i-1}\right)-2F(y|x)\mathbb{E}\left(K_i^2(x)H_i(y)|\mathcal{F}_{i-1}\right)\\
&+ \left(F(y|x)\right)^2\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\Big].
\end{aligned}$$

We put

$$\begin{aligned}
I_1 &= \sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)H_i^2(y)|\mathcal{F}_{i-1}\right),\\
I_2 &= \sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)H_i(y)|\mathcal{F}_{i-1}\right),\\
I_3 &= \sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right).
\end{aligned}$$

We write

$$\begin{aligned}
I_1 &= F(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)+\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)H_i^2(y)|\mathcal{F}_{i-1}\right)-F(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\\
&= F(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)+\sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}\left(H_i^2(y)|X_i\right)K_i^2(x)|\mathcal{F}_{i-1}\right]\\
&\quad -F(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\\
&\leq \sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}\left(H_i^2(y)|X_i\right)K_i^2(x)|\mathcal{F}_{i-1}\right]-F(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right).
\end{aligned}$$

Using the same argument as those used in proof of the part (1), we obtain

$$\frac{1}{n\varphi_n(x,a_i)}I_2 = o(1).$$

For I_3, we get

$$\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right) = K^2(1)\phi_i(x,a_i)-\int_0^1\left(K^2(u)\right)'\phi_i(x,a_i)du$$

so under (H1) we have

$$\begin{aligned}
\frac{1}{n\varphi_n(x,a_i)}\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right) &= \frac{1}{n\varphi_n(x,a_i)}K^2(1)\sum_{i=1}^{n}\phi_i(x,a_i)\\
&\quad -\frac{1}{n\varphi_n(x,a_i)}\int_0^1\left(K^2(1)\right)'\sum_{i=1}^{n}\phi_i(x,ua_i)du\\
&= K^2(u)-\int_0^1\left(K^2(u)\right)'\frac{\sum_{i=1}^{n}\phi_i(x,a_i)}{n\varphi_n(x,a_i)}du\\
&= o(1).
\end{aligned}$$

By combining this results, we deduce that $\lim_{n\longrightarrow\infty}\sum_{i=1}^{n}\mathbb{E}\left(\eta_{ni}^2|\mathcal{F}_{i-1}\right)=1$, which complete the proof of part (a).

Proof of part (b)

The Lindeberg condition in which implies that

$$\mathbb{E}\left(\xi_{ni}^2\mathbb{I}_{\xi_{ni}>\epsilon n}\right)\leq 4\mathbb{E}\left(\eta_{ni}^2\mathbb{I}_{\eta_{ni}>n\epsilon/2}\right).$$

Let $a>1$ and $b>1$ such that $\frac{1}{a}+\frac{1}{b}=1$ making use the Holder's and Markov's inequalities one can write for all $\epsilon > 0$

$$\mathbb{E}\left(\eta_{ni}^2 \mathbb{I}_{\eta_{ni}>n\epsilon/2}\right) \leq \frac{\mathbb{E}\left(\eta_{ni}\right)^{2a}}{(n\epsilon/2)^{2a/b}}.$$

Taking $C_0 \in \mathbb{R}_+^*$ and $2a = 2+\delta$, from some $\delta > 0$, such that
$\mathbb{E}\left(|H_i(y)|^{2+\delta}\right) < \infty$ and $\mathbb{E}\left(|H_i(y) - F(y|x)|^{2+\delta} |X_i = u\right) = \overline{W}_{2+\delta}(u)$ is a continuous function, we obtain

$$\begin{aligned}
4\mathbb{E}\left(\eta_{ni}^2 \mathbb{I}_{\eta_{ni}>n\epsilon}\right) &\leq C_0 \left(\frac{n\varphi_n(x,a_i)}{\sigma_F^2}\right)^{2+\delta} \frac{1}{(n\varphi_n(x,a_i))^{2+\delta}} \left(|H_i(y) - F(y|x)|^{2+\delta} K_i^{2+\delta}(x)\right) \\
&\leq C_0 \left(\frac{n\varphi_n(x,a_i)}{n\sigma_F^2}\right)^{2+\delta} \frac{\mathbb{E}\left(\mathbb{E}\big(|H_i(y) - F(y|x)|^{2+\delta} K_i^{2+\delta}(x)|X_i\big)\right)}{(n\varphi_n(x,a_i))^{2+\delta}} \\
&\leq C_0 \left(\frac{n\varphi_n(x,a_i)}{n\sigma_F^2}\right)^{2+\delta} \frac{\mathbb{E}\left(K_i^{2+\delta}(x)\overline{W}_{2+\delta}(x)\right)}{\left((\varphi_n(x,a_i))^{2+\delta}\right)}
\end{aligned}$$

$$\begin{aligned}
&\leq C_0 \left(\frac{n\varphi_n(x,a_i)}{n\sigma_F^2}\right)^{2+\delta} \frac{\mathbb{E}\left(K_i^{2+\delta}(x)\left|\overline{W}_{2+\delta}(x) - \overline{W}_{2+\delta}(x)\right| + \left|\overline{W}_{2+\delta}(x)\right|\right)\mathbb{E}\left(K_i^{2+\delta}(x)\right)}{(\varphi_n(x))^{2+\delta}} \\
&\leq C_0 \left(\frac{n}{\sigma_F^2}\right)^{2+\delta} \mathbb{E}\left(K_i^{2+\delta}(x)\right)\left(\left|\overline{W}_{2+\delta}(x)\right| + o(1)\right).
\end{aligned}$$

Consequently $\frac{1}{n}\sum_{i=1}^{n} n\mathbb{E}\left(\xi_{ni}^2 \mathbb{I}_{\xi_{ni}>\epsilon n}\right) \longrightarrow 0$ as $n \longrightarrow \infty$ which completes the proof of lemma.

Proof of Lemma 4.5

Observe that

$$\begin{aligned}
\widehat{F}_D(x) - 1 &= \frac{1}{\varphi_n(x,a_i)} \sum_{i=1}^{n} \left(K_i(x) - \mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) + \mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) - 1\right) \\
&= \underbrace{\frac{1}{\varphi_n(x,a_i)} \sum_{i=1}^{n} \left(K_i(x) - \mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\right)}_{T_1} + \underbrace{\frac{1}{\varphi_n(x,a_i)} \sum_{i=1}^{n} \left(\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) - 1\right)}_{T_2}.
\end{aligned}$$

The proof of the lemma follows then if show that
$T_1 = o(1)$ as $n \longrightarrow \infty$ and $T_2 \longrightarrow 0$ in probability as $n \longrightarrow \infty$.
For T_2,
Under (H3) and (H1) we prove that

$$\frac{1}{\varphi_n(x,a_i)} \sum_{i=1}^{n} \mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) = o(1) \text{ as } n \longrightarrow \infty.$$

So, it is easily seen that $T_2 \longrightarrow 0$ in probability as $n \longrightarrow \infty$.
For T_1,
observe that $T_1(x) = \sum_{i=1}^{n} L_{ni}(x)$, where $\{L_{ni}(x)\}$ is a triangular array of martingale differences with respect to the σ-algebra $\mathfrak{F}_{i-1}$.
Combining the Burkholder inequality(see P.H. All and C.Heyde p(23), 1980) and Jensen's inequality (see

Laib and Laouni p(365),2011), we obtain for any $\epsilon > 0$, there exists a constant $C_0 > 0$ such that

$$\mathbb{P}\big(|T_1(x)| > \epsilon\big) \le C_0 \frac{\mathbb{E}\left(K_1^2(x)\right)}{\epsilon^2 n \left(\varphi_n(x)\right)^2} = o\left(\frac{1}{\epsilon^2 n\varphi_n(x,a_i)} + o(1)\right)$$

since $n\varphi_n(x) \longrightarrow \infty$ as $n \longrightarrow \infty$, we conclude then that $T_1 = o(1)$ in probability as $n \longrightarrow \infty$. Which completes the proof of lemma 4.5.

Proof of Lemma 4.6

We have

$$\widehat{B}_{n,1}(x,y) = \frac{\overline{F}_N(y|x)}{\overline{F}_D(x)}.$$

We write

$$\begin{aligned}
\left|\widehat{B}_{n,1}(x,y)\right| &= \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\left[\mathbb{E}\big[K_i(x)\mathbb{E}\big[H_i(y)|\mathfrak{B}_{i-1}\big]|\mathcal{F}_{i-1}\right]\\
&- F(y|x)\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\big]\\
&= \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\left[\mathbb{E}\big[K_i(x)\mathbb{E}\big[H_i(y)|X_i\big]|\mathcal{F}_{i-1}\right]\\
&- F(y|x)\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\big]\\
&\le \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\left[\mathbb{E}\big[K_i(x)\mathbb{E}\big[H_i(y)|X_i\big] - F(y|x)|\mathcal{F}_{i-1}\big]\right].
\end{aligned}$$

Next, an integration by parts and change of variable allow to get

$$\mathbb{E}\big[H_i(y)|X_i\big] = \int_{\mathbb{R}} H^{(1)}(t)F(y - b_i t|X_i)dt$$

thus, we have

$$\big|\mathbb{E}\big[H_i(y)|X_i\big] - F(y|x)\big| \le \int_{\mathbb{R}} H^{(1)}(t)\left|F(y - b_i t|X_i) - F(y|x)\right| dt \tag{9}$$

under $(H2)$, we obtain that

$$\mathbb{I}_{B(x,a_i)}(X_i)\left|\mathbb{E}\big[H_i(y)|X_i\big] - F(y|x)\right| \le C\int_{\mathbb{R}} H^{(1)}(t)(a_i^{\beta_1} + |t|\beta_2 b_i^{\beta_2})dt \tag{10}$$

and under (H4) we prove that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) = o(1).$$

We achieve the proof of lemma 4.6.

Proof of Lemma 4.7

We write

$$\begin{aligned}
\widehat{R}_{n,1}(x,y) &= -\big(\widehat{B}_{n,1}(x,y) - F(y|x)\big)\big(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\big)\\
&= -\left(\frac{\overline{F}_N(x,y) - F(y|x)\overline{F}_D(x,y)}{\overline{F}_D(x,y)}\right)\left(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\right).
\end{aligned}$$

Clearly, it is suffices to show that

(a) $\left(\frac{\overline{F}_N(x,y)-F(y|x)\overline{F}_D(x,y)}{\overline{F}_D(x,y)}\right) = o(1)$,

(b) $\left(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\right) = o(1)$.

The proof of the first hand uses arguments similar to those used in the proof of lemma 4.6 of the second part, will be established
(i) $\mathbb{E}\left(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\right) = 0$, $Var\left(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\right) \longrightarrow 0$ as $n \longrightarrow \infty$.
For all $i = 1, ..., n$, we put

$$\delta_i(x,y) = \frac{1}{n\varphi_n(x,a_i)}\left[K_i(x)H_i(y) - \mathbb{E}\left(K_i(x)H_i(y)|\mathcal{F}_{i-1}\right)\right]$$

where $\delta_i(x,y)$ is a triangular array of martingale differences according to the $\sigma-$algebra $\mathcal{F}_{i-1}$ next by $(H1)(ii)$ and $(H4)$ we obtain

$$\widehat{F}_N(x,y) - \overline{F}_N(x,y) = \sum_{i=1}^{n}\delta_i(x,y)$$

$\mathbb{E}\left(\delta_i(x,y)\right) = 0$ by definition of $\delta_i(x,y)$, we write

$$\sum_{i=1}^{n}\left(\delta_i^2(x,y)\right) \leq \sum_{i=1}\mathbb{E}\left(\delta_i^2(x,y)\right).$$

Furthermore, by Jensen's inequality we have

$$\begin{aligned}
\mathbb{E}\left(\delta_i^2(x,y)\right) &\leq \frac{1}{(n\varphi_n(x,a_i))^2}\mathbb{E}\left(K_i^2(x)H_i^2(y)|\mathcal{F}_{i-1}\right)\\
&\leq \frac{1}{(n\varphi_n(x,a_i)^2}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\\
&\leq \frac{1}{(n\varphi_n(x,a_i))^2}\mathbb{P}\left(X_i(x,a_i)|\mathcal{F}_{i-1}\right)\\
&\leq \frac{1}{(n\varphi_n(x,a_i))^2}\phi_i(x,a_i).
\end{aligned}$$

So, we obtain

$$\sum_{i=1}\mathbb{E}\left(\delta_i^2(x,y)\right) \leq \frac{\sum_{i=1}^{n}\phi_i(x,a_i)}{n^2\varphi_n^2(x,a_i))}.$$

We deduce under (H1)(ii) that $var\left(\widehat{F}_N(x,y) - \overline{F}_N(x,y)\right) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof of lemma 4.8

We use the ideas in Laib and Louani For all $i = 1, ..., n$, we define

$$\eta'_{ni} = \left(\frac{\varphi_n(x)}{n\sigma_f^2}\right)^{\frac{1}{2}}\left(b_i^{-1}H'_i(y) - f(y|x)\right)\frac{K_i(x)}{n\varphi_n(x)}$$

and, we define $\xi'_{ni} = \eta'_{n_i} - \mathbb{E}\left(\eta'_{n_i}|\mathcal{F}_{i-1}\right)$.
It is easily seen that

$$\left(\frac{n\varphi_n(x)}{\sigma_f^2}\right)^{\frac{1}{2}}\widehat{Q}_{n,2}(x,y) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi'_{ni}.$$

We follow the same idea in the proof of lemma 4.2 to prove this, we establish the following statements:

a) $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\xi_{ni}^{'2}|\mathcal{F}_{i-1}\right)\longrightarrow 1$ in probability,

b) $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\xi_{ni}^{'2}\mathbb{I}_{\xi_{ni}^{'2}>\epsilon n}\right)\longrightarrow 0$ holds for any $\epsilon>0$ (Lindeberg condition).

Proof of part (a)

$$\begin{aligned}\mathbb{E}\left(\xi_{ni}^{'2}|\mathcal{F}_{i-1}\right) &= \mathbb{E}\left(\left(\eta'_{ni}-\mathbb{E}\big(\eta'_{ni}|\mathcal{F}_{i-1}\big)\right)^2\right)\\ &= \mathbb{E}\left(\eta_{ni}^{'2}|\mathcal{F}_{i-1}\right)-\mathbb{E}^2\big(\eta'_{ni}|\mathcal{F}_{i-1}\big).\end{aligned}$$

The statement (a) follows then if we show that

(1) $\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^2\big(\eta'_{ni}|\mathcal{F}_{i-1}\big)=0$ in probability,

(2) $\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\eta_{ni}^{'2}|\mathcal{F}_{i-1}\right)=1$ in probability.

To prove (1), we put $A=\left(\frac{n\varphi_n(x)}{\sigma_f^2}\right)^{\frac{1}{2}}$, we have

$$\begin{aligned}\mathbb{E}\left(\eta'_{ni}|\mathcal{F}_{i-1}\right) &= \frac{A}{n\varphi_n(x)}\mathbb{E}\left(\big(b_i^{-1}H'_i\left(y\right)-f\left(y|x\right)\big)K_i(x)|\mathcal{F}_{i-1}\right)\\ \left|\mathbb{E}\left(\eta'_{ni}|\mathcal{F}_{i-1}\right)\right| &= \frac{A}{n\varphi_n(x)}\left|\mathbb{E}\left[\left[\big(\mathbb{E}\big(b_i^{-1}H'_i(y)|\mathfrak{B}_{i-1}\big)-f(y|x)\big)K_i(x)\right]|\mathcal{F}_{i-1}\right]\right|\\ &= \frac{A}{n\varphi_n(x)}\left|\mathbb{E}\left[\left[\mathbb{E}\big(b_i^{-1}H'_i(y)|X_i\big)-f(y|x)K_i(x)\right]|\mathcal{F}_{i-1}\right]\right|\end{aligned}$$

under (H1) and (H4), we have

$$C\phi_i(x,a_i)\le\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\le C'\phi_i(x,a_i).$$

Next, an integration by parts and a change of variable allow to get

$$\mathbb{E}\big(H'_i(y)|X_i\big)=b_i\int_{\mathbb{R}}H'(t)f(y-b_it|X_i)dt.$$

Thus, we have

$$\left|\mathbb{E}\left(H'_i(y)|X_i\right)-b_if(y|x)\right|\le C'a_i^{B_1}$$

combining this results, we have

$$\begin{aligned}\left|\mathbb{E}\left(\eta'_{ni}|\mathcal{F}_{i-1}\right)\right| &\le \frac{A}{\varphi_n(x,a_i)}C''\phi_i(x,a_i)a_i^{\beta_1}\\ \left|\mathbb{E}\left(\eta'_{ni}|\mathcal{F}_{i-1}\right)\right| &\le \left(\frac{\varphi_n(x)}{\sigma_f}\right)^{1/2}\frac{\phi_i(x,a_i)}{\varphi_n(x,a_i)}a_i\beta_1\\ \frac{1}{n}\left(\mathbb{E}\left(\eta'_{ni}|\mathcal{F}_{i-1}\right)\right)^2 &\le \frac{a_i^{2\beta_1}}{\sigma_f}\frac{\sum_{i=1}^{n}\left(a_i^{\beta_1}\phi_i(x,a_i)\right)^2}{n\varphi_n(x,a_i)}\longrightarrow 0\text{ (Under (H3)).}\end{aligned}$$

Now, we have to prove (2), to do this, we observe that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\left(\eta'_{ni}\right)^2|\mathcal{F}_{i-1}\right) &= \frac{A^2}{n\varphi_n^2(x,a_i)}\sum_{i=1}^{n}\mathbb{E}\left(\left(b_i^{-1}H_i'(y)-f(y|x)\right)^2K_i^2(x)|\mathcal{F}_{i-1}\right)\\ &= \frac{A^2}{n\varphi_n^2(x,a_i)}\sum_{i=1}^{n}\Big[\mathbb{E}\left(b_i^{-2}K_i^2(x)H_i^{'2}(y)|\mathcal{F}_{i-1}\right)\\ &\quad -2f(y|x)\mathbb{E}\left(b_i^{-1}K_i^2(x)H_i'(y)|\mathcal{F}_{i-1}\right)+\left(f(y|x)\right)^2\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\Big].\end{aligned}$$

We put

$$\begin{aligned}I_1 &= \sum_{i=1}^{n}\mathbb{E}\left(b_i^{-2}K_i^2(x)H_i^{'2}(y)|\mathcal{F}_{i-1}\right),\\ I_2 &= \sum_{i=1}^{n}\mathbb{E}\left(b_i^{-1}K_i^2(x)H_i'(y)|\mathcal{F}_{i-1}\right),\\ I_3 &= \sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right).\end{aligned}$$

We write

$$\begin{aligned}I_1 &= f(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)+\sum_{i=1}^{n}\mathbb{E}\left(b_i^{-2}K_i^2(x)H_i^{'2}(y)|\mathcal{F}_{i-1}\right)\\ &\quad -f(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\\ &= f(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)+\sum_{i=1}^{n}\mathbb{E}\left[b_i^{-2}\mathbb{E}\left(H_i^{'2}(y)|X_i\right)K_i^2(x)|\mathcal{F}_{i-1}\right]\\ &\quad -f(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\\ &\leq \sum_{i=1}^{n}\mathbb{E}\left[b_i^{-2}\mathbb{E}\left(H_i^{'2}(y)|X_i\right)K_i^2(x)|\mathcal{F}_{i-1}\right]-f(y|x)\sum_{i=1}^{n}\mathbb{E}\left(K_i^2(x)|\mathcal{F}_{i-1}\right)\end{aligned}$$

using the same argument as those used in proof of the part (1), we have

$$\frac{1}{n\varphi_n(x,a_i)}I_2 = o(1).$$

For I_3, it is the same I_3 proved in proof of lemma 4.2 and by the combining this results, we deduce that $\lim_{n\longrightarrow\infty}\sum_{i=1}^{n}\mathbb{E}\left(\eta_{ni}^{'2}|\mathcal{F}_{i-1}\right)=1$ which complete the proof of part (a).

Proof of part (b)

The Lindeberg condition in which implies that

$$\mathbb{E}\left(\xi_{ni}^{'2}\mathbb{I}_{\xi'_{ni}>\epsilon n}\right)\leq 4\mathbb{E}\left(\eta_{ni}^{'2}\mathbb{I}_{\eta'_{ni}>n\epsilon/2}\right).$$

Let $a>1$ and $b>1$ such that $\frac{1}{a}+\frac{1}{b}=1$ making use the Holder's and Markov's inequalities one can write for all $\epsilon > 0$

$$\mathbb{E}\left(\eta_{ni}^{'2}\mathbb{I}_{\eta'_{ni}>n\epsilon/2}\right) \leq \frac{\mathbb{E}\left(\eta'_{ni}\right)^{2a}}{(n\epsilon/2)^{2a/b}}.$$

Taking $C_0 \in \mathbb{R}_+^*$ and $2a = 2+\delta$, from some $\delta > 0$, such that $\mathbb{E}\left(|Y_i(y)|^{2+\delta}\right) < \infty$ and $\mathbb{E}\left(b_i^{-(2+\delta)}\left|H_i'(y) - F(y|x)\right|^{2+\delta}|X_i = u\right)$ $\overline{W}_{2+\delta}(u)$ is a continuous function, we obtain

$$\begin{aligned}
4\mathbb{E}\left(\eta_{ni}^{'2}\mathbb{I}_{\eta'_{ni}>n\epsilon}\right) &\leq C_0\left(\frac{n\varphi_n(x,a_i)}{\sigma_f^2}\right)^{2+\delta}\frac{1}{(n\varphi_n(x,a_i))^{2+\delta}}\left(\left|b_i^{-1}H_i'(y) - f(y|x)\right|^{2+\delta}K_i^{2+\delta}(x)\right)\\
&\leq C_0\left(\frac{n\varphi_n(x,a_i)}{\sigma_f^2}\right)^{2+\delta}\frac{\mathbb{E}\left(\mathbb{E}\big(\left|b_i^{-1}H_i'(y) - f(y|x)\right|^{2+\delta}K_i^{2+\delta}(x)|X_i\big)\right)}{(n\varphi_n(x,a_i))^{2+\delta}}\\
&\leq C_0\left(\frac{n\varphi_n(x,a_i)}{\sigma_f^2}\right)^{2+\delta}\frac{\mathbb{E}\left(K_i^{2+\delta}(x)\overline{W}_{2+\delta}(x)\right)}{\left((\varphi_n(x,a_i))^{2+\delta}\right)}
\end{aligned}$$

$$\begin{aligned}
&\leq C_0\left(\frac{n\varphi_n(x,a_i)}{\sigma_f^2}\right)^{2+\delta}\frac{\mathbb{E}\left(K_i^{2+\delta}(x)\left|\overline{W}_{2+\delta}(x) - \overline{W}_{2+\delta}(x)\right| + \left|\overline{W}_{2+\delta}(x)\right|\right)\mathbb{E}\left(K_i^{2+\delta}(x)\right)}{(\varphi_n(x))^{2+\delta}}\\
&\leq C_0\left(\frac{n}{\sigma_f^2}\right)^{2+\delta}\mathbb{E}\left(K_i^{2+\delta}(x)\right)\left(\left|\overline{W}_{2+\delta}(x)\right| + o(1)\right).
\end{aligned}$$

Consequently $\frac{1}{n}\sum_{i=1}^{n} n\mathbb{E}\left(\xi_{ni}^{'2}\mathbb{I}_{\xi'_{ni}>\epsilon n}\right) \longrightarrow 0$ as $n \longrightarrow \infty$ which completes the proof of lemma.

Proof of Lemma 4.9

We have

$$\widehat{B}_{n,2}(x,y) = \frac{\overline{f}_N(y|x)}{\overline{F}_D(x)}.$$

We write

$$\begin{aligned}
\left|\widehat{B}_{n,2}(x,y)\right| &= \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\big[\mathbb{E}\big[K_i(x)\mathbb{E}\big[b_i^{-1}H_i'(x)|\mathfrak{B}_{i-1}\big]|\mathcal{F}_{i-1}\big]\\
&- f(y|x)\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\big]\\
&= \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\big[\mathbb{E}\big[K_i(x)\mathbb{E}\big[b_i^{-1}H_i'(x)|X_i\big]|\mathcal{F}_{i-1}\big]\\
&- f(y|x)\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)\big]\\
&\leq \frac{1}{\sum_{i=1}^{n}\mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big)}\sum_{i=1}^{n}\big[\mathbb{E}\big[K_i(x)\mathbb{E}\big[b_i^{-1}H_i'(x)|X_i\big] - b_i f(y|x)|\mathcal{F}_{i-1}\big]\big].
\end{aligned}$$

Next, an integration by parts and change of variable allow to get

$$\mathbb{E}\big[H_i'(y)|X_i\big] = b_i \int_{\mathbb{R}} H'(t) f(y - b_i t|X_i) dt$$

thus, we have

$$\big|\mathbb{E}\big[b_i^{-1} H_i'(y)|X_i\big] - f(y|x)\big| \leq \int_{\mathbb{R}} H'(t) \left|F(y - b_i t|X_i) - f(y|x)\right| dt. \tag{11}$$

Under $(H2)$, we obtain that

$$\mathbb{I}_{B(x,a_i)}(X_i) \left|\mathbb{E}\big[H_i'(y)|X_i\big] - b_i f(y|x)\right| \leq C b_i \int_{\mathbb{R}} H'(t)(a_i^{\beta_1} + |t|\beta_2 b_i^{\beta_2}) dt \tag{12}$$

and under (H4) we prove that

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\big(K_i(x)|\mathcal{F}_{i-1}\big) = o(1).$$

We achieve the proof of lemma.

Proof of Lemma 4.10

We write

$$\begin{aligned}
\widehat{R}_{n,2}(x,y) &= -\widehat{B}_{n,2}(x,y)\big(\widehat{f}_N(x,y) - \overline{f}_N(x,y)\big) \\
&= -\left(\frac{\overline{f}_N(x,y) - f(y|x)\overline{F}_D(x,y)}{\overline{F}_D(x,y)}\right)\left(\widehat{f}_N(x,y) - \overline{f}_N(x,y)\right).
\end{aligned}$$

Clearly, it is suffices to show that:

(a) $\left(\frac{\overline{f}_N(x,y) - f(y|x)\overline{F}_D(x,y)}{\overline{F}_D(x,y)}\right) = o(1)$,

(b) $\left(\widehat{f}_N(x,y) - \overline{f}_N(x,y)\right) = o(1)$.

The proof of the first hand uses arguments similar to those used in the proof of lemma 4.6. Of the second part, will be established

(i)$\mathbb{E}\left(\widehat{f}_N(x,y) - \overline{f}_N(x,y)\right) = 0$, $Var\left(\widehat{f}_N(x,y) - \overline{f}_N(x,y)\right) \longrightarrow 0$ as $n \longrightarrow \infty$ For all $i = 1, ..., n$, we put

$$\delta_i'(x,y) = \frac{1}{n\varphi_n(x,a_i)}\left[b_i^{-1} K_i(x) H_i'(y) - \mathbb{E}\left(b_i^{-1} K_i(x) H_i'(y)|\mathcal{F}_{i-1}\right)\right]$$

where $\delta_i(x,y)$ is a triangular array of martingale differences according to the $\sigma-$algebra $\mathcal{F}_{i-1}$ next by $(H1)(ii)$ and $(H4)$ we obtain

$$\widehat{f}_N(x,y) - \overline{f}_N(x,y) = \sum_{i=1}^{n} \delta_i'(x,y)$$

$\mathbb{E}\left(\delta_i'(x,y)\right) = 0$ by definition of $\delta_i'(x,y)$,
we write

$$\sum_{i=1}^{n}\left(\delta_i'^2(x,y)\right) \leq \sum_{i=1} \mathbb{E}\left(\delta_i'^2(x,y)\right).$$

Furthermore, by Jensen's inequality we have

$$\begin{aligned}\mathbb{E}\left(\delta_i^{'2}(x,y)\right) &\leq \frac{1}{(n\varphi_n(x,a_i))^2}\mathbb{E}\left(b_i^{-2}K_i^2(x)H_i^{'2}(y)|\mathcal{F}_{i-1}\right)\\ &\leq \frac{1}{(n\varphi_n(x,a_i)^2}\mathbb{E}\left(b_i^{-2}K_i^2(x)|\mathcal{F}_{i-1}\right)\\ &\leq \frac{1}{(n\varphi_n(x,a_i))^2}\mathbb{P}\left(X_i(x,a_i)\mathbb{E}\left(H_i^{'2}(y)|\mathfrak{B}_{i-1}\right)|\mathcal{F}_{i-1}\right)\\ &\leq \frac{Cb_i^{-1}\phi_i(x,a_i)}{(n\varphi_n(x,a_i))^2}\end{aligned}$$

so, we obtain

$$\sum_{i=1}\mathbb{E}\left(\delta_i^{\prime 2}(x,y)\right) \leq \frac{C\sum_{i=1}^n b_i^{-1}\phi_i(x,a_i)}{n^2\varphi_n^2(x,a_i))}.$$

We deduce that $var\left(\widehat{f}_N(x,y)-\overline{f}_N(x,y)\right) \longrightarrow 0$ as $n \longrightarrow \infty$ since (H1)(ii).

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References

[1] Ardjoun, F.Z., Ait Hennani, L. and Laksaci, A. (2016). A recursive kernel estimate of the functional modal regression under ergodic dependence condition, *Journal of Statistical Theory and Practice*, **10**, 475–496.

[2] Attouch, M.k., Belabed, F.Z. (2014). The k Nearest Neighbors estimation of the conditional hazard function for functional data, *REVSTATStatistical Journal*, **12**, 273–297.

[3] Benziadi, F., Laksaci, A. and Tebboune, F. (2016). Note on conditional quantiles for functional ergodic data, *C. R. Acad.Sci.Paris,Ser.I*, **354**, 628–633.

[4] Ferraty, F., and Vieu, P. (2006). Nonparametric functional data analysis, *Springer Series in Statistics*, Springer New York.

[5] Ferraty, F., Rabhi, A., and Vieu, P. (2008). Estimation non-paramétrique de la fonction de hasard avec variable explicative fonctionnelle,*Rev. Roumaine Math. Pures Appl*, **53**, 1–18.

[6] Gheriballah, A., Laksaci, A., and Sekkal, S. (2013). Nonparametric M-regression for functional ergodic data, *Statistics and Probability Letters* **83**, 902–908.

[7] Laïb, N., and Louani, D. (2010). Nonparametric Kernel regression estimation for functional stationary ergodic data: Asymptotic proprties, *J. Mult. Anal.*, **101**, 2266–2281.

[8] Laïb, N., and Louani, D. (2011). Rates of strong consistencies of the regression function estimator for functional stationary ergodic data, *J. of Stat. Plan. and Inf.*, **141**, 359–372.

[9] Laksaci, A., and Mechab, B. (2010). Estimation non-paramtrique de la fonction de hasard avec variable explicative fonctionnelle : cas des données spatiales, *Rev. Roumaine Math. Pures Appl*, **55**, 35–51.

[10] Laksaci, A., and Mechab, B. (2014). Conditional hazard estimate for functional random fields, *Journal of Statistical Theory and Practice*, **8**, 192–200.

[11] Massim, I., and Mechab, B. (2016). Local linear estimation of the conditional hazard function, *International Journal of Statistics & Economics* **17**, 1-11.

[12] Quintela-del-Río, A. (2008). Hazard function given a functional variable: Non-parametric estimation under strong mixing conditions, *J. Nonparametr. Stat.*, **20**, 413–430.

[13] Ramsay, J.O., and Silverman, B.W. (2002). Applied functional data analysis; Methods and case studies, *Springer-Verlag*, New York.