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Extreme value theory of mixture generalized order statistics

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Abstract. Most statistical approaches assume that all of the data points come from the same distribution. However, in real-life applications the data points come from more than one distribution with no information to identify which observation goes with which distribution. In such cases, the classical extreme value theory can not help us. In this paper, we investigate the extreme value theory of data from more than one distribution based on generalized order statistics under continuous strictly monotone normalization. This paper investigates the asymptotic behaviors of upper and lower extremes generalized order statistics based on a random sample drawn from a finite mixture of distributions normalized by the same continuous strictly monotonic sequence or a mixture of continuous strictly monotonic sequences. For illustration of the usage of our theoretical results, these asymptotic behaviors under linear and power normalization are studied with examples.

1. Introduction

1.1. Generalized order statistics

Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables with common probability density function f and distribution function $\mathcal{F}$. If the first n random variables are arranged in ascending order of magnitude and written as $X_{1:n} < X_{2:n} < ... < X_{n:n}$, we call them ordinary order statistics. Generalized order statistics have been introduced by Kamps (1995) as a unification of several models of ascendingly ordered random variables. The generalized order statistics $X_{1:n}^{(\tilde{m},k)}$, $X_{2:n}^{(\tilde{m},k)}$,..., $X_{n:n}^{(\tilde{m},k)}$ are defined by their probability density function, which is given on the cone $\{(x_1, ..., x_n) : x_0 = \mathcal{F}^{-1}(0) < x_1 \leq ... \leq x_n < \mathcal{F}^{-1}(1) = x^0\}$ as follows

$$f_{1,2,...,n:n}^{(\tilde{m},k)}(x_1, ..., x_n) = kf(x_n)(1 - \mathcal{F}(x_n))^{k-1} \prod_{i=1}^{n-1} \theta_i f(x_i)(1 - \mathcal{F}(x_i))^{m_i},$$

where $\theta_1, ..., \theta_n$ are defined by $\theta_n = k > 0$ and $\theta_j = k + n - j + \sum_{i=j}^{n-1} m_i > 0$, $j = 1, 2, ..., n-1$ and $\tilde{m} = (m_1, ..., m_{n-1}) \in \Re^{n-1}$.

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Generalized order statistics have been widely used and they attracted much attention since Kamps (1995) first introduced them. Generalized order statistics have the advantage of including important well-known models which are discussed separately in literature, such as the ordinary order statistics, sequential order statistics, Progressive type II censored order statistics, record values, kth record values and Pfeifer's records. Particular choice of the parameters $\theta_1, ..., \theta_n$ leads to different models, e.g., ordinary order statistics ($m_1 = ... = m_{n-1} = 0, k = 1$); order statistics with non-integral sample size ($m_1 = ... = m_{n-1} = 0, k = \tau - n + 1$, and $\tau > n - 1$); kth record values ($m_1 = ... = m_{n-1} = -1$ and k is any positive integer) and sequential order statistics ($m_i = (n-i+1)\phi_i - (n-i)\phi_{i+1} - 1, 1 \leq i \leq n-1, k = \phi_n$ and $\phi_1, \phi_2, ..., \phi_n > 0$).

In this paper, we consider a wide subclass of generalized order statistics by assuming $\theta_j - \theta_{j+1} = m+1 > 0$, i.e., $m_1 = ... = m_{n-1} = m$. This subclass is known as m-generalized order statistics, we call it symmetric generalized order statistics throughout this paper. Many important practical models, such as ordinary order statistics, record values, sequential order statistics and order statistics with non-integral sample size, are included in the symmetric generalized order statistics. The distribution function of the ℓth lower and the ℓth upper symmetric generalized order statistics are represented by

$$\Xi_{\ell:n}^{(m,k)}(x) = I_{1-(1-\mathcal{F}(x))^{m+1}}\left(\ell, \frac{k}{m+1} + n - \ell\right) \tag{1}$$

and

$$\Xi_{n-\ell+1:n}^{(m,k)}(x) = I_{1-(1-\mathcal{F}(x))^{m+1}}\left(n - \ell + 1, \frac{k}{m+1} + \ell - 1\right), \tag{2}$$

respectively, where $I_x(n,m) = \frac{1}{B(n,m)} \int_0^x t^{n-1}(1-t)^{m-1}dt$, $B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$ and $\Gamma(.)$ is the gamma function (c.f. Nasri-Roudsari, 1996).

1.2. Classical extreme value theory

The main problem of the classical extreme value theory of ordinary order statistics is to find conditions on the distribution function $\mathcal{F}$ under which there is a strongly monotone continuous transformation $\mathcal{B}_n(x)$, such that for the ℓth upper extreme ordinary order statistic the distribution function of $\mathcal{B}_n^{-1}(X_{n-\ell+1:n})$ (for the ℓth lower extreme ordinary order statistic the distribution function of $\mathcal{B}_n^{-1}(X_{\ell:n})$) converges weakly to a non-degenerate distribution (i.e., the distribution is not degenerate at 0 or 1), to obtain the classes of possible limit distribution functions and to give methods for calculating the normalizing transformation $\mathcal{B}_n(x)$. Several authors have studied this problem for the linear transformation $\mathcal{B}_n(x) = a_n x + b_n$, such as Gnedenko (1943), Balkema and de Haan (1972) and Smirnov (1952). The central result of the classical extreme value theory turns out that the class of possible limit distributions has essentially three different types of distributions. For some suitable normalizing constants $c_n, a_n > 0$ and $d_n, b_n \in \Re$, we have

$$\Xi_{\ell:n}^{(0,1)}(c_n x + d_n) \xrightarrow[n]{w} \mathcal{H}_{i,\alpha}^{l(0,1)}(x) = 1 - \frac{1}{\Gamma(\ell)} \int_{\mathcal{U}_{i,\alpha}(x)}^{\infty} t^{\ell-1} e^{-t} dt \tag{3}$$

and

$$\Xi_{n-\ell+1:n}^{(0,1)}(a_n x + b_n) \xrightarrow[n]{w} \mathcal{H}_{j,\beta}^{u(0,1)}(x) = \frac{1}{\Gamma(\ell)} \int_{\mathcal{V}_{j,\beta}(x)}^{\infty} t^{\ell-1} e^{-t} dt, \tag{4}$$

if, and only if, $n\mathcal{F}(c_n x + d_n) \xrightarrow[n]{} \mathcal{U}_{i,\alpha}(x)$ and $n(1 - \mathcal{F}(a_n x + b_n)) \xrightarrow[n]{} \mathcal{V}_{j,\beta}(x)$, respectively at all continuity points of $\mathcal{U}_{i,\alpha}(x)$ and $\mathcal{V}_{j,\beta}(x)$, where $\xrightarrow[n]{w}$ denotes the weak convergence, as $n \to \infty$, $\xrightarrow[n]{}$ means the limit as $n \longrightarrow \infty$,

$$\mathcal{U}_{1,\alpha}(x) = \begin{cases} (-x)^{-\alpha}, & x < 0,\ \alpha > 0, \\ \infty, & x \geq 0, \end{cases} \quad \mathcal{U}_{2,\alpha}(x) = \begin{cases} x^{\alpha}, & x \geq 0,\ \alpha > 0, \\ 0, & x < 0, \end{cases}$$

$$\mathcal{V}_{1,\beta}(x)=\begin{cases} x^{-\beta}, & x>0,\ \beta>0,\\ \infty, & x\le 0,\end{cases}\quad \mathcal{V}_{2,\beta}(x)=\begin{cases} (-x)^{\beta}, & x\le 0,\ \beta>0,\\ 0, & x>0\end{cases}$$

and $\mathcal{U}_{3,0}(x)=\mathcal{V}_{3,0}(-x)=e^x, -\infty\le x\le\infty$. In this case, we say that the distribution function $\mathcal{F}$ belongs to the domain of attraction of each of the limits $\mathcal{H}^{l(0,1)}_{i,\alpha}$ and $\mathcal{H}^{u(0,1)}_{j,\beta}$ under linear normalization, written $\mathcal{F}\in\mathbb{LDA}(\mathcal{H}^{l(0,1)}_{i,\alpha})$ and $\mathcal{F}\in\mathbb{LDA}(\mathcal{H}^{u(0,1)}_{j,\beta})$, respectively.

The following theorem, due to Nasri-Roudsari (1996) (cf. also Nasri-Roudsari and Cramer, 1999), extends the above results to the symmetric generalized order statistics.

Theorem 1.1. *For any symmetric generalized order statistics, let $m>-1,\ k,c_n,a_n>0,\ d_n,b_n\in\Re,\ \mathcal{A}_{n(m+1)}(x)=c_{n(m+1)}x+d_{n(m+1)},\ \mathcal{B}_{n^{1/(m+1)}}(x)=a_{n^{1/(m+1)}}x+b_{n^{1/(m+1)}}$ and ℓ be a fixed integer $1\le\ell\le n$. Then, we have*

$$\Xi^{(m,k)}_{\ell:n}\big(\mathcal{A}_{n(m+1)}(x)\big)\ \xrightarrow[n]{w}\ \mathcal{H}^{l(m,k)}_{i,\alpha}(x)=1-\frac{1}{\Gamma(\ell)}\int_{\mathcal{U}_{i,\alpha}(x)}^{\infty}t^{\ell-1}e^{-t}dt$$

and

$$\Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x))\ \xrightarrow[n]{w}\ \mathcal{H}^{u(m,k)}_{j,\beta}(x)=\frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{\mathcal{V}^{m+1}_{j,\beta}(x)}^{\infty}t^{\frac{k}{m+1}+\ell-2}e^{-t}dt,$$

if, and only if, $\mathcal{F}\in\mathbb{LDA}(\mathcal{H}^{l(0,1)}_{i,\alpha})$ and $\mathcal{F}\in\mathbb{LDA}(\mathcal{H}^{u(0,1)}_{j,\beta})$, respectively.

Pantcheva (1985) (cf. also Mohan and Ravi, 1992) introduced a non-linear normalization for the maximal ordinary order statistics $\mathcal{B}_n(x)=\alpha_n|x|^{\beta_n}\mathcal{S}(x)$ known as the power normalization, where $\mathcal{S}(x)=-1,0,1$, if $x<0, x=0, x>0$, respectively. The limit distributions obtained with non-linear normalization attract more distributions than the linear normalization. The following slight generalization of the results in Pantcheva (1985) for the ℓth upper extreme ordinary order statistics under power normalization is given by Barakat and Nigm (2002). For some suitable normalizing constants $\alpha_n>0$ and $\beta_n>0$, we have

$$\Xi^{(0,1)}_{n-\ell+1:n}(\alpha_n|x|^{\beta_n}\mathcal{S}(x))=I_{\mathcal{F}(\mathcal{B}_n(x))}(n-\ell+1,\ell)\ \xrightarrow[n]{w}\ Q^{(0,1)}_{j,\delta}=\frac{1}{\Gamma(\ell)}\int_{\varphi_{j,\delta}(x)}^{\infty}t^{\ell-1}e^{-t}dt,$$

if, and only if, $n\left(1-\mathcal{F}\left(\alpha_n|x|^{\beta_n}\mathcal{S}(x)\right)\right)\ \xrightarrow[n]{}\ \varphi_{j,\delta}(x)$ at all continuity points of $\varphi_{j,\delta}(x)$, where

$$\varphi_{1,\delta}(x)=\begin{cases}\infty, & x\le 1,\\ (\log x)^{-\delta}, & x>1,\delta>0,\end{cases}\quad \varphi_{2,\delta}(x)=\begin{cases}\infty, & x\le 0,\\ (-\log x)^{\delta}, & 0<x\le 1,\delta>0,\\ 0, & x>1,\end{cases}$$

$$\varphi_{3,\delta}(x)=\begin{cases}\infty, & x\le -1,\\ (-\log(-x))^{-\delta}, & -1<x\le 0,\delta>0,\\ 0, & x>0,\end{cases}\quad \varphi_{4,\delta}(x)=\begin{cases}(\log(-x))^{\delta}, & x\le -1,\delta>0,\\ 0, & x>-1,\end{cases}$$

$\varphi_{5,\delta}(x)=\varphi_5(x)=\mathcal{V}_{1,1}(x)$ and $\varphi_{6,\delta}(x)=\varphi_6(x)=\mathcal{V}_{2,1}(x)$. In this case, we say that the distribution function $\mathcal{F}$ belongs to the domain of attraction of $Q^{(0,1)}_{j,\delta}$ under power normalization, written $\mathcal{F}\in\mathbb{PDA}(Q^{(0,1)}_{j,\delta})$.

The following theorem, due to Nasri-Roudsari (1999), extends the above results to the symmetric generalized order statistics.

Theorem 1.2. *For any symmetric generalized order statistics, let $m>-1,\ k,\alpha_n,\beta_n>0$ and ℓ be a fixed integer $1\le\ell\le n$. Then, we have*

$$\Xi^{(m,k)}_{n-\ell+1:n}\left(\alpha_{n^{1/(m+1)}}|x|^{\beta_{n^{1/(m+1)}}}\mathcal{S}(x)\right)\ \xrightarrow[n]{w}\ Q^{(m,k)}_{j,\delta}=\frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{\varphi^{m+1}_{j,\delta}(x)}^{\infty}t^{\frac{k}{m+1}+\ell-2}e^{-t}dt,$$

if, and only if, the distribution function $\mathcal{F}$ satisfies $\mathcal{F}\in\mathbb{PDA}(Q^{(0,1)}_{j,\delta})$.

1.3. Finite mixture distribution

The classical extreme value theory assumes that all the observations of a sample come from the same distribution (i.e., homogeneous population). However, in real-life projects the observations come from more than one distribution (i.e., heterogeneous population) with no information to identify which observation goes with which distribution. In this mixture, the ith component (distribution function of the ith sub-population) is $\mathcal{F}_i(x)$ and the mixing proportions $\mathcal{P}_i > 0, 1 \leq i \leq \zeta$ are such that $\sum_{i=1}^{\zeta} \mathcal{P}_i = 1$. In such cases, the distribution function is called a finite mixture distribution and is given by $\mathcal{F}(x) = \sum_{i=1}^{\zeta} \mathcal{P}_i \mathcal{F}_i(x)$. The finite mixture distribution provides a natural representation of heterogeneity in many real-life data from a mixture of distributions. The finite mixture distribution has received increasing attention in recent years and has proven to be a useful approach in modeling heterogeneous data. The reader can refer to Everitt and Hand (1981), Titterington et al. (1985), Maclachlan and Basford (1988), Lindsay (1995) and Elsawah (2014) for some properties and applications of finite mixture distribution.

In the last few years, much attention has been paid in exploring the potential application of the extreme value theory to the finite mixture distribution. AL-Hussaini and El-Adll (2004) studied the asymptotic distribution of the maximum ordinary order statistics under finite mixture distribution. Subsequently, Sreehari and Ravi (2010) gave a closer look at the asymptotic distribution of the maximum ordinary order statistics under finite mixture distribution with some generalizations. Gwak et al. (2016) studied the asymptotic distribution of the maximum ordinary order statistics under finite mixture distribution and a statistical method to control the possible bias. Finally, Alawady et al. (2016) investigated the asymptotic behavior of the appropriately linear normalized coordinatewise maximum and minimum under multivariate finite mixture distribution from independent, but not obligatory identically distributed random vectors.

In this paper, we extend the classical extreme value theory to the case of finite mixture distribution based on symmetric generalized order statistics under continuous strictly monotone normalization. This paper provides the possible limit distributions of upper and lower extremes symmetric generalized order statistics based on a random sample drawn from a finite of mixture distributions normalized by the same continuous strictly monotonic sequence or different continuous strictly monotonic sequences. For illustration of the usage of our theoretical results, the asymptotic behaviors of mixture symmetric generalized order statistics under linear and power normalization are studied with some examples.

2. Mixture extreme value theory

The possible non-degenerate limit distribution functions of the ℓth upper extreme and the ℓth lower extreme symmetric generalized order statistics based on a random sample drawn from a finite of mixture distributions normalized by the same continuous strictly monotonic sequence, as well as sufficient conditions for the existence of these limit distribution functions are given in Theorems 2.6 and 2.7, respectively. Theorems 2.13 and 2.14 are intended to extend the results of Theorems 2.6 and 2.7, by considering a finite of mixture distributions where the mixing distributions for the purpose of limiting distributions could be normalized by different continuous strictly monotonic sequences. The proof of the results for the ℓth upper extreme depends on the following Lemmas 2.1 and 2.2, which are simple applications of Theorem 1.5.1 in Leadbetter et al. (1983) (cf. also Lemma 3.1 and Corollary 3.2 in Nasri-Roudsari, 1996) and Theorem 3 in Smirnov (1952) (cf. also Lemma 3.1 in Barakat, 1997), respectively. The proof of the results for the ℓth lower extreme depends on the following Lemmas 2.3 and 2.4, which are simple extensions of Lemmas 2.1 and 2.2 respectively.

Lemma 2.1. *For any distribution function $\mathbb{F}$ and non-degenerate distribution function $\mathcal{G}$, let $a_n > 0,\ b_n \in \Re,\ n \in \mathbb{N}$, $a(t) = a_{\lfloor t \rfloor},\ b(t) = b_{\lfloor t \rfloor}$ and $\lfloor t \rfloor$ be the integral part of t. Then, the following statements are equivalent:*

- $\mathbb{F}^n(a_n x + b_n) \xrightarrow[n]{w} \mathcal{G}(x)$ *for all continuity points of $\mathcal{G}$.*

- $n[1-\mathbb{F}(a_nx+b_n)] \underset{n}{\longrightarrow} -log(\mathcal{G}(x))$ *for all continuity points of* $\mathcal{G}$.
- $\mathbb{F}^t(a(t)x+b(t)) \underset{t}{\xrightarrow{w}} \mathcal{G}(x)$ *for all continuity points of* $\mathcal{G}$.
- $t[1-\mathbb{F}(a(t)x+b(t))] \underset{t}{\longrightarrow} -log(\mathcal{G}(x))$ *for all continuity points of* $\mathcal{G}$.

Lemma 2.2. *For any distribution function* $\mathbb{F}$*, let* $\mathcal{R}$ *be a fixed integer* $1 \leq \mathcal{R} \leq \mathcal{N}$*,* $\mathcal{N}(1-\mathbb{F}) \underset{\mathcal{N}}{\longrightarrow} \lambda < \infty$*,* $\Gamma(x,y) = \int_y^\infty t^{x-1}e^{-t}dt$ *and* $\sigma_\mathcal{N}$ *and* $\rho_\mathcal{N} \underset{\mathcal{N}}{\longrightarrow} 0$*. Then, for large values of* $\mathcal{N}$ *we have*

$$\frac{1}{\Gamma(\mathcal{R})}\Gamma\left(\mathcal{R},\mathcal{N}(1-\mathbb{F}(x))\right) - \sigma_\mathcal{N} \leq I_{\mathbb{F}(x)}(\mathcal{N}-\mathcal{R}+1,\mathcal{R}) \leq \frac{1}{\Gamma(\mathcal{R})}\Gamma\left(\mathcal{R},\mathcal{N}(1-\mathbb{F}(x))\right) + \rho_\mathcal{N}.$$

Lemma 2.3. *For any distribution function* $\mathbb{F}$ *and non-degenerate distribution function* $\mathcal{J}$*, let* $c_n > 0$*,* $d_n \in \Re$*,* $n \in \mathbb{N}$*,* $c(t) = c_{\lfloor t \rfloor}$*,* $d(t) = d_{\lfloor t \rfloor}$ *and* $\lfloor t \rfloor$ *is the integral part of* t*. Then, the following statements are equivalent:*

- $1-[1-\mathbb{F}(c_nx+d_n)]^n \underset{n}{\xrightarrow{w}} \mathcal{J}(x)$ *for all continuity points of* $\mathcal{J}$.
- $n\mathbb{F}(c_nx+d_n) \underset{n}{\longrightarrow} -\log(\mathcal{J}(x))$ *for all continuity points of* $\mathcal{J}$.
- $1-[1-\mathbb{F}(c(t)x+d(t))]^t \underset{t}{\xrightarrow{w}} \mathcal{J}(x)$ *for all continuity points of* $\mathcal{J}$.
- $t\mathbb{F}(c(t)x+d(t)) \underset{t}{\longrightarrow} -\log(\mathcal{J}(x))$ *for all continuity points of* $\mathcal{J}$.

Lemma 2.4. *For any distribution function* $\mathbb{F}$*, let* $\mathcal{N}\mathbb{F} \underset{\mathcal{N}}{\longrightarrow} \gamma < \infty$ *and* $\Theta_\mathcal{N}$ *and* $\Omega_\mathcal{N} \underset{\mathcal{N}}{\longrightarrow} 0$*. Then, for large values of* $\mathcal{N}$ *we have*

$$1-\frac{1}{\Gamma(\mathcal{R})}\Gamma\left(\mathcal{R},\mathcal{N}\mathbb{F}(x)\right) - \Theta_\mathcal{N} \leq I_{\mathbb{F}(x)}(\mathcal{R},\mathcal{N}-\mathcal{R}+1) \leq 1-\frac{1}{\Gamma(\mathcal{R})}\Gamma\left(\mathcal{R},\mathcal{N}\mathbb{F}(x)\right) + \Omega_\mathcal{N}.$$

Remark 2.5. *The same results in Lemmas 2.1 and 2.3 hold with* a_nx+b_n *and* c_nx+d_n *replaced by any continuous strictly monotonic sequence* $\mathcal{B}_n(x)$.

2.1. Mixture extreme value theory via the same normalization

Theorem 2.6. *For any symmetric generalized order statistics from a mixture of* ζ *distributions* $\mathcal{F}_i(x)$ *with mixture distribution function* $\mathcal{F}(x) = \sum_{i=1}^{\zeta} \mathcal{P}_i\mathcal{F}_i(x)$*,* $0 < \mathcal{P}_i < 1$ *and* $\sum_{i=1}^{\zeta} \mathcal{P}_i = 1$*, let* $m > -1$*,* $k > 0$*,* ℓ *be a fixed integer* $1 \leq \ell \leq n$ *and* $\mathbb{L}(\mathcal{K}_i(x))$ *and* $\mathbb{R}(\mathcal{K}_i(x))$ *be the left and the right end points of a non-degenerate distribution function* $\mathcal{K}_i(x)$*, respectively. If* $\mathcal{F}_i(x)$ *are normalized by the same strongly monotone continuous sequence* $\{\mathcal{B}_n(x)\}_{n=1}^\infty$ *and*

$$\mathcal{F}_i^n(\mathcal{B}_n(x)) \underset{n}{\xrightarrow{w}} \mathcal{K}_i(x), \tag{5}$$

then when $\max_{1\leq i\leq\zeta} \mathbb{L}(\mathcal{K}_i(x)) < x < \max_{1\leq i\leq\zeta} \mathbb{R}(\mathcal{K}_i(x))$ *we get*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \quad \underset{n}{\xrightarrow{w}} \quad \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{(-1)^{m+1}\left(\sum_{i=1}^{\zeta}\log(\mathcal{K}_i(x))^{\mathcal{P}_i}\right)^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$$

Proof. Since (5) holds, from Lemma 2.1 and Remark 2.5 for $1 \leq i \leq \zeta$ we have

$$n^{1/(m+1)}(1-\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x))) \underset{n}{\longrightarrow} -\log\mathcal{K}_i(x). \tag{6}$$

It then follows, since $\sum_{i=1}^{\zeta} \mathcal{P}_i = 1$ and $\mathcal{F}(x) = \sum_{i=1}^{\zeta} \mathcal{P}_i\mathcal{F}_i(x)$ that

$$1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)) = \sum_{i=1}^{\zeta} \mathcal{P}_i\left(1-\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x))\right). \tag{7}$$

Therefore, from (6) and (7) we get

$$\begin{aligned}\left(\frac{k}{m+1}+n-1\right)(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))^{m+1} &\sim n(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))^{m+1}\\ &\sim \left[n^{1/(m+1)}(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))\right]^{m+1}\\ &\sim \left[\sum_{i=1}^{\zeta}\mathcal{P}_i n^{1/(m+1)}\left(1-\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x))\right)\right]^{m+1}\\ &\underset{n}{\longrightarrow} \left(\sum_{i=1}^{\zeta}-\mathcal{P}_i\log\mathcal{K}_i(x)\right)^{m+1}. \qquad (8)\end{aligned}$$

From (2) and (8) and by taking $\mathbb{F}=1-(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))^{m+1}$, $\mathcal{N}=\frac{k}{m+1}+n-1$ and $\mathcal{R}=\frac{k}{m+1}+\ell-1$ in Lemma 2.2, the proof can be completed. $\square$

Theorem 2.7. *For any symmetric generalized order statistics from a mixture of ζ distributions $\mathcal{F}_i(x)$ with mixture distribution function $\mathcal{F}(x)=\sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(x)$, $0<\mathcal{P}_i<1$ and $\sum_{i=1}^{\zeta}\mathcal{P}_i=1$, let $m>-1$, $k>0$, ℓ be a fixed integer $1\le\ell\le n$ and $\mathbb{L}(\mathcal{E}_i(x))$ and $\mathbb{R}(\mathcal{E}_i(x))$ be the left and the right end points of a non-degenerate distribution function $\mathcal{E}_i(x)$, respectively. If $\mathcal{F}_i(x)$ are normalized by the same strongly monotone continuous sequence $\{\mathcal{A}_n(x)\}_{n=1}^{\infty}$ and*

$$1-(1-\mathcal{F}_i(\mathcal{A}_n(x)))^n \xrightarrow[n]{w} \mathcal{E}_i(x), \qquad (9)$$

then when $\max_{1\le i\le\zeta}\mathbb{L}(\mathcal{E}_i(x))<x<\max_{1\le i\le\zeta}\mathbb{R}(\mathcal{E}_i(x))$ we have

$$\Xi_{\ell:n}^{(m,k)}(\mathcal{A}_{n(m+1)}(x)) \xrightarrow[n]{w} 1-\frac{1}{\Gamma(\ell)}\int_{\sum_{i=1}^{\zeta}\log(\mathcal{E}_i(x))^{-\mathcal{P}_i}}^{\infty} t^{\ell-1}e^{-t}dt.$$

Proof. Since (9) holds, from Lemma 2.3 and Remark 2.5 we get

$$n(m+1)\mathcal{F}_i(\mathcal{A}_{n(m+1)}(x)) \underset{n}{\longrightarrow} -\log\mathcal{E}_i(x). \qquad (10)$$

It then follows, since $\sum_{i=1}^{\zeta}\mathcal{P}_i=1$ and $\mathcal{F}(\mathcal{A}_{n(m+1)}(x))=\sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(\mathcal{A}_{n(m+1)}(x))$ that

$$\mathcal{F}(\mathcal{A}_{n(m+1)}(x))=1-\sum_{i=1}^{\zeta}\mathcal{P}_i\left(1-\mathcal{F}_i(\mathcal{A}_{n(m+1)}(x))\right). \qquad (11)$$

From Maclaurin series, we have

$$(1-\mathcal{F}(\mathcal{A}_{n(m+1)}(x)))^{m+1}=1-(m+1)\mathcal{F}(\mathcal{A}_{n(m+1)}(x))(1+\circ(1))\sim 1-(m+1)\mathcal{F}(\mathcal{A}_{n(m+1)}(x)), \qquad (12)$$

where $\circ(1) \underset{n}{\longrightarrow} 0$. Therefore, from (10), (11) and (12) we get

$$\begin{aligned}\left(\frac{k}{m+1}+n-1\right)\left(1-(1-\mathcal{F}(\mathcal{A}_{n(m+1)}(x)))^{m+1}\right) &\sim n\left(1-(1-\mathcal{F}(\mathcal{A}_{n(m+1)}(x)))^{m+1}\right)\\ &\sim n(m+1)\mathcal{F}(\mathcal{A}_{n(m+1)}(x))\\ &\sim \sum_{i=1}^{\zeta}\mathcal{P}_i n(m+1)\mathcal{F}_i(\mathcal{A}_{n(m+1)}(x))\\ &\underset{n}{\longrightarrow} \sum_{i=1}^{\zeta}-\mathcal{P}_i\log\mathcal{E}_i(x). \qquad (13)\end{aligned}$$

From (1) and (13) and by taking $\mathbb{F}=1-(1-\mathcal{F}(\mathcal{A}_{n(m+1)}(x)))^{m+1}$, $\mathcal{N}=\frac{k}{m+1}+n-1$ and $\mathcal{R}=\ell$ in Lemma 2.4, the proof can be completed. $\square$

Corollary 2.8. *From Theorems 2.6 and 2.7, we have*

$$\Xi^{(0,1)}_{n-\ell+1:n}(\mathcal{B}_n(x)) \text{ and } \Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \begin{cases} 0, & \text{if } x \leq \max_{1\leq i\leq\zeta} \mathbb{L}(\mathcal{K}_i(x)), \\ 1, & \text{if } x \geq \max_{1\leq i\leq\zeta} \mathbb{R}(\mathcal{K}_i(x)) \end{cases}$$

and

$$\Xi^{(0,1)}_{\ell:n}(\mathcal{A}_n(x)) \text{ and } \Xi^{(m,k)}_{\ell:n}(\mathcal{A}_{n(m+1)}(x)) \xrightarrow[n]{w} \begin{cases} 0, & \text{if } x \leq \max_{1\leq i\leq\zeta} \mathbb{L}(\mathcal{E}_i(x)), \\ 1, & \text{if } x \geq \max_{1\leq i\leq\zeta} \mathbb{R}(\mathcal{E}_i(x)). \end{cases}$$

Corollary 2.9. *From Theorems 2.6 and 2.7, we have the following special cases. It is worth noting that, the special cases (iii) in (I) and (II) can be found in AL-Hussaini and El-Adll (2004) and Sreehari and Ravi (2010).*

(I) *For ordinary order statistics, i.e., $m = 0$ and $k = 1$, we get*

- **(i)** $\Xi^{(0,1)}_{n-\ell+1:n}(\mathcal{B}_n(x)) \xrightarrow[n]{w} \frac{1}{\Gamma(\ell)}\int_{\sum_{i=1}^{\zeta}\log(\mathcal{K}_i(x))^{-\mathcal{P}_i}}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(ii)** $\Xi^{(0,1)}_{\ell:n}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \frac{1}{\Gamma(\ell)}\int_{\sum_{i=1}^{\zeta}\log(\mathcal{E}_i(x))^{-\mathcal{P}_i}}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(iii)** $\Xi^{(0,1)}_{n:n}(\mathcal{B}_n(x)) \xrightarrow[n]{w} \prod_{i=1}^{\zeta}(\mathcal{K}_i(x))^{\mathcal{P}_i}.$
- **(iv)** $\Xi^{(0,1)}_{1:n}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \prod_{i=1}^{\zeta}(\mathcal{E}_i(x))^{\mathcal{P}_i}.$

(II) *If the observations come from the same distribution, i.e., $\mathcal{K}_i(x) = \mathcal{K}(x)\ \forall i$ and $\mathcal{E}_i(x) = \mathcal{E}(x)\ \forall i$, then*

- **(i)** $\Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{(-1)^{m+1}(\log\mathcal{K}(x))^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$
- **(ii)** $\Xi^{(0,1)}_{\ell:n}(\mathcal{A}_n(x))$ *and* $\Xi^{(m,k)}_{\ell:n}(\mathcal{A}_{n(m+1)}(x)) \xrightarrow[n]{w} 1 - \frac{1}{\Gamma(\ell)}\int_{-\log\mathcal{E}(x)}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(iii)** $\Xi^{(0,1)}_{n:n}(\mathcal{B}_n(x)) \xrightarrow[n]{w} \mathcal{K}(x).$
- **(iv)** $\Xi^{(0,1)}_{1:n}(\mathcal{A}_n(x))$ *and* $\Xi^{(m,k)}_{1:n}(\mathcal{A}_{n(m+1)}(x)) \xrightarrow[n]{w} 1 - \mathcal{E}(x).$

Corollary 2.10. *For any symmetric generalized order statistics from a mixture of ζ distributions normalized by the same linear sequence $\mathcal{B}_n(x) = a_nx + b_n$, let $m > -1,\ k, a_n > 0,\ b_n \in \Re,\ \ell$ be a fixed integer $1 \leq \ell \leq n$ and $\mathbb{L}(\mathcal{K}_i(x))$ and $\mathbb{R}(\mathcal{K}_i(x))$ be the left and the right end points of a non-degenerate distribution function $\mathcal{K}_i(x)$, respectively. Then, when $\max_{1\leq i\leq\zeta}\mathbb{L}(\mathcal{K}_i(x)) < x < \max_{1\leq i\leq\zeta}\mathbb{R}(\mathcal{K}_i(x))$, we get*

(I) *Let $\mathcal{F}(x) = \mathcal{P}\mathcal{F}_1(x)+(1-\mathcal{P})\mathcal{F}_2(x)$, where $0 < \mathcal{P} < 1,\ \mathcal{F}_1 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{1,\beta}(x))$ and $\mathcal{F}_2 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{2,\beta}(x))$. Then, we get $\mathcal{F}(x) \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{1,\beta}(x))$. That is, for $x > 0$ we get*

$$\Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{(\mathcal{P}x^{-\beta})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$$

(II) *Let $\mathcal{F}(x) = \mathcal{P}\mathcal{F}_1(x)+(1-\mathcal{P})\mathcal{F}_2(x)$, where $0 < \mathcal{P} < 1,\ \mathcal{F}_1 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{1,\beta}(x))$ and $\mathcal{F}_2 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{3,\beta}(x))$. Then for $x > 0$, we get the following non-degenerate distribution*

$$\Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{(\mathcal{P}(x^{-\beta}-e^{-x})+e^{-x})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$$

(III) *Let $\mathcal{F}(x) = \mathcal{P}\mathcal{F}_1(x)+(1-\mathcal{P})\mathcal{F}_2(x)$, where $0 < \mathcal{P} < 1,\ \mathcal{F}_1 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{2,\beta}(x))$ and $\mathcal{F}_2 \in \mathbb{LDA}(\mathcal{H}^{u(0,1)}_{3,\beta}(x))$. Then, we have $\Xi^{(m,k)}_{n-\ell+1:n}(\mathcal{B}_{n^{1/(m+1)}}(x))$*

$$\xrightarrow[n]{w} \begin{cases} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{(e^{-x}+\mathcal{P}((-x)^{\beta}-e^{-x}))^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt, & x \leq 0, \\ \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{((1-\mathcal{P})e^{-x})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt, & x > 0. \end{cases}$$

(IV) *Let* $\mathcal{F}(x) = \mathcal{P}_1\mathcal{F}_1(x) + \mathcal{P}_2\mathcal{F}_2(x) + \mathcal{P}_3\mathcal{F}_3(x)$, *where* $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = 1,\ 0 < \mathcal{P}_1,\ \mathcal{P}_2,\ \mathcal{P}_3 < 1,\ \mathcal{F}_1 \in \mathbb{LDA}(\bar{\mathcal{H}}_{1,\beta}^{(m,k)}(x)),\ \mathcal{F}_2 \in \mathbb{LDA}(\mathcal{H}_{2,\beta}^{u(0,1)}(x))$ *and* $\mathcal{F}_3 \in \mathbb{LDA}(\mathcal{H}_{3,\beta}^{u(0,1)}(x))$.

Then for $x > 0$, $\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{(\mathcal{P}_1 x^{-\beta}+\mathcal{P}_3 e^{-x})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt.$

Proof. The proof is obvious from Theorems 1.1 and 2.6 with simple algebra. We will provide the proof for the first case (I) and the proof for the other cases will be by the same technique. Let $\mathcal{F}(x) = \mathcal{P}\mathcal{F}_1(x) + (1-\mathcal{P})\mathcal{F}_2(x)$, where $0 < \mathcal{P} < 1,\ \mathcal{F}_1 \in \mathbb{LDA}(\mathcal{H}_{1,\beta}^{u(0,1)}(x))$ and $\mathcal{F}_2 \in \mathbb{LDA}(\mathcal{H}_{2,\beta}^{u(0,1)}(x))$. Then, from Theorems 1.1 and 2.6 we get

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \quad \xrightarrow[n]{w} \quad \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{(-1)^{m+1}(-\mathcal{P}\mathcal{V}_{1,\beta}(x)-(1-\mathcal{P})\mathcal{V}_{2,\beta}(x))^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt.$$

From (4), we get

$$\mathcal{P}\mathcal{V}_{1,\beta}(x) + (1-\mathcal{P})\mathcal{V}_{2,\beta}(x) = \begin{cases} \mathcal{P}x^{-\beta}, & x > 0,\ \beta > 0, \\ \infty, & x \le 0, \end{cases} + \begin{cases} (1-\mathcal{P})(-x)^{\beta}, & x \le 0,\ \beta > 0, \\ 0, & x > 0 \end{cases}$$

Thus, we get

$$\mathcal{P}\mathcal{V}_{1,\beta}(x) + (1-\mathcal{P})\mathcal{V}_{2,\beta}(x) = \begin{cases} \mathcal{P}x^{-\beta}, & x > 0,\ \beta > 0, \\ \infty, & x \le 0. \end{cases}$$

□

Corollary 2.11. *For any symmetric generalized order statistics from a mixture of* ζ *distributions normalized by the same strongly monotone continuous power sequence* $\mathcal{B}_n(x) = \alpha_n|x|^{\beta_n}\mathcal{S}(x)$, *let* $m > -1,\ k, \alpha_n, \beta_n > 0,\ \ell$ *be a fixed integer* $1 \le \ell \le n$ *and* $\mathbb{L}(\mathcal{K}_i(x))$ *and* $\mathbb{R}(\mathcal{K}_i(x))$ *be the left and the right end points of a non-degenerate distribution function* $\mathcal{K}_i(x)$, *respectively. Then, when* $\max_{1\le i\le\zeta} \mathbb{L}(\mathcal{K}_i(x)) < x < \max_{1\le i\le\zeta} \mathbb{R}(\mathcal{K}_i(x))$, *we get*

(I) *Let* $\mathcal{F}(x) = \sum_{i=1}^{5} \mathcal{P}_i\mathcal{F}_i(x)$, *where* $\sum_{i=1}^{5} \mathcal{P}_i = 1,\ 0 < \mathcal{P}_i < 1,\ \mathcal{F}_1 \in \mathbb{PDA}(Q_{1,\delta}^{(0,1)}(x))$ *and* $\mathcal{F}_j \in \mathbb{PDA}(Q_{t,\delta}^{(0,1)}(x)),\ t \in \{2,3,4,6\},\ 2 \le j \le 5$. *Then for* $x > 1$, *we get*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{(\mathcal{P}_1(\log x)^{-\delta})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt.$$

(II) *Let* $\mathcal{F}(x) = \sum_{i=1}^{6} \mathcal{P}_i\mathcal{F}_i(x)$, *where* $\sum_{i=1}^{6} \mathcal{P}_i = 1,\ 0 < \mathcal{P}_i < 1,\ \mathcal{F}_1 \in \mathbb{PDA}(Q_{1,\delta}^{(0,1)}(x)),\ \mathcal{F}_5 \in \mathbb{PDA}(Q_{5,\delta}^{(0,1)}(x))$ *and* $\mathcal{F}_j \in \mathbb{PDA}(Q_{t,\delta}^{(0,1)}(x)),\ t \in \{2,3,4,6\},\ j = 2,3,4,6$. *Then for* $x > 1$, *we get*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{\left(\mathcal{P}_1(\log x)^{-\delta}+\mathcal{P}_5\frac{1}{x}\right)^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt.$$

(III) *Let* $\mathcal{F}(x) = \sum_{i=1}^{4} \mathcal{P}_i\mathcal{F}_i(x)$, *where* $\sum_{i=1}^{4} \mathcal{P}_i = 1,\ 0 < \mathcal{P}_i < 1,\ \mathcal{F}_2 \in \mathbb{PDA}(Q_{2,\delta}^{(0,1)}(x))$ *and* $\mathcal{F}_j \in \mathbb{PDA}(Q_{t,\delta}^{(0,1)}(x)),\ t \in \{3,4,6\},\ j = 1,3,4$. *Then for* $0 < x \le 1$, *we get*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{((-1)^{\delta}\mathcal{P}_2(\log x)^{\delta})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt.$$

(IV) *Let* $\mathcal{F}(x) = \sum_{i=1}^{5} \mathcal{P}_i\mathcal{F}_i(x)$, *where* $\sum_{i=1}^{5} \mathcal{P}_i = 1,\ 0 < \mathcal{P}_i < 1, \mathcal{F}_2 \in \mathbb{PDA}(Q_{2,\delta}^{(0,1)}(x)), \mathcal{F}_5 \in \mathbb{PDA}(Q_{5,\delta}^{(0,1)}(x))$ *and* $\mathcal{F}_j \in \mathbb{PDA}(Q_{t,\delta}^{(0,1)}(x)),\ t \in \{3,4,6\},\ j = 1,3,4$. *Then,*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \begin{cases} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{\left(\mathcal{P}_2(-\log x)^{\delta}+\frac{\mathcal{P}_5}{x}\right)^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt, & 0 < x < 1, \\ \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{\left(\frac{\mathcal{P}_5}{x}\right)^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2} e^{-t} dt, & x > 1. \end{cases}$$

(V) *Let* $\mathcal{F}(x) = \sum_{i=1}^{4}\mathcal{P}_i\mathcal{F}_i(x)$, *where* $\sum_{i=1}^{4}\mathcal{P}_i = 1$, $0 < \mathcal{P}_i < 1$, $\mathcal{F}_1 \in \mathbb{PDA}(Q_{5,\delta}^{(0,1)}(x))$ *and* $\mathcal{F}_j \in \mathbb{PDA}(Q_{t,\delta}^{(0,1)}(x))$, $t \in \{3,4,6\}$, $j = 2,3,4$. *Then for* $x > 0$, *we get*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma(\frac{k}{m+1}+\ell-1)}\int_{(\frac{\mathcal{P}_1}{x})^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$$

Proof. The proof is obvious from Theorems 1.2 and 2.6 by the same technique as that in the proof of Corollary 2.10. □

Remark 2.12. *If* $\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma(\frac{k}{m+1}+\ell-1)}\int_{z}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt$, *then* $\Xi_{n-\ell+1:n}^{(0,1)}(\mathcal{B}_n(x)) \xrightarrow[n]{w} e^{-z}$. *Therefore, the discussion in Corollaries 2.10 and 2.11 can be immediately given for the ordinary order statistics.*

2.2. Mixture extreme value theory via different normalizations

Theorem 2.13. *For any symmetric generalized order statistics from a mixture of* ζ *distributions* $\mathcal{F}_i(x)$ *with mixture distribution function* $\mathcal{F}(x) = \sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(x)$, $0 < \mathcal{P}_i < 1$, $\sum_{i=1}^{\zeta}\mathcal{P}_i = 1$, *let* $m > -1$, $k > 0$, ℓ *be a fixed integer* $1 \le \ell \le n$ *and* $\mathbb{L}(\mathcal{K}_i(x))$ *and* $\mathbb{R}(\mathcal{K}_i(x))$ *be the left and the right end points of a non-degenerate distribution function* $\mathcal{K}_i(x)$, *respectively. Suppose* $\mathcal{F}_i(x)$ *are normalized by different strongly monotone continuous sequences* $\{\mathcal{B}_{i,n}(x)\}_{n=1}^{\infty}$ *such that, for* $1 \le i \le \zeta$, *the following conditions hold*

($\mathcal{C}_1^u$) $\mathcal{F}_i^n(\mathcal{B}_{i,n}(x)) \xrightarrow[n]{w} \mathcal{K}_i(x)$, $0 < \mathcal{K}_i(x) < \infty$,

($\mathcal{C}_2^u$) $n[\mathcal{F}_i(\mathcal{B}_n(x)) - \mathcal{F}_i(\mathcal{B}_{i,n}(x))] \xrightarrow[n]{} \eta_i(x)$, $0 < \eta_i(x) < \infty$, *for some strongly monotone continuous sequence* $\{\mathcal{B}_n(x)\}_{n=1}^{\infty}$.

Then, when $\max_{1\le i\le\zeta}\mathbb{L}(\mathcal{K}_i(x)) < x < \max_{1\le i\le\zeta}\mathbb{R}(\mathcal{K}_i(x))$, *we have*

$$\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{\left(-\sum_{i=1}^{\zeta}\left[\log(\mathcal{K}_i(x))^{\mathcal{P}_i}+\mathcal{P}_i\eta_i(x)\right]\right)^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$$

Proof. Since $\mathcal{F}(x) = \sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(x)$, we get

$$\begin{aligned}\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)) &= \sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x)) + \sum_{i=1}^{\zeta}\mathcal{P}_i\left[\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x)) - \mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\right]\\ &= \mathcal{P}_\zeta\mathcal{F}_\zeta(\mathcal{B}_{\zeta,n^{1/(m+1)}}(x)) + \sum_{i=1}^{\zeta-1}\mathcal{P}_i\mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\\ &\quad+\sum_{i=1}^{\zeta}\mathcal{P}_i\left[\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x)) - \mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\right]. \end{aligned} \quad (14)$$

Since $\mathcal{P}_\zeta = 1 - \sum_{i=1}^{\zeta-1}\mathcal{P}_i$, from (14) we have

$$\begin{aligned}\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)) &= \mathcal{F}_\zeta(\mathcal{B}_{\zeta,n^{1/(m+1)}}(x)) - \sum_{i=1}^{\zeta-1}\mathcal{P}_i\left[\mathcal{F}_\zeta(\mathcal{B}_{\zeta,n^{1/(m+1)}}(x)) - \mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\right]\\ &\quad+\sum_{i=1}^{\zeta}\mathcal{P}_i\left[\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x)) - \mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\right]. \end{aligned} \quad (15)$$

Thus, from (15) we get

$$\begin{aligned} n^{1/(m+1)}(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x))) &= n^{1/(m+1)}(1-\mathcal{F}_\zeta(\mathcal{B}_{\zeta,n^{1/(m+1)}}(x))) \\ &\quad -\sum_{i=1}^{\zeta}\mathcal{P}_i n^{1/(m+1)}\left[\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x))-\mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))\right] \\ &\quad +\sum_{i=1}^{\zeta-1}\mathcal{P}_i\Big[n^{1/(m+1)}(1-\mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))) \\ &\quad - n^{1/(m+1)}(1-\mathcal{F}_\zeta(\mathcal{B}_{\zeta,n^{1/(m+1)}}(x)))\Big]. \end{aligned} \tag{16}$$

Suppose that $(\mathcal{C}_1^u)$ is satisfied. Then, from Lemma 2.1 and Remark 2.5 for $i=1,2,...,\zeta$ we have $n(1-\mathcal{F}_i(\mathcal{B}_n(x)))\ \underset{n}{\longrightarrow}\ -\log\mathcal{K}_i(x)$. Therefore, $(\mathcal{C}_1^u)$ and $(\mathcal{C}_2^u)$ become

$$n^{1/(m+1)}(1-\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x)))\ \underset{n}{\longrightarrow}\ -\log\mathcal{K}_i(x) \tag{17}$$

and

$$n^{1/(m+1)}[\mathcal{F}_i(\mathcal{B}_{n^{1/(m+1)}}(x))-\mathcal{F}_i(\mathcal{B}_{i,n^{1/(m+1)}}(x))]\ \underset{n}{\longrightarrow}\ \eta_i(x), \tag{18}$$

respectively. Then, from (16), (17) and (18) we get

$$\begin{aligned} n^{1/(m+1)}(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x))) &\underset{n}{\longrightarrow} -\log\mathcal{K}_\zeta(x)-\sum_{i=1}^{\zeta}\mathcal{P}_i\eta_i(x) \\ &\quad +\sum_{i=1}^{\zeta-1}\mathcal{P}_i(-\log\mathcal{K}_i(x)+\log\mathcal{K}_\zeta(x)). \end{aligned} \tag{19}$$

It then follows, since $\sum_{i=1}^{\zeta-1}\mathcal{P}_i=1-\mathcal{P}_\zeta$, that

$$n^{1/(m+1)}(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))\ \underset{n}{\longrightarrow}\ -\sum_{i=1}^{\zeta}\mathcal{P}_i(\log\mathcal{K}_i(x)+\eta_i(x)). \tag{20}$$

Therefore, from (20) we get

$$\begin{aligned} \left(\frac{k}{m+1}+n-1\right)(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))^{m+1} &\sim n\left[(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))\right]^{m+1} \\ &\sim \left[n^{1/(m+1)}(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))\right]^{m+1} \\ &\underset{n}{\longrightarrow} \left[-\sum_{i=1}^{\zeta}\mathcal{P}_i(\log\mathcal{K}_i(x)+\eta_i(x))\right]^{m+1}. \end{aligned} \tag{21}$$

From (2) and (21) and by taking $\mathbb{F}=1-(1-\mathcal{F}(\mathcal{B}_{n^{1/(m+1)}}(x)))^{m+1}$, $\mathcal{N}=\frac{k}{m+1}+n-1$ and $\mathcal{R}=\frac{k}{m+1}+\ell-1$ in Lemma 2.2, the proof can be completed. □

Theorem 2.14. *For any symmetric generalized order statistics from a mixture of ζ distributions $\mathcal{F}_i(x)$ with mixture distribution function $\mathcal{F}(x)=\sum_{i=1}^{\zeta}\mathcal{P}_i\mathcal{F}_i(x)$, $0<\mathcal{P}_i<1$, $\sum_{i=1}^{\zeta}\mathcal{P}_i=1$, let $m>-1$, $k>0$, ℓ be a fixed integer $1\le\ell\le n$ and $\mathbb{L}(\mathcal{E}_i(x))$ and $\mathbb{R}(\mathcal{E}_i(x))$ be the left and the right end points of a non-degenerate distribution function $\mathcal{E}_i(x)$, respectively. Suppose $\mathcal{F}_i(x)$ are normalized by different strongly monotone continuous sequences $\{\mathcal{A}_{i,n}(x)\}_{n=1}^{\infty}$ such that, for $1\le i\le\zeta$, the following conditions hold*

$(\mathcal{C}_1^l)$ $1-(1-\mathcal{F}_i(\mathcal{A}_{i,n}(x)))^n\ \underset{n}{\xrightarrow{w}}\ \mathcal{E}_i(x),\ 0<\mathcal{E}_i(x)<\infty,$

$(\mathcal{C}_2^l)$ $n[\mathcal{F}_i(\mathcal{A}_n(x)) - \mathcal{F}_i(\mathcal{A}_{i,n}(x))] \underset{n}{\longrightarrow} \mu_i(x),\ 0 < \mu_i(x) < \infty$, *for some strongly monotone continuous sequence* $\{\mathcal{A}_n(x)\}_{n=1}^{\infty}$.

Then, when $\max_{1\le i\le\zeta} \mathbb{L}(\mathcal{E}_i(x)) < x < \max_{1\le i\le\zeta} \mathbb{R}(\mathcal{E}_i(x))$, *we have*

$$\Xi_{\ell:n}^{(m,k)}(\mathcal{A}_{n(m+1)}(x)) \xrightarrow[n]{w} 1 - \frac{1}{\Gamma(\ell)} \int_{\sum_{i=1}^{\zeta}\left[\log(\mathcal{E}_i(x))^{-\mathcal{P}_i} - \mathcal{P}_i\mu_i(x)\right]}^{\infty} t^{\ell-1}e^{-t}dt.$$

Proof. The proof is similar to the proof of Theorems 2.7 and 2.13, except the obvious changes. □

Corollary 2.15. *From Theorems 2.13 and 2.14, we have the following special cases*

(I) *For ordinary order statistics, i.e.,* $m = 0$ *and* $k = 1$, *we get*

- **(i)** $\Xi_{n-\ell+1:n}^{(0,1)}(\mathcal{B}_n(x)) \xrightarrow[n]{w} \frac{1}{\Gamma(\ell)} \int_{\sum_{i=1}^{\zeta}\left[\log(\mathcal{K}_i(x))^{-\mathcal{P}_i} - \mathcal{P}_i\eta_i(x)\right]}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(ii)** $\Xi_{\ell:n}^{(0,1)}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \frac{1}{\Gamma(\ell)} \int_{\left[\log(\mathcal{E}_i(x))^{-\mathcal{P}_i} - \mathcal{P}_i\mu_i(x)\right]}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(iii)** $\Xi_{n:n}^{(0,1)}(\mathcal{B}_n(x)) = \mathcal{F}^n(\mathcal{B}_n(x)) \xrightarrow[n]{w} \prod_{i=1}^{\zeta}\left[\mathcal{K}_i(x)\exp(\eta_i(x))\right]^{\mathcal{P}_i}.$
- **(iv)** $\Xi_{1:n}^{(0,1)}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \prod_{i=1}^{\zeta}\left[\mathcal{E}_i(x)\exp(\mu_i(x))\right]^{\mathcal{P}_i}.$

(II) *If the observations come from the same distribution, i.e.,* $\mathcal{K}_i(x) = \mathcal{K}(x),\ \eta_i(x) = \eta(x),\ \mathcal{E}_i(x) = \mathcal{E}(x)$ *and* $\mu_i(x) = \mu(x)$ *for* $1 \le i \le \zeta$, *then we have*

- **(i)** $\Xi_{n-\ell+1:n}^{(m,k)}(\mathcal{B}_{n^{1/(m+1)}}(x)) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{(-1)^{m+1}(\log\mathcal{K}(x)+\eta(x))^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt.$
- **(ii)** $\Xi_{\ell:n}^{(m,k)}(\mathcal{A}_{n(m+1)}(x))$ *and* $\Xi_{\ell:n}^{(0,1)}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \frac{1}{\Gamma(\ell)} \int_{-\log\mathcal{E}(x)-\mu(x)}^{\infty} t^{\ell-1}e^{-t}dt.$
- **(iii)** $\Xi_{n:n}^{(0,1)}(\mathcal{B}_n(x)) \xrightarrow[n]{w} \mathcal{K}(x)\exp(\eta(x)).$
- **(iv)** $\Xi_{1:n}^{(m,k)}(\mathcal{A}_{n(m+1)}(x))$ *and* $\Xi_{1:n}^{(0,1)}(\mathcal{A}_n(x)) \xrightarrow[n]{w} 1 - \mathcal{E}(x)\exp(\mu(x)).$

3. Applications

In this section some illustrative examples of the most practically important distributions are obtained, which lend further support to our theoretical results.

Example 3.1. (logistic, Exponential(1) and Gumbel distributions). Let $X_1, X_2, ..., X_n$ be a random sample drawn from a mixed population whose distribution function $\mathcal{F}(x) = \mathcal{P}_1\mathcal{F}_1(x) + \mathcal{P}_2\mathcal{F}_2(x) + \mathcal{P}_3\mathcal{F}_3(x)$, where $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = 1,\ 0 < \mathcal{P}_i < 1,\ i = 1, 2, 3,\ \mathcal{F}_1(x) = \frac{1}{1+e^{-x}},\ -\infty < x < \infty,\ \mathcal{F}_2(x) = 1 - e^{-x}, x > 0$ and $\mathcal{F}_3(x) = \exp(-e^{-x}),\ -\infty < x < \infty$. One can choose the common linear normalization, $\mathcal{B}_n(x) = x + \log n$. Then $\mathcal{B}_{n^{1/(m+1)}} = x + \frac{1}{m+1}\log n$. It is not difficult to show that $n(1 - \mathcal{F}_i(x + \log n)) \underset{n}{\longrightarrow} e^{-x}$, or, equivalently, $\mathcal{F}_i^n(x + \log n) \xrightarrow[n]{w} \exp(-e^{-x}), i = 1, 2, 3$. Then, we have

$$\Xi_{n-\ell+1:n}^{(m,k)}(x + \frac{1}{m+1}\log n) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)} \int_{e^{-x(m+1)}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt$$

$$\text{and}\ \ \Xi_{n:n}^{(0,1)}(x + \log n) \xrightarrow[n]{w} \exp(-e^{-x}).$$

Example 3.2. (Power(1) and Exponential(1) distributions). Let $X_1, X_2, ..., X_n$ be a random sample drawn from a mixed population whose distribution function $\mathcal{F}(x) = \mathcal{P}\mathcal{F}_1(x) + (1-\mathcal{P})\mathcal{F}_2(x)$, where $0 < \mathcal{P} < 1$, $\mathcal{F}_1(x) = x,\ 0 < x < 1$ and $\mathcal{F}_2(x) = 1 - e^{-x}, x > 0$. One can choose the common linear normalization, $\mathcal{A}_n(x) = \frac{x}{n}$. Then, we get $\mathcal{A}_{n(m+1)}(x) = \frac{x}{n(m+1)}$. It is not difficult to show that $n\mathcal{F}_i(\frac{x}{n}) \underset{n}{\longrightarrow} x, i = 1, 2$.

or, equivalently, $1-(1-\mathcal{F}_i(\frac{x}{n}))^n \xrightarrow[n]{w} 1-e^{-x}, i=1,2$. Then, we have $\Xi^{(m,k)}_{\ell:n}\left(\frac{x}{n(m+1)}\right)$ and $\Xi^{(0,1)}_{\ell:n}\left(\frac{x}{n}\right)$ $\xrightarrow[n]{w} 1-\frac{1}{\Gamma(\ell)}\int_x^\infty t^{\ell-1}e^{-t}dt$ and $\Xi^{(0,1)}_{1:n}\left(\frac{x}{n}\right) \xrightarrow[n]{w} 1-e^{-x}$.

Example 3.3. Let $X_1, X_2, ..., X_n$ be a random sample drawn from a mixed population whose distribution function $\mathcal{F}(x)=\mathcal{P}\mathcal{F}_1(x)+(1-\mathcal{P})\mathcal{F}_2(x)$, where $0<\mathcal{P}<1$, $\mathcal{F}_1(x)=\begin{cases}0, & x\le 0,\\ \exp[-(-\log x)^\alpha], & 0<x\le 1\\ 1, & x>1,\end{cases}$ and $\mathcal{F}_2(x)=\begin{cases}\exp[-(\log(-x))^\alpha], & x\le -1,\\ 1, & x>-1.\end{cases}$ One can choose the common power normalization, $\mathcal{B}_n(x)=|x|^{n^{\frac{-1}{\alpha}}}\mathcal{S}(x)$. Then $\mathcal{B}_{n^{1/(m+1)}}(x)=|x|^{n^{\frac{-1}{\alpha(m+1)}}}\mathcal{S}(x)$. It is not difficult to show that

$\left[\mathcal{F}_i\left(|x|^{n^{-1/\alpha}}\mathcal{S}(x)\right)\right]^n \xrightarrow[n]{w} \mathcal{F}_i(x),\ i=1,2$. Then, we have

$$\Xi^{(m,k)}_{n-\ell+1:n}\left(|x|^{n^{\frac{-1}{\alpha(m+1)}}}\mathcal{S}(x)\right) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{(-1)^{m+1}(\log([\mathcal{F}_1(x)]^{\mathcal{P}}[\mathcal{F}_2(x)]^{1-\mathcal{P}}))^{m+1}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt$$

$$\text{and}\quad \Xi^{(0,1)}_{n:n}\left(|x|^{n^{\frac{-1}{\alpha}}}\mathcal{S}(x)\right) \xrightarrow[n]{w} [\mathcal{F}_1(x)]^{\mathcal{P}}[\mathcal{F}_2(x)]^{1-\mathcal{P}}.$$

Example 3.4. (Pareto and Frechet distributions). Let $X_1, X_2, ..., X_n$ be a random sample drawn from a mixed population whose distribution function $\mathcal{F}(x)=\mathcal{P}\mathcal{F}_1(x)+(1-\mathcal{P})\mathcal{F}_2(x)$, where $0<\mathcal{P}<1$, $\mathcal{F}_1(x)=\begin{cases}0, & x<1,\\ 1-x^{-\alpha}, & x\ge 1, \alpha>0\end{cases}$ and $\mathcal{F}_2(x)=\begin{cases}\exp[-x^{-\alpha}], & x\ge 0, \alpha>0\\ 0, & x<0.\end{cases}$ One can choose the common power normalization, $\mathcal{B}_n(x)=(nx)^{1/\alpha}$. Then, we get $\mathcal{B}_{n^{1/(m+1)}}(x)=(n^{\frac{1}{m+1}}x)^{1/\alpha}$. It is not difficult to show that $n(1-\mathcal{F}_i((nx)^{1/\alpha})) \xrightarrow[n]{} \frac{1}{x}, i=1,2$. Then, we get

$$\Xi^{(m,k)}_{n-\ell+1:n}\left((n^{\frac{1}{m+1}}x)^{1/\alpha}\right) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{x^{-(m+1)}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt$$

$$\text{and}\quad \Xi^{(0,1)}_{n:n}\left((nx)^{1/\alpha}\right) = \xrightarrow[n]{w} e^{-1/x}.$$

Similarly, one can choose the common linear normalization, $\mathcal{B}_n(x)=n^{1/\alpha}x$. Then, $\mathcal{B}_{n^{1/(m+1)}}(x)=n^{1/\alpha(m+1)}x$. It is not difficult to show that $n(1-\mathcal{F}_i(n^{1/\alpha}x)) \xrightarrow[n]{} x^{-\alpha},\ i=1,2$. Then,

$$\Xi^{(m,k)}_{n-\ell+1:n}\left(n^{1/\alpha(m+1)}x\right) \xrightarrow[n]{w} \frac{1}{\Gamma\left(\frac{k}{m+1}+\ell-1\right)}\int_{x^{-\alpha(m+1)}}^{\infty} t^{\frac{k}{m+1}+\ell-2}e^{-t}dt$$

$$\text{and}\quad \Xi^{(0,1)}_{n:n}\left(n^{1/\alpha}x\right) = \xrightarrow[n]{w} e^{-x^{-\alpha}}.$$

Remark 3.1. *It is worth mentioning that at this stage we are unable to find a suitable example where the normalizing constants are different to support the assumptions* ($\mathcal{C}_2^u$) *and* ($\mathcal{C}_2^l$) *of Theorems 2.13 and 2.14, respectively. This will be investigated in our future work.*

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