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Relationships for moments of generalized order statistics from Erlang-truncated exponential distribution and related inference

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Abstract. In this paper some recurrence relations satisfied by single and product moments of generalized order statistics from Erlang-truncated exponential distribution have been obtained. Then we use these results to compute the first four moments of order statistics, record values and second record values for some specific values of the parameters. Further, we use the results on order statistics to obtain BLUEs of location and scale parameters based on type-II right censored samples. In addition, we carry out numerical illustration through Monte Carlo simulations to show the usefulness of the findings.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x).
Assume that $k > 0,\ n \in N,\ n \geq 2,\ \tilde{m} = (m_1, m_2,\ ...,\ m_{n-1}) \in R^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + (n-r) + M_r > 0$ for all $r \in \{1, 2, ..., n-1\}$. Then $X(r, n, \tilde{m}, k),\ r = 1, 2, ..., n,$ are called generalized order statistics if their joint pdf is given by

$$f_{X(1,n,\tilde{m},k),...,X(n,n,\tilde{m},k)}(x_1, ..., x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{i=1}^{n-1} (1 - F(x_i))^{m_i} f(x_i)\right) \times (1 - F(x_n))^{k-1} f(x_n), \quad F^{-1}(0+) < x_1 \leq ... \leq x_n < F^{-1}(1). \tag{1}$$

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n - i + 1;\ i = 1, 2, ..., n,\ i.e.,\ m_1 = m_2 = ...m_{n-1} = 0,\ k = 1$), k-th records values ($\gamma_i = k,\ i.e.,\ m_1 = m_2 = ... = m_{n-1} = -1,\ k \in N$), sequential order statistics ($\gamma_i = (n - i + 1)\alpha_i;\ \alpha_1, \alpha_2, ..., \alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = \alpha - i + 1;\ \alpha > 0$), Pfeifer's record values ($\gamma_i = \beta_i;\ \beta_1, \beta_2, ..., \beta_n > 0$) and progressively type-II right censored order statistics ($m_i \in N_0,\ k \in N$) are obtained [cf. Kamps (1995a, b), Kamps and Cramer (2001)].

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The joint pdf of first r generalized order statistics is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1,...,x_r) = c_{r-1}\left(\prod_{i=1}^{r-1}(1-F(x_i))^{m_i}f(x_i)\right)$$
$$\times (1-F(x_r))^{k+(n-r)+M_r-1}f(x_r),$$
$$F^{-1}(0+) < x_1 \leq ... \leq x_r < F^{-1}(1). \quad (2)$$

We may consider two cases here:
Case I: $m_1 = m_2 = ... = m_{n-1} = m$.
Case II: $\gamma_i \neq \gamma_j;\ i \neq j,\ i,\ j = 1,\ 2, ...,\ n-1.$

For **Case I**, the r-th generalized order statistic will be denoted by $X(r,n,m,k)$. The pdf of $X(r,n,m,k)$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \qquad x \in R \quad (3)$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k), 1 \leq r < s \leq n$, is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[1-F(x)]^m f(x)g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s-1}f(y), x < y, \quad (4)$$

where
$c_{r-1} = \prod_{j=1}^{r}\gamma_j, \qquad \gamma_r = k+(n-r)(m+1), \qquad r = 1,2,...,n$

$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1)$

$$h_m(x) = \begin{cases} \frac{-1}{m+1}(1-x)^{m+1} & , m \neq -1 \\ -log(1-x) & , m = -1 \end{cases}$$

(cf. Kamps (1995a, b)).

For **case II**, $X(r,n,\tilde{m},k)$ denotes the r-th generalized order statistic. The pdf of $X(r,n,\tilde{m},k)$ is given by

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1}f(x)\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i-1}, \qquad x \in R, \quad (5)$$

and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq$ n, is given by

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)$$
$$= c_{s-1}\left\{\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\left\{\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right\}$$
$$\times \frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}, \qquad x < y, \quad (6)$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + n - i + M_i, \qquad r = 1, 2, ..., n$$

$$a_i(r) = \prod_{j(\neq i)=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \qquad 1 \le i \le r \le n$$

and

$$a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \qquad r+1 \le i \le s \le n$$

(cf. Kamps and Cramer(2001)).

Further, it can be easily proved that

$$\left.\begin{array}{rc} a_i(r) = & (\gamma_{r+1} - \gamma_i)\, a_i(r+1), \\ c_{r-1} = & \frac{c_r}{\gamma_{r+1}}, \\ \sum_{i=1}^{r+1} a_i(r+1) = & 0, \\ \sum_{i=r+1}^{s} a_i^{(r)}(s) = & 0. \end{array}\right\} \tag{7}$$

Also, for $m_i = m_j = m$, it can be shown that

$$\sum_{i=1}^{r} a_i(r)\, (1 - F(x))^{\gamma_i} = \frac{(1 - F(x))^{\gamma_r}}{(r-1)!} g_m^{r-1}\left(F(x)\right), \tag{8}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{1 - F(y)}{1 - F(x)}\right)^{\gamma_i} = \frac{1}{(s-r-1)!} \left(\frac{1 - F(y)}{1 - F(x)}\right)^{\gamma_s} \times \left(\frac{1}{1 - F(x)}\right)^{(m+1)(s-r-1)} \left[h_m\left(F(y)\right) - h_m\left(F(x)\right)\right]^{s-r-1}. \tag{9}$$

In this paper, in Section 3, we have established recurrence relations for single and product moments of generalized order statistics from Erlang-truncated exponential distribution for Case II only, i.e., for $\gamma_i \neq \gamma_j;\ i \neq j,\ i, j = 1, 2, ..., n-1$. The corresponding results for Case I, i.e., when $m_1 = m_2 = ... = m_{n-1} = m$, can be deduced from the results of Case II as special cases, which have been obtained by Khan, Kumar and Athar (2010).Then we use these results to compute the first four moments of order statistics, record values and second record values for some specific values of the parameters. Next, in Section 4, we use the results on order statistics to obtain BLUEs of location and scale parameters based on type-II right censored samples. In addition, we carry out numerical illustration through Monte Carlo simulations to show the usefulness of the findings.

2. Erlang-truncated exponential distribution

A random variable X is said to have Erlang-truncated exponential distribution if its probability density function (pdf) is of the form

$$f(x) = \beta\left(1 - e^{-\lambda}\right) e^{-\beta x\left(1 - e^{-\lambda}\right)}, \qquad x \ge 0,\ \beta,\ \lambda > 0 \tag{10}$$

and the cumulative distribution function (cdf) is of the form

$$F(x) = 1 - e^{-\beta x\left(1-e^{-\lambda}\right)}. \tag{11}$$

Its characterizing differential equation is given by

$$f(x) = \beta\left(1 - e^{-\lambda}\right)(1 - F(x)). \tag{12}$$

More details on this distribution can be found in El-Alosey (2007).
The pdf of the location-scale parameter Erlang-truncated exponential distribution is

$$f(x) = \beta\left(1 - e^{-\lambda}\right) e^{-\beta\left(\frac{x-\mu}{\sigma}\right)\left(1-e^{-\lambda}\right)}, \qquad x > \mu,\ \beta,\ \lambda,\ \mu,\ \sigma > 0, \tag{13}$$

and the corresponding cdf is of the form

$$F(x) = 1 - e^{-\beta\left(\frac{x-\mu}{\sigma}\right)\left(1-e^{-\lambda}\right)}. \tag{14}$$

3. Recurrence relations for single and product moments of generalized order statistics from Erlang-truncated exponential distribution for Case II

Theorem 3.1. *For the distribution given in* (10) *and* $n \in N,\ k \geq 1$,

$$E\left[X^{j+1}(1,n,\tilde{m},k)\right] = \frac{j+1}{\beta\left(1-e^{-\lambda}\right)\gamma_1} E\left[X^{j}(1,n,\tilde{m},k)\right], \tag{15}$$

and, for $2 \leq r \leq n$,

$$\begin{aligned} E\left[X^{j+1}(r,n,\tilde{m},k)\right] &= \frac{j+1}{\beta\left(1-e^{-\lambda}\right)\gamma_r} E\left[X^{j}(r,n,\tilde{m},k)\right] \\ &\quad + E\left[X^{j+1}(r-1,n,\tilde{m},k)\right]. \end{aligned} \tag{16}$$

Proof. From (5) and (12), we have

$$\begin{aligned} E[X^j(r,n,\tilde{m},k)] &= c_{r-1}\int_0^\infty x^j f(x)\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1}dx \\ &= c_{r-1}\beta(1-e^{-\lambda})\int_0^\infty x^j \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}dx \\ &= c_{r-1}\beta(1-e^{\lambda})\int_0^\infty \frac{x^{j+1}}{(j+1)} \sum_{i=1}^{r} a_i(r)\gamma_i(1-F(x))^{\gamma_i-1}f(x)dx \\ &= \frac{c_{r-1}\beta(1-e^{\lambda})}{j+1}\int_0^\infty x^{j+1} \sum_{i=1}^{r-1} a_i(r)\gamma_i(1-F(x))^{\gamma_i-1}f(x)dx \\ &\quad + \frac{c_{r-1}\beta(1-e^{\lambda})}{j+1}\int_0^\infty x^{j+1} a_r(r)\gamma_r(1-F(x))^{\gamma_r-1}f(x)dx \end{aligned}$$

$$= \frac{-c_{r-1}\beta(1-e^{\lambda})}{j+1}\int_0^{\infty} x^{j+1}\sum_{i=1}^{r-1} a_i(r)(\gamma_r-\gamma_i-\gamma_r)$$
$$\times (1-F(x))^{\gamma_i-1}f(x)dx$$
$$+\frac{c_{r-1}\beta(1-e^{\lambda})}{j+1}\int_0^{\infty} x^{j+1}a_r(r)\gamma_r(1-F(x))^{\gamma_r-1}f(x)dx$$
$$= \frac{-c_{r-1}\beta(1-e^{\lambda})}{j+1}\int_0^{\infty} x^{j+1}\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i-1}f(x)dx$$
$$+\frac{c_{r-1}\beta(1-e^{\lambda})\gamma_r}{j+1}\int_0^{\infty} x^{j+1}\sum_{i=1}^{r-1} a_i(r)(1-F(x))^{\gamma_i-1}f(x)dx$$
$$+\frac{c_{r-1}\beta(1-e^{\lambda})\gamma_r}{j+1}\int_0^{\infty} x^{j+1}a_r(r)(1-F(x))^{\gamma_r-1}f(x)dx.$$

Using (5), we have

$$E[X^j(r,n,\tilde{m},k)] = \frac{-\gamma_r\beta(1-e^{-\lambda})}{(j+1)}E[X^{j+1}(r-1,n,\tilde{m},k)]$$
$$+\frac{\gamma_r\beta(1-e^{-\lambda})}{j+1}E[X^{j+1}(r,n,\tilde{m},k)].$$

Rearranging the terms in the above expression, we get the required result as given in (16). Proceeding in a similar manner, the result given in (15) can easily be established. It may be noted that the relation (15) follows from (16) by taking therein $X(0,n,\tilde{m},k)=0$. □

Remark 3.2. *Putting $m_i = m_j = m$ in* (5) *and using* (8)*, the recurrence relations established in Theorem 3.1 reduce to the recurrence relations for single moments of generalized order statistics from Erlang-truncated exponential distribution for **Case I**, i.e., when $m_1 = m_2 = ... = m_{n-1} = m$, as obtained by Khan, Kumar and Athar (2010).*

Remark 3.3. *If we take $k = 1$ and $m_i = m_j = m = 0$ in* (15) *and* (16)*, we get recurrence relations for single moments of order statistics from Erlang-truncated exponential distribution. Numerical computations for first four moments of order statistics from Erlang-truncated exponential distribution for arbitrarily chosen values of λ and β and for various sample sizes n=1,2,3,4 are given in Tables 1 - 4.*

Remark 3.4. *If we take $k = 1$ and $m_i = -1$, $\forall i = 1, 2, ..., n-1$ in* (15) *and* (16)*, we get recurrence relations for single moments of record values from Erlang-truncated exponential distribution. Numerical computations for first four moments of record values from Erlang-truncated exponential distribution for arbitrarily chosen values of λ and β and for various sample sizes n=1,2,3 are given in Tables 5 - 8.*

Remark 3.5. *If we take $m_i = -1\ \forall\ i = 1, 2, ..., n-1$, $\gamma_r = k$ and $k \in N$ in* (15) *and* (16)*, we get recurrence relations for single moments of k-th record values from Erlang-truncated exponential distribution. Numerical computations for first four moments of 2^{nd} record values from Erlang-truncated exponential distribution for arbitrarily chosen values of λ and β and for various sample sizes n=1,2,3 are given in Tables 9 - 12.*

Table 1: First four moments of order statistics taking $\beta = 1$ and $\lambda = 1$

		i	1	2	3	4
n=1						
r	1	$E(X^i(1,1,0,1))$	1.581977	5.005301	23.75481	150.3182
n=2						
r	1	$E(X^i(1,2,0,1))$	0.790988	1.251325	2.969351	9.394888
	2	$E(X^i(2,2,0,1))$	2.372965	8.759276	44.54026	291.2415
n=3						
r	1	$E(X^i(1,3,0,1))$	0.527326	0.556145	0.879808	1.85578
	2	$E(X^i(2,3,0,1))$	1.318314	2.641686	7.148437	24.4731
	3	$E(X^i(3,3,0,1))$	2.900291	11.81807	63.23618	424.6257
n=4						
r	1	$E(X^i(1,4,0,1))$	0.395494	0.312831	0.371169	0.587181
	2	$E(X^i(2,4,0,1))$	0.92282	1.286084	2.405724	5.66158
	3	$E(X^i(3,4,0,1))$	1.713808	3.997289	11.89115	43.28463
	4	$E(X^i(4,4,0,1))$	3.295785	14.425	80.35118	551.7394

Table 2: First four moments of order statistics taking $\beta = 1$ and $\lambda = 2$

		i	1	2	3	4
n=1						
r	1	$E(X^i(1,1,0,1))$	1.156518	2.675066	9.281283	42.93587
n=2						
r	1	$E(X^i(1,2,0,1))$	0.578259	0.668767	1.16016	2.683492
	2	$E(X^i(2,2,0,1))$	1.734776	4.681366	17.40241	83.18826
n=3						
r	1	$E(X^i(1,3,0,1))$	0.385506	0.29723	0.343751	0.530073
	2	$E(X^i(2,3,0,1))$	0.963765	1.41184	2.792979	6.990331
	3	$E(X^i(3,3,0,1))$	2.120282	6.316128	24.70712	121.2872
n=4						
r	1	$E(X^i(1,4,0,1))$	0.289129	0.167192	0.14502	0.167718
	2	$E(X^i(2,4,0,1))$	0.674635	0.687343	0.939945	1.617135
	3	$E(X^i(3,4,0,1))$	1.252894	2.136338	4.646013	12.36353
	4	$E(X^i(4,4,0,1))$	2.409412	7.709393	31.39416	157.5951

Table 3: First four moments of order statistics taking $\beta = 2$ and $\lambda = 1$

		i	1	2	3	4
n=1						
r	1	$E(X^i(1,1,0,1))$	0.790988	1.251325	2.969352	9.394891
n=2						
r	1	$E(X^i(1,2,0,1))$	0.395494	0.312831	0.371169	0.587181
	2	$E(X^i(2,2,0,1))$	1.186483	2.189819	5.567534	18.2026
n=3						
r	1	$E(X^i(1,3,0,1))$	0.263663	0.139036	0.109976	0.115986
	2	$E(X^i(2,3,0,1))$	0.659157	0.660422	0.893555	1.529569
	3	$E(X^i(3,3,0,1))$	1.450145	2.954518	7.904524	26.53912
n=4						
r	1	$E(X^i(1,4,0,1))$	0.197747	0.078208	0.046396	0.036699
	2	$E(X^i(2,4,0,1))$	0.321521	0.46141	0.300716	0.353849
	3	$E(X^i(3,4,0,1))$	0.856904	0.999322	1.486394	2.70529
	4	$E(X^i(4,4,0,1))$	1.647893	3.60625	10.0439	34.48373

Table 4: First four moments of order statistics taking $\beta = 2$ and $\lambda = 2$

		i	1	2	3	4
n=1						
r	1	$E(X^i(1,1,0,1))$	0.578259	0.668767	1.160161	2.683495
n=2						
r	1	$E(X^i(1,2,0,1))$	0.28913	0.167192	0.14502	0.167718
	2	$E(X^i(2,2,0,1))$	0.867388	1.170342	2.175302	5.199271
n=3						
r	1	$E(X^i(1,3,0,1))$	0.192753	0.074307	0.042969	0.03313
	2	$E(X^i(2,3,0,1))$	0.481883	0.35296	0.349123	0.436896
	3	$E(X^i(3,3,0,1))$	1.060141	1.579033	3.088392	7.580458
n=4						
r	1	$E(X^i(1,4,0,1))$	0.144565	0.041798	0.018128	0.010482
	2	$E(X^i(2,4,0,1))$	0.337318	0.171836	0.117493	0.101071
	3	$E(X^i(3,4,0,1))$	0.626447	0.534085	0.580752	0.772721
	4	$E(X^i(4,4,0,1))$	1.204706	1.927349	3.924272	9.849704

Table 5: First four moments of record values taking $\beta = 1$ and $\lambda = 1$.

	i	1	2	3	4
r=1	$E(X^i(1,n,-1,1))$	1.581977	5.005301	23.75481	150.3182
r=2	$E(X^i(2,n,-1,1))$	3.163953	15.0159	95.01923	751.591
r=3	$E(X^i(3,n,-1,1))$	4.74593	30.03181	237.5481	2254.773

Table 6: First four moments of record values taking $\beta = 1$ and $\lambda = 2$.

	i	1	2	3	4
r=1	$E(X^i(1,n,-1,1))$	1.156517	2.675064	9.281274	42.93582
r=2	$E(X^i(2,n,-1,1))$	2.313035	8.025193	37.1251	214.6791
r=3	$E(X^i(3,n,-1,1))$	3.469552	16.050385	92.81275	644.037

Table 7: First four moments of record values taking $\beta = 2$ and $\lambda = 1$.

	i	1	2	3	4
r=1	$E(X^i(1,n,-1,1))$	0.790988	1.251325	2.969352	9.394891
r=2	$E(X^i(2,n,-1,1))$	1.581977	3.753976	11.87741	46.97446
r=3	$E(X^i(3,n,-1,1))$	2.372966	7.50795	29.69352	140.9234

Table 8: First four moments of record values taking $\beta = 2$ and $\lambda = 2$

	i	1	2	3	4
r=1	$E(X^i(1,n,-1,1))$	0.578259	0.668767	1.160161	2.683495
r=2	$E(X^i(2,n,-1,1))$	1.156518	2.006301	4.640645	13.41747
r=3	$E(X^i(3,n,-1,1))$	1.734777	4.012601	11.60162	40.25242

Table 9: First four moments of 2^{nd} record values taking $\beta = 1$ and $\lambda = 1$.

k=2	i	1	2	3	4
r=1	$E(X^i(1,n,-1,2))$	0.790988	1.251325	2.969351	9.394888
r=2	$E(X^i(2,n,-1,2))$	1.581977	3.753975	11.8774	46.97444
r=3	$E(X^i(3,n,-1,2))$	2.372965	7.50795	29.69351	140.9233

Table 10: First four moments of 2^{nd} record values taking $\beta = 1$ and $\lambda = 2$.

k=2	i	1	2	3	4
r=1	$E(X^i(1,n,-1,2))$	0.578259	0.668766	1.160159	2.683489
r=2	$E(X^i(2,n,-1,2))$	1.156517	2.006298	4.640637	13.41744
r=3	$E(X^i(3,n,-1,2))$	3.469552	8.025193	23.20319	80.50466

Table 11: First four moments of 2^{nd} record values taking $\beta = 2$ and $\lambda = 1$.

k=2	i	1	2	3	4
r=1	$E(X^i(1,n,-1,2))$	0.395494	0.312831	0.371169	0.587181
r=2	$E(X^i(2,n,-1,2))$	0.790988	0.938494	1.484676	2.935904
r=3	$E(X^i(3,n,-1,2))$	2.372965	3.753976	7.423379	17.61542

Table 12: First four moments of 2^{nd} record values taking $\beta = 2$ and $\lambda = 2$.

k=2	i	1	2	3	4
r=1	$E(X^i(1,n,-1,2))$	0.28913	0.167192	0.14502	0.167718
r=2	$E(X^i(2,n,-1,2))$	0.578259	0.501575	0.580081	0.838592
r=3	$E(X^i(3,n,-1,2))$	1.734777	2.006301	2.900403	5.031553

Theorem 3.6. *For* $1 \leq r < s \leq n$ *and* $i\ ,\ j \geq 0$,

$$E\left[X^i(r,n,\tilde{m},k)X^{j+1}(s,n,\tilde{m},k)\right] \\ = \frac{j+1}{\beta\left(1-e^{-\lambda}\right)\gamma_s} E\left[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)\right] \\ + E\left[X^i(r,n,\tilde{m},k)X^{j+1}(s-1,n,\tilde{m},k)\right]. \tag{17}$$

Proof. Using (6), we have

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] \\ = c_{s-1}\int_0^\infty\int_x^\infty x^i y^j \left\{\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\left\{\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right\} \\ \times \frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}dydx \\ = \int_0^\infty x^i \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\frac{f(x)}{1-F(x)}I(x)dx, \tag{18}$$

where

$$I(x) = c_{s-1}\int_x^\infty y^j \sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy.$$

Using (12), we obtain

$$I(x) = \beta(1-e^{-\lambda})c_{s-1}\int_x^\infty y^j \sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} dy.$$

Integrating by parts taking y^j for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned}
I(x) &= \beta(1-e^{-\lambda})c_{s-1}\int_x^\infty \frac{y^{j+1}}{j+1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy\\
&= \frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}\sum_{i=r+1}^{s-1} a_i^{(r)}(s)\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy\\
&\quad+\frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}a_s^{(r)}(s)\gamma_s\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\frac{f(y)}{1-F(y)}dy\\
&= -\frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}\sum_{i=r+1}^{s-1} a_i^{(r)}(s)(\gamma_s-\gamma_i-\gamma_s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\\
&\quad\times\frac{f(y)}{1-F(y)}dy+\frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}a_s^{(r)}(s)\gamma_s\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\\
&\quad\times\frac{f(y)}{1-F(y)}dy\\
&= \frac{-\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}\sum_{i=1}^{s-1} a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy\\
&\quad+\frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}\sum_{i=1}^{s-1} a_i^{(r)}(s)\gamma_s\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy\\
&\quad+\frac{\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}a_s^{(r)}(s)\gamma_s\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\frac{f(y)}{1-F(y)}dy\\
&= \frac{-\beta(1-e^{-\lambda})c_{s-1}}{j+1}\int_x^\infty y^{j+1}\sum_{i=1}^{s-1} a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy\\
&\quad+\frac{\beta(1-e^{-\lambda})c_{s-1}\gamma_s}{j+1}\int_x^\infty y^{j+1}\sum_{i=1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\frac{f(y)}{1-F(y)}dy.
\end{aligned}$$

Putting the value of $I(x)$, so obtained, in (18), we get

$$\begin{aligned}
&E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]\\
&=\frac{\beta(1-e^{-\lambda})\gamma_s}{j+1}\left\{-E[X^i(r,n,\tilde{m},k)X^{j+1}(s-1,n,\tilde{m},k)]\right.\\
&\quad\left.+E[X^i(r,n,\tilde{m},k)X^{j+1}(s,n,\tilde{m},k)]\right\}.
\end{aligned}$$

Rearranging the terms, we get the required result as given in (17). □

Remark 3.7. *Putting $m_i = m_j = m$ in (6) and using (9), the recurrence relation obtained in Theorem 3.6 reduces to the recurrence relation for product moments of generalized order statistics from Erlang-truncated exponential distribution for **Case I**, i.e., when $m_1 = m_2 = ... = m_{n-1} = m$, as obtained by Khan, Kumar and Athar (2010).*

4. BLUEs of μ and σ

Let $X_{1:n} \le X_{2:n} \le ... \le X_{n-c:n}$, $c = 0, 1, ..., n-1$, denote Type-II right-censored sample from the location-scale parameter Erlang-truncated exponential distribution in (13). Let us denote $Z_{r:n} = (X_{r:n} -$

$\mu)/\sigma,\ E(Z_{r:n}) = \mu_{r:n}^{(1)},\ 1 \le r \le (n-c)$, and $Cov(Z_{r:n}, Z_{s:n}) = \sigma_{r,s:n} = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)}\mu_{s:n}^{(1)}, 1 \le r < s \le (n-c)$. We shall use the following notations:

$$X = (X_{1:n}, X_{2:n}, ..., X_{n-c:n})^T,$$

$$\mu = (\mu_{1:n}, \mu_{2:n}, ..., \mu_{n-c:n})^T,$$

$$\mathbf{1} = (1, 1, ..., 1)^T$$

$$and \sum = (\sigma_{r,s:n}),\ 1 \le r,\ s \le n-c.$$

The BLUEs of μ and σ are given by

$$\mu^* = \left\{ \frac{\mu^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1} - \mu^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} X \ = \sum_{r=1}^{n-c} a_r X_{r:n} \tag{19}$$

and

$$\sigma^* = \left\{ \frac{\mathbf{1}^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1} - \mathbf{1}^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} X \ = \sum_{r=1}^{n-c} b_r X_{r:n}, \tag{20}$$

and the variances and covariance of these BLUEs are given by

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\mu^T \sum^{-1} \mu}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1, \tag{21}$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2 \tag{22}$$

and

$$Cov(\mu^*,\ \sigma^*) = \sigma^2 \left\{ \frac{-\mu^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3, \tag{23}$$

respectively, (cf. Arnold et al., 1992).

Tables 13 and 14 display the coefficients of the BLUEs for Type-II right censored samples of sizes n=7, 10 with $\alpha = \beta = 1$ and different censoring cases $c = 0(1)([n/2]-1)$. The coefficients of the BLUEs in Tables 13 and 14 are checked by using the conditions

$$\sum_{r=1}^{n-c} a_r = 1$$

and

$$\sum_{r=1}^{n-c} b_r = 0.$$

Illustrative example: In this example, we show the usefulness of the coefficients of the BLUEs in Tables 13 and 14 by simulating a random sample of order statistics from the Erlang-truncated exponential distribution of size n=10 when $\mu = 0, \alpha = \beta = 1$ as: 0.00556 0.05840 0.06346 0.13194 0.20618 0.21803

Table 13: Coefficients of the BLUE of location parameter .

$\alpha,\ \beta$	n	c	$a_i,\ i = 1, 2, 3, ..., (n-c)$				
1, 1	7	0	0.568117	0.45632	0.352111	0.275443	0.397996
			-1.6765	0.626511			
		1	0.429793	0.346651	0.264901	0.189213	0.160267
			-0.39083				
		2	0.589969	0.453052	0.263837	0.070801	-0.37766
	10	0	0.295909	0.255621	0.215686	0.17638	0.138413
			0.103673	0.080978	0.16285	-0.51903	0.089517
		1	0.288645	0.249365	0.210314	0.171655	0.13385
			0.098044	0.069667	0.10546	-0.327	
		2	0.359782	0.303432	0.246629	0.188934	0.129271
			0.0644	-0.01966	-0.27278		
		3	0.436929	0.35603	0.27236	0.183434	0.08314
			-0.0468	-0.2851			
		4	0.536244	0.413318	0.281946	0.134547	-0.0473
			-0.31876				

Table 14: Coefficients of the BLUE of scale parameter .

$\alpha,\ \beta$	n	c	$b_i, i = 1, 2, 3,\ ...,\ (n-c)$				
1, 1	7	0	0.327467	0.249086	0.1789	0.137805	0.305227
			-1.82211	0.623626			
		1	-0.18978	-0.13992	-0.09209	-0.05197	
			-0.06859	0.542359			
		2	-0.41089	-0.27917	-0.10092	0.107341	0.683656
	10	0	0.12123	0.098041	0.075225	0.053092	0.032436
			0.015355	0.011721	0.125359	-0.618	0.085539
		1	0.114289	0.092063	0.070093	0.048576	0.028075
			0.009976	0.000912	0.070519	-0.4345	
		2	-0.20881	-0.16391	-0.11835	-0.07154	-0.02199
			0.034729	0.117788	0.432077		
		3	-0.33101	-0.24722	-0.1591	-0.06282	0.051078
			0.210859	0.53822			
		4	-0.5185	-0.35537	-0.1772	0.029467	0.297328
			0.724277				

Table 15: Variances and covariance of the BLUEs when $\mu = 0$ and $\sigma = 1$

$\alpha,\ \beta$	n	c	Var (μ^*)	Var(σ^*)	$Cov(\mu^*, \sigma^*)$
1, 1	7	0	0.682576	0.43417285	0.40886318
		1	0.51322	0.26637256	0.24028678
		2	0.707872	0.5297302	0.68075363
	10	0	0.336412	0.16716826	0.14464194
		1	0.328068	0.15954928	0.13666862
		2	0.415885	0.96311522	0.25335595
		3	0.516251	0.56640735	0.41233099
		4	0.654313	1.05845596	0.67297104

0.25672 0.31311 0.85642 1.00280. By using the entries of Tables 13 and 14,we calculate the BLUEs of μ and σ, for the case c=0, to be

$$\begin{aligned}\mu^* &= \sum_{r=1}^{n} a_r X_{r:n}\\ &= 0.29591 \times 0.00556 + 0.25562 \times 0.05840 + 0.21569 \times 0.06346\\ &\quad + 0.17638 \times 0.13194 + 0.13841 \times 0.20618 + 0.10367 \times 0.21803\\ &\quad + 0.08098 \times 0.25672 + 0.16285 \times 0.31311 + (-0.51903) \times 0.85642\\ &\quad + 0.08952 \times 1.00280\\ &= 0.00165 + 0.01493 + 0.01369 + 0.02327 + 0.02854\\ &\quad + 0.02260 + 0.02079 + 0.05099 - 0.44450 + 0.08977\\ &= -0.17828\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= \sum_{r=1}^{n} b_r X_{r:n}\\ &= 0.00556 \times 0.12123 + 0.05840 \times 0.098041 + 0.06346 \times 0.075225\\ &\quad + 0.13194 \times .053092 + 0.20618 \times 0.032436 + 0.21803 \times .015255\\ &\quad + 0.25672 \times .011721 + 0.31311 \times .125359 + 0.85642 \times (-0.618)\\ &\quad + 1.00280 \times .085539\\ &= .030828\end{aligned}$$

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