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A Heterogeneous two-server queueing system with reneging and no waiting line

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Abstract. This paper deals with a heterogeneous two-server queueing system with reneging and no waiting line. In this work, reneging involves the situation where customers begin receiving service may disengage before service completion, and via certain mechanism, these latter are retained in the system. After the completion of service, each customer may rejoin the system as a feedback customer for receiving another regular service with some probability or can leave the system. We obtain the stationary state probabilities and deduce the explicit expressions of different performance measures of the system. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.

1. Introduction

Motivated by the telephone call center applications, this work focuses on the study of a heterogeneous two-server queueing system queueing systems with impatient customers and no waiting line. Recently, there has been a great interest in multiserver queueing systems with impatient customer. The majority of call centers can be classified into two categories: revenue-generating and service-oriented. An important aspect for modeling of service-oriented call centers is the impatience behavior of the customers.

Two usual modes in which customers advertise their impatience are balking and reneging; a customer refuses to enter the queue if the wait is too long or the queue is too big, this is the balking behavior. On the other hand, a customer who is waiting to be served might hang up (renege) before getting service if the wait in line becomes too long; this is the reneging behavior. Of course, there can be a combination of the two.

The first who considered this type of queues was Haight (13), where he studied an $M/M/1$ queue with balking. An $M/M/1$ queue with reneged customers was also proposed by Haight in (14). The combined effects of balking and reneging in an $M/M/1/N$ queue have been investigated by Ancker and Gafarian in (2; 3), after that multiple works were given, let's cite for instance (1) considered the multiple servers queueing system $M/M/c/N$ with balking and reneging, then (27) extended this work to study an $M/M/c/N$ queue with balking, reneging and server breakdowns. (28) studied call centers with impatient customers authors considered a $M/M/n+G$ queue, the model was characterized by Poisson arrivals, exponential service times, n service agents and generally distributed patience times of customers, (8) studied many-server queues with

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customer abandonment, where their diffusion model has been analyzed. (26) focused on the multi-channel queueing system with heterogeneous servers, regenerative input flow and balking, where servers times are random variables but not necessary exponential. (15) dealt with Markovian queueing system with random balking, after that the optimum system capacity for Markovian queueing system with an adaptive balking was presented in (15).

Feedback represents the case where customer after getting incomplete or dissatisfying service comes back to the system for another regular service, for instance, in computer communication, the transmission of a protocol data unit may be sometimes repeated because of occurrence of an error, this generally occurs due to dissatisfying quality of service. Another example of feedback queue is a rework in industrial operations. (9; 22; 24) considered a single server feedback queue with impatient and feedback customers, they studied $M/M/1$ queueing model for queue length at arrival epochs and obtained result for stationary distribution, mean and variance of queue length, (25) obtained transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions, the steady-state solution, moments under steady state and busy period analysis were calculated. (4) studied $M/M/1$ retrial queueing system with loss and feedback under non preemptive priority service by matrix geometric method, (18) dealt with a single server queueing system with retention of reneged customers. (19) studied a single server queueing system with retention of reneged customers and balking, (23) considered a single server, finite capacity Markovian feedback queue with reneging, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. (11) treated a $M/M/2/N$ queue with general balk function, reneging and two heterogeneous servers.

Most of the earlier works on multiserver queueing models deal with homogeneous servers, that is, the individual service rates are same for all servers in the system. But this assumption may be valid only when the service process is mechanically or electronically controlled. In a queueing system with human servers the above assumption is highly unrealistic. Often servers providing identical service, serves at different rates. This motivated the researchers to study multiserver queueing system with heterogeneous servers (17).

Heterogeneity of service is a common feature of many real multi-server queueing situations. The heterogeneous service mechanisms are invaluable scheduling methods that allow customers to receive different quality of service. Heterogeneous service is clearly a main feature of the operation of almost any manufacturing system. The role of quality and service performance are crucial aspects in customer perceptions and firms must dedicate special attention to them when designing and implementing their operations (10). The queueing systems with heterogeneous servers have received a considerable attention in the literature, let's cite for instance (7; 12; 20; 21).

In this work we consider an heterogeneous two-server queueing model with Bernoulli feedback, reneged customers, retention of reneged customers and no waiting line. In our model customers being receiving service may disengage before service completion and using certain mechanism they can be retained with some probability. Using a recursive method, we obtain the stationary state probabilities, then we deduce useful performance measures. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.

2. Model description

Consider an $M/M/2$ queue with Bernoulli feedback, reneged customers, retention of reneged customers and no waiting space. Customers arrive at the service station one by one according to Poisson process with arrival rate λ. There exist two heterogeneous servers which provide services to all arriving customers. Service times are independently and identically distributed exponential random variables with rates μ_i; $i = 1, 2$.

After completion of each service, with probability β', the customer can either rejoin the system as a Bernoulli feedback customer in order to get another regular service, or he can leave the system definitively with probability β where $\beta + \beta' = 1$. Note that there is no distinguishing between the regular arrival and feedback arrival.

The regular or feedback arrivals may become impatient when the service is so long and can disengage before service completion. In fact, each customer at the beginning of services activates an individual timer (reneging time). The impatience timers follow an exponential distribution with parameters ν_1, ν_2 for services 1 and 2 respectively. Then, if the customer's service has not been finished before the customer's timer expires, the customer may leave the system. And using certain mechanism, each reneged customer may be retained in the system with probability α', otherwise he abandons the system without getting service with complimentary probability α.

The customers are served according to the following discipline:

✓ If the two servers are busy, the customers leave the system.

✓ If one server is free, the first customer who comes to the system goes to it.

✓ If both servers are free, the head customer chooses server 1 with probability π_1 and server 2 with probability π_2, where $\pi_1 + \pi_2 = 1$.

Let P_n be the probability that are n customers in the system in steady state, such that
$P_{0,0}$= $\mathbb{P}$(there is no customer in the system),
$P_{1,0} = \mathbb{P}$(there is one customer being served by server 1),
$P_{0,1} = \mathbb{P}$(there is one customer being served by server 2),
$P_2 = \mathbb{P}$(there are 2 customers in the system).
Also, $P_0 = P_{0,0}$; $P_1 = P_{1,0} + P_{0,1}$ and $P_2 = P_{1,1}$.

3. Steady-State Solution

In this section, we derive the steady state probabilities, using the Markov process theory, the differential-difference equations of the model are as

$$\frac{dP_{0,0}(t)}{dt} = -\lambda P_{0,0}(t) + (\beta\mu_1 + \alpha\nu_1)P_{1,0} + (\beta\mu_2 + \alpha\nu_2)P_{0,1}(t), \tag{1}$$

$$\frac{dP_{1,0}(t)}{dt} = -(\lambda + \beta\mu_1 + \alpha\nu_1)P_{1,0}(t) + (\beta\mu_2 + \alpha\nu_2)P_{1,1}(t) + \lambda\pi_1 P_{0,0}(t), \tag{2}$$

$$\frac{dP_{0,1}(t)}{dt} = -(\lambda + \beta\mu_2 + \alpha\nu_2)P_{0,1}(t) + (\beta\mu_1 + \alpha\nu_1)P_{1,1}(t) + \lambda\pi_2 P_{0,0}(t), \tag{3}$$

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + (\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))P_2(t). \tag{4}$$

Theorem 3.1. *If we have a heterogeneous two-server queueing system with Bernoulli feedback, reneging, retention of reneged customers and no waiting line, then*
1. The steady-state equations are

$$\lambda P_{0,0} = (\beta\mu_1 + \alpha\nu_1)P_{1,0} + (\beta\mu_2 + \alpha\nu_2)P_{0,1}, \tag{5}$$

$$(\lambda + \beta\mu_1 + \alpha\nu_1)P_{1,0} = (\beta\mu_2 + \alpha\nu_2)P_{1,1} + \lambda\pi_1 P_{0,0}, \tag{6}$$

$$(\lambda + \beta\mu_2 + \alpha\nu_2)P_{0,1} = (\beta\mu_1 + \alpha\nu_1)P_{1,1} + \lambda\pi_2 P_{0,0}, \tag{7}$$

$$\lambda P_1 = (\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))P_2. \tag{8}$$

2. The steady-state-probabilities P_n of system size are given by

$$P_{1,0} = \left\{ \frac{\lambda + (\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))\pi_1}{2\lambda + \beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2)} \frac{\lambda}{\beta\mu_1 + \alpha\nu_1} \right\} P_{0,0}. \tag{9}$$

$$P_{0,1} = \left\{ \frac{\lambda + (\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))\pi_2}{2\lambda + \beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2)} \frac{\lambda}{\beta\mu_2 + \alpha\nu_2} \right\} P_{0,0}. \tag{10}$$

$$P_1 = \frac{\lambda(\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))}{(\beta\mu_1 + \alpha\nu_1)(\beta\mu_2 + \alpha\nu_2)} \left\{ \frac{\lambda + (\beta\mu_1 + \alpha\nu_1)\pi_2 + (\beta\mu_2 + \alpha\nu_2)\pi_1}{2\lambda + \beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2)} \right\} P_{0,0}. \tag{11}$$

And

$$P_2 = \frac{\lambda^2}{(\beta\mu_1 + \alpha\nu_1)(\beta\mu_2 + \alpha\nu_2)} \left\{ \frac{\lambda + (\beta\mu_1 + \alpha\nu_1)\pi_2 + (\beta\mu_2 + \alpha\nu_2)\pi_1}{2\lambda + \beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2)} \right\} P_{0,0}. \tag{12}$$

With

$$P_{0,0} = \left(1 + \left\{ \frac{(\lambda + (\beta\mu_1 + \alpha\nu_1)\pi_2 + (\beta\mu_2 + \alpha\nu_2)\pi_1)\left(\lambda^2 + \lambda(\beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))\right)}{(2\lambda + \beta(\mu_1 + \mu_2) + \alpha(\nu_1 + \nu_2))((\beta\mu_1 + \alpha\nu_1)(\beta\mu_2 + \alpha\nu_2))} \right\} \right)^{-1}. \tag{13}$$

Proof. We obtain the steady-state-probabilities by using iterative method. By solving equations (6)-(7) we get easily equation (9) and (10). Then, by summing equations (9) and (10) we obtain (11). After that it suffices to substitute equation (11) in equation (8) to obtain equation (13).

Finally, by using the normalizing condition we find equation(13).

4. Performance measures

This part of paper is devoted to present some of performance measures that are of general interest for the evaluation of the characteristic of the existing queueing system.

- The mean number of customers in the system.

$$L_s = P_1 + 2P_2. \tag{14}$$

Customers arrive into the system at the rate of λ. But not all the customers who arrive can join the system because of finite capacity of the system. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda' = \lambda(1 - P_2). \tag{15}$$

- The expected number of customers served.

$$\mathbb{E}(C.S) = \beta\mu_1 P_{1,0} + \beta\mu_2 P_{0,1} + 2\beta(\mu_1 + \mu_2)P_2. \tag{16}$$

We assumed that each customer has a random patience time. Thus, the reneging rate of the system would depend on the state of the system. Consequently

- The average reneging rate.

$$R_{ren} = \alpha\nu_1 P_{1,0} + \alpha\nu_2 P_{0,1} + 2\alpha(\nu_1 + \nu_2)P_2. \tag{17}$$

- The average retention rate.

$$R_{ret} = (1-\alpha)\nu_1 P_{1,0} + (1-\alpha)\nu_2 P_{0,1} + 2(1-\alpha)(\nu_1 + \nu_2)P_2. \tag{18}$$

- Rate of abandonment

$$R_a = \lambda - \beta\mu_1 P_{1,0} - \beta\mu_2 P_{0,1} - 2\beta(\mu_1 + \mu_2)P_2. \tag{19}$$

In system management, customers who leave the system (renege) represent business lost. Consequently, It is interesting to present the proportion of customers lost. So, using (15) and (17), we get

- Proportion of customers lost due to reneging out of those arriving and joining the system.

$$P_{Clost} = \text{Average reneging rate}/\lambda'. \tag{20}$$

5. Numerical Solution and Graphical Representation

In this section some numerical examples are carried out in order to show the impact of different parameters .

5.1. Impact of service rates.

Let us vary λ and take $\alpha = 0.6$, $\beta = 0.2$, $\nu_1 = 5$, $\nu_2 = 4$, $\pi_1 = 0.3$, $\pi_2 = 0.7$, then consider two cases of service rates, $\mu_1 = 2$, $\mu_2 = 6$ and $\mu_1 = 14$, $\mu_2 = 20$. The numerical results obtained for these situations are stored in Tables 1-2 and illustrated in Figure 1.

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
2	0,371279513	1,756538024	2,435957461	0,769179319	1,230820681	1,623971641	1,386794608
3	0,608540925	2,323843416	3,824436536	1,181494662	1,818505338	2,549624357	1,645737621
4	0,824300374	2,694177625	5,040981992	1,538017198	2,461982802	3,360654662	1,871065198
5	1,006726854	2,919985838	6,041423261	1,828996283	3,171003717	4,027615507	2,068990603
6	1,155998128	3,048515418	6,8416619	2,060631272	3,939368728	4,561107933	2,244260226
7	1,276712204	3,114354161	7,630226247	2,264251435	4,735748565	5,086817498	2,450018801

Table 1: Variation of different characteristics of the system vs. λ when $\mu_1 = 2$, and $\mu_2 = 6$

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
2	0,176749399	1,907702664	1,236867581	1,627958459	0,372041541	0,824578387	0,648354487
3	0,30437317	2,716422512	2,033185326	2,643702886	0,356297114	1,355456884	0,748479045
4	0,440945682	3,394928739	2,852432032	3,679791196	0,320208804	1,901621355	0,840203801
5	0,577095155	3,943822922	3,646288028	4,678557412	0,321442588	2,430858685	0,924556731
6	0,706765667	4,37525134	4,385754491	5,605711384	0,394288616	2,923836328	1,002400582
7	0,826633358	4,706396685	5,149674857	6,531774886	0,468225114	3,433116571	1,094186317

Table 2: Variation of different characteristics of the system vs. λ when $\mu_1 = 14$, $\mu_2 = 20$

According to Tables 1–2 and Figure 1 we observe that along the increase of λ, the mean number of customer in the system L_s, the effective arrival rate λ', the mean number of customers served $\mathbb{E}(C.S)$, the average reneging rate R_{ren}, the average retention rate R_{ret} and the average abandonment rate R_a increase.

Further, by comparing the results when $\mu_1 = 2$, $\mu_2 = 6$ and $\mu_1 = 14$, $\mu_2 = 20$, we remark that when $\mu_1 = 14$, $\mu_2 = 20$ (the mean service rates small), the size of the system L_s, the average reneging rate R_{ren}, the average retention rate R_{ret}, the abandonment rate R_a and the proportion of customers lost due to reneging out of those who arrive and join the system P_{Clost} are smaller than the case where $\mu_1 = 2$, $\mu_2 = 6$. On the other side the effective arrival rate λ', and the mean number of customers served $\mathbb{E}(C.S)$ are bigger in the first case. The obtained results are as it should be expected.

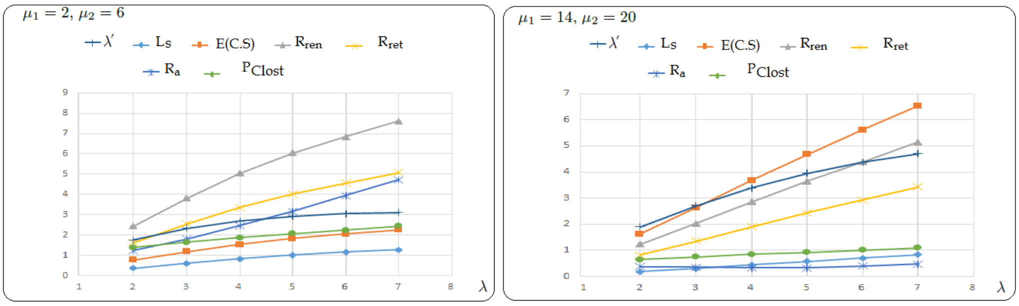

Figure 1: Variation of different characteristics of the system vs. λ for different values of service rates.

5.2. Impact of reneging rates.

Let's vary ν_1, and take $\lambda = 15$, $\beta = 0.3$, $\alpha = 0.7$, $\mu_1 = 2$, $\mu_2 = 1$, $\nu_2 = 3$, $\pi_1 = 0.6$, and $\pi_2 = 0.4$. Graphical representation is presented in Figure 2.

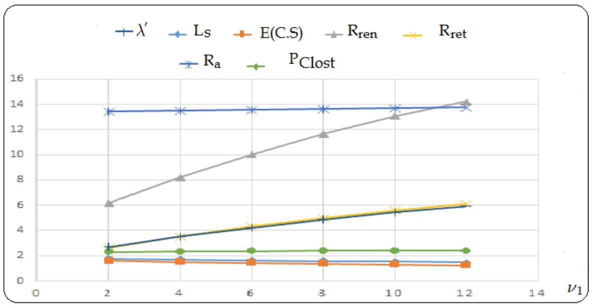

Figure 2: Variation of different characteristics of the system vs. ν_1

From Figure 2 we observe that along the increase of ν_1, the average reneging rate R_{ren} increases, that is, customers become more impatient and consequently they leave the system, this implies that R_a and R_{ret} increase, more customers abandon the system, more the probability of retention becomes high. On the other side, the increase of ν_1 leads to a decrease of L_s and of the mean number of customers served $\mathbb{E}(C.S)$ which is absolutely explicable; more the average rate of impatience becomes small, more customers leave the system without getting their complete service.

• Now, let's vary μ_1 and put $\lambda = 3$, $\beta = 0.2$, $\alpha = 0.8$, $\mu_2 = 6$,$\pi_1 = 0.3$, $\pi_2 = 0.7$ then we consider $\nu_1 = 1$, $\nu_2 = 8$, and $\nu_1 = 10$, $\nu_2 = 12$. The numerical results obtained for these situations are stored in Tables 3-4 and illustrated in Figure 3.

μ_1	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
2	0,369715288	2,636020152	8,04310153	0,749702825	2,250297175	2,010775383	3,051229151
2,8	0,365806725	2,64089654	7,969750771	0,811497248	2,188502752	1,992437693	3,017820143
3,6	0,362015439	2,645630354	7,89862888	0,871494209	2,128505791	1,97465722	2,985537593
4,4	0,358336114	2,650228	7,829634391	0,929774284	2,070225716	1,957408598	2,954324832
5,2	0,354763754	2,654695497	7,762671934	0,986413236	2,013586764	1,940667983	2,924128941
6	0,351293656	2,659038511	7,694063604	1,040944141	1,959055859	1,923515901	2,893551023

Table 3: Variation of different characteristics of the system vs. μ_1 when $\nu_1 = 10$, $\nu_2 = 12$

μ_1	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
2	0,665043366	2,307246493	4,641297567	1,120479746	1,879520254	1,160324392	2,011617562
2,8	0,622413367	2,355234771	4,420160941	1,208761818	1,791238182	1,105040235	1,876738996
3,6	0,585777863	2,396520059	4,230185869	1,285118525	1,714881475	1,057546467	1,76513685
4,4	0,553948815	2,432429492	4,065194573	1,351899029	1,648100971	1,016298643	1,671248678
5,2	0,526032918	2,463961021	3,920541936	1,410870262	1,589129738	0,980135484	1,591154204
6	0,501345522	2,491879539	3,785710874	1,45295136	1,54704864	0,946427719	1,519219054

Table 4: Variation of different characteristics of the system vs. μ_1 when $\nu_1 = 1$, $\nu_2 = 8$

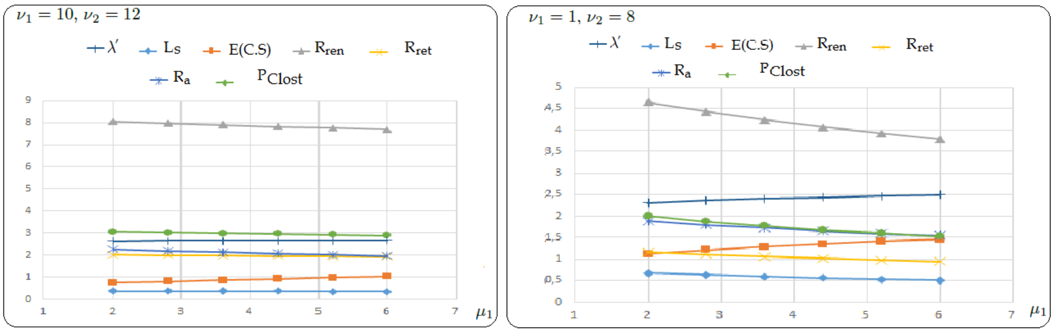

Figure 3: Variation of different characteristics of the system vs. μ_1 for different values of reneging rates.

✓ According to Tables 3-4 and Figure 3 we observe that when μ_1 increases, customers are served faster, on the other hand when the mean service rates are small, R_a, $\mathbb{E}(C.S)$ is high while R_{ren}, R_{ret} are small, thus the mean number of customers in the system decreases which agree absolutely with our intuition.

✓ Comparing results given in Figure 3, we remark that when the reneging rates are $\nu_1 = 1, \ \nu_2 = 8$ (mean reneging rates large) the mean number of customers in the systems L_s, the average reneging rate R_{ren}, the average retention rate R_{ret} and rate of abandonment R_a are higher than the case where $\nu_1 = 10, \ \nu_2 = 12$ (mean reneging rates small), contrariwise the mean number of customers served in the first case is smaller than the second one.

5.3. Impact of retention rate.

Let's vary λ and put $(\alpha = 0.3, \ 0.8)$, $\beta = 0.6$, $\nu_1 = 5$, $\nu_2 = 7$, $\mu_1 = 6$, $\mu_2 = 4$, $\pi_1 = 0.6$ and $\pi_2 = 0.4$. The numerical results obtained for these situations are presented in Tables 5-6 and illustrated in Figure 4.

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
1	0,146029931	0,98118171	0,455030188	0,77906934	0,22093066	1,061737105	0,463757307
3	0,486380425	2,627770083	1,600761773	2,694598338	0,305401662	3,735110803	0,609171169
5	0,788135941	3,749784357	2,659321115	4,45104289	0,54895711	6,205082602	0,709193079
7	1,021154082	4,46778593	3,494735375	5,834901569	1,165098431	8,154382542	0,782207436
9	1,195034526	4,92601866	4,127273206	6,882557588	2,117442412	9,630304148	0,83785172
11	1,325146102	5,223966917	4,704770168	7,912987472	3,087012528	10,97779706	0,90061255

Table 5: Variation of different characteristics of the system vs. λ when $\alpha = 0.3$

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
1	0,086447147	0,992390216	0,696744941	0,461842959	0,538157041	0,174186235	0,702087676
3	0,291219262	2,829363713	2,471669759	1,599926121	1,400073879	0,61791744	0,873577952
5	0,501499359	4,352402687	4,378916652	2,802191098	2,197808902	1,094729163	1,0060918
7	0,693491872	5,545329549	6,164505918	3,919395438	3,080604562	1,541126479	1,111657272
9	0,85932298	6,455951705	7,73279672	4,896712181	4,103287819	1,93319918	1,197777969
11	0,999010066	7,145401211	9,295857363	5,931182967	5,068817033	2,323964341	1,30095667

Table 6: Variation of different characteristics of the system vs. λ when $\alpha = 0.8$

Comparing the evolution of different performance measures given in Figure 4 and Tables 5-6, we remark that

✓ with the increase in the probability of non-retention α, the size of the system decreases and consequently we get a decrease in the number of customers served, this leads to a lose in potential customers which has a negative impact.

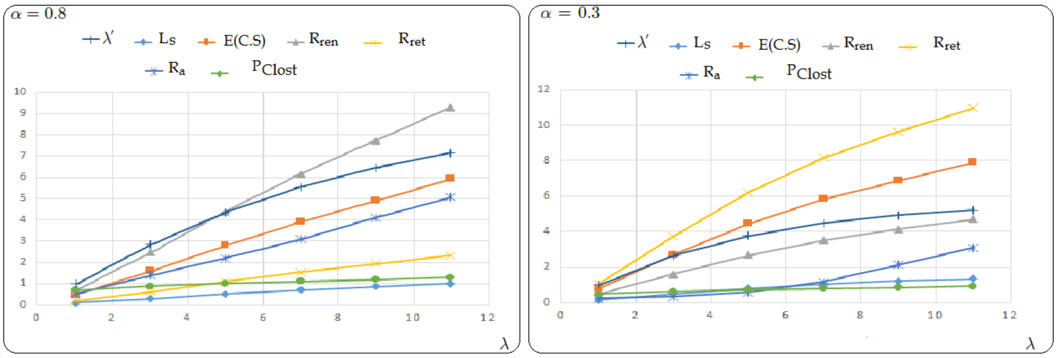

Figure 4: Variation of different characteristics of the system vs. λ for different values of retention rate.

✓ when the probability of retention is big $\alpha' = 0.7$ ($\alpha = 0.3$), the size of the system L_s, the mean number of customers served $\mathbb{E}(C.S)$, the average retention rate are bigger compared to the case where $\alpha' = 0.2$ ($\alpha = 0.8$). Then when the probability of retaining customers is big $\alpha' = 0.7$ ($\alpha = 0.3$) the average reneging rate, the rate of abandonment, the proportion of lost customers and the effective arrival rate are smaller compared to the case where $\alpha' = 0.2$ ($\alpha = 0.8$).

5.4. Impact of feedback probability.

Let's vary λ and put ($\beta = 0.2, 0.8$), $\alpha = 0.4$, $\nu_1 = 8$, $\nu_2 = 7$, $\mu_1 = 6$, $\mu_2 = 4$, $\pi_1 = 0.3$ and $\pi_2 = 0.7$. The numerical results obtained for these situations are stored in Tables 7-8 and illustrated in Figure 5.

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
1	0,129041059	0,970672487	1,036693493	0,332833174	0,667166826	1,55504024	1,068015739
1,2	0,16431637	1,150705089	1,292867362	0,416513386	0,783486614	1,939301042	1,12354362
1,4	0,201727176	1,323964372	1,558892247	0,50370394	0,89629606	2,33833837	1,177442747
1,6	0,240860215	1,489892473	1,832258065	0,593548387	1,006451613	2,748387097	1,229792148
1,8	0,281326704	1,64808358	2,110642583	0,685253949	1,114746051	3,165963874	1,280664773
2	0,322766571	1,798270893	2,405249942	0,783092648	1,216907352	3,607874913	1,337534824

Table 7: Variation of different characteristics of the system in the case $\beta = 0.2$

λ	L_s	λ'	R_{ren}	$\mathbb{E}(C.S)$	R_a	R_{ret}	P_{Clost}
1	0,064163217	0,989651094	0,545239503	0,690715553	0,309284447	0,817859255	0,550941141
1,2	0,080804054	1,182370024	0,674676841	0,857354365	0,342645635	1,012015261	0,570613959
1,4	0,098515865	1,372415558	0,80962129	1,031729785	0,368270215	1,214431934	0,589924302
1,6	0,117212249	1,559450898	0,949524815	1,213093981	0,386906019	1,424287223	0,608884074
1,8	0,136807321	1,743172344	1,0938485	1,400711744	0,399288256	1,64077275	0,627504506
2	0,157216285	1,923309129	1,245822806	1,599500302	0,400499698	1,86873421	0,647749645

Table 8: Variation of different characteristics of the system in the case $\beta = 0.8$

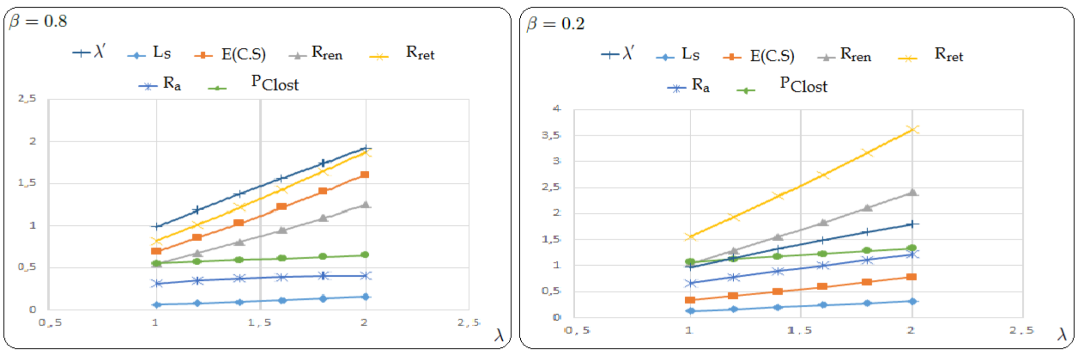

Figure 5: Variation of different characteristics of the system vs. λ for different values of feedback probability.

Comparing the results in Figure 5 and Tables 7-8 we observe that

✓ with the increase of λ, the mean number of customers in the system L_s, average reneging rate R_{ren}, average retention rate R_{ret}, proportion of customer lost due to reneging out of those who arrived and join the system, P_{CLost}, the expected number of customers served $\mathbb{E}(C.S)$ increase. When $\beta = 0.2$, the rate of abandonment R_a increases, while when $\beta = 0.8$, it decreases.

✓ when the probability of feedback $\beta' = 0.8$, $(\beta = 0.2)$ is big, average reneging rate, average retention rate, proportion of customer lost due to reneging out of those who arrived and join the system, rate of abandonment and the mean size of the system are bigger than the case where $\beta' = 0.2$, while the mean number of customers served is smaller in the first case, this is due to the high number of lost customers.

6. Conclusion

In this paper we analyzed a heterogeneous two-server queueing system with Bernoulli feedback, reneging, retention of reneged customers and no waiting line. The stationary state probabilities were obtained and the explicit expressions of different characteristics of the system were deduced. Finally, some numerical examples have been presented to demonstrate how the various parameters of the model influence the behavior of the system.

For further work, this model can be studied under the provision of time dependent arrival and service rates. The cost-profit analysis of the model can also be carried out to study its economic analysis.

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