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# On Limit Laws of Joint Distribution of Two Normalized Upper Order Statistics

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**Abstract.** For a fixed positive integer $k$, limit laws of linearly normalized $k$-th upper order statistics are well known. In this article, we study the joint limit distributions of two normalized upper order statistics under fixed and random sample sizes. We also look at some bivariate stochastic orders.

## 1. Introduction with motivation

Limit laws of linearly (or power) normalized partial maxima $M_n = \max\{X_1, \ldots, X_n\}$ of independent and identically distributed (iid) random variables (rvs) $X_1, X_2, \ldots,$ with common distribution function (df) $F$ are called extreme value laws (p-max stable laws). Namely, if

$$\lim_{n\to\infty} P(M_n \le g_n(x)) = \lim_{n\to\infty} F^n(g_n(x)) = G(x), \;\; x \in \mathcal{C}(G), \tag{1}$$

holds for some $g_n(x)$, with $g_n(x)$ as linear normalization of the form $a_n x + b_n$ for some norming constants $a_n > 0$, $b_n \in \mathbb{R}$, or $g_n(x)$ as power normalization of the form $\alpha_n |x|^{\beta_n} \operatorname{sign}(x)$ for some norming constants $\alpha_n > 0$, $\beta_n > 0$, $\operatorname{sign}(x) = -1, 0, +1$ accordingly as $x < 0, x = 0, x > 0$, $G$, a non-degenerate df, $\mathcal{C}(G)$, the set of all continuity points of $G$, then $G$ is some extreme value law or $l$-max stable law when $g_n(x)$ is linear norming, and $G$ is some $p$-max stable law when $g_n(x)$ is power norming. We write this as $F \in D_l(G)$ under linear norming and $F \in D_p(G)$ under power norming. Note that (1) is equivalent to

$$\lim_{n\to\infty} n\{1 - F(g_n(x))\} = -\ln G(x), \; x \in \{y \in \mathbb{R} : G(y) > 0\}.$$

Criteria for $F \in D_l(G)$ are well known (see, for example, Galambos, 1987; Resnick, 1987; Embrechts et al., 1997). Further when $F$ satisfies (1), we can write $G(x) = \exp\left(-\nu_{r,\alpha}(x)\right), r = l$ or $p$. For $l-$max stable laws $G$, $\nu_{l,\alpha}(x)$ has three forms, viz., for parameter $\alpha > 0$,

$$x^{-\alpha}, \; x > 0; \quad (-x)^{\alpha}, \; x < 0; \quad e^{-x}, \; x \in \mathbb{R};$$

(for more details, see Leadbetter et al., 1983, Resnick, 1987 and de Haan and Ferreira, 2006) and for $p-$max stable laws $G$, $\nu_{p,\alpha}(x)$ has six forms, viz., for parameter $\alpha > 0$,

$$(\ln x)^{-\alpha}, \; x > 1; \quad (-\ln x)^{\alpha}, \; 0 < x < 1; \quad x^{-1}, \; x > 0;$$

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$(\ln(-x))^{-\alpha},\ x<-1;\quad (-\ln(-x))^{-\alpha},\ -1<x<0;\quad x,\ x<0;$

(for more details, see Pancheva, 1985, and Mohan and Ravi, 1993).

Suppose that $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the increasing order statistics from a random sample $\{X_1, \ldots, X_n\}$ from an absolutely continuous df $F$ with probability density function (pdf) $f$. Let $X_{n-k+1:n}$ denote the $k$-th upper order statistic for a fixed integer $k \geq 1$, independent of $n$. Note that for $k = 1$, $X_{n:n}$ corresponds to the maximum of $X_1, X_2, \ldots, X_n$. Further, for fixed positive integers $k > l$, the joint pdf of $X_{n-k+1:n}$ and $X_{n-l+1:n}$ is easily computed as

$$g_{k:n,l:n}(u,v) = c_{kl}(n)F^{n-k}(u)(F(v)-F(u))^{k-l-1}(1-F(u))^{l-1}f(u)f(v), u \in \mathbb{R}, v \in \mathbb{R},$$

where $c_{kl}(n) = \dfrac{n!}{(n-k)!(k-l-1)!(l-1)!}$. With the convention that $\begin{pmatrix} n \\ a,b \end{pmatrix} = \dfrac{n!}{a!\ b!\ (n-a-b)!}$, the joint df of $X_{n-k+1:n}$ and $X_{n-l+1:n}$ is given by

$$\begin{aligned} F_{k:n,l:n}(x,y) &= P(X_{n-k+1:n} \leq x, X_{n-l+1:n} \leq y), \\ &= \begin{cases} \sum\limits_{j=0}^{l-1} \binom{n}{j} F^{n-j}(y)(1-F(y))^j & \text{if } x \geq y, \\ \sum\limits_{j=0}^{l-1} \sum\limits_{i=0}^{k-1-j} \binom{n}{i,j} F^{n-i-j}(x)(F(y)-F(x))^i(1-F(y))^j & \text{if } x \leq y; \end{cases} \end{aligned} \tag{2}$$

since, if $x \geq y$, we have that $\{X_{n-l+1:n} \leq y\} \subseteq \{X_{n-l+1:n} \leq x\} \subset \{X_{n-k+1:n} \leq x\}$, and hence $F_{k:n,l:n}(x,y) = P(X_{n-l+1:n} \leq y)$, the df of the $(n-l+1)$-th upper order statistic, see, for example, page 12 in David and Nagaraja (2003). Replacing $x$ and $y$ in (2) respectively by the norming constants $g_n(x)$ and $g_n(y)$, and taking limit as $n \to \infty$, we get

$$\begin{aligned} G_{k,l}(x,y) &= \lim_{n\to\infty} F_{k:n,l:n}(g_n(x), g_n(y)), \\ &= \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=0}^{l-1} \binom{n}{j} F^{n-j}(g_n(y))(1-F(g_n(y)))^j & \text{if } x \geq y, \\ \lim\limits_{n\to\infty} \sum\limits_{j=0}^{l-1} \sum\limits_{i=0}^{k-1-j} \binom{n}{i,j} F^{n-i-j}(g_n(x))(F(g_n(y))-F(g_n(x)))^i(1-F(g_n(y)))^j & \text{if } x \leq y. \end{cases} \end{aligned} \tag{3}$$

Assuming that $F \in D_l(G)$, the tail behaviour of the limit

$$G_l(y) = \lim_{n\to\infty} \sum_{j=0}^{l-1} \binom{n}{j} F^{n-j}(y)(1-F(y))^j = G(y) \sum_{j=0}^{l-1} \frac{(-\ln G(y))^j}{j!}$$

was studied in Ravi and Manohar (2017) including the tail behaviour of dfs $F_l$ obtained by replacing the extreme value df $G$ in $G_l$ by any df $F$, max domains of attraction through tail equivalence, and univariate stochastic orderings. It was observed in Ravi and Manohar (2017) that the results obtained hold also under power norming. In this article, we study $G_{k,l}(x,y) = \lim\limits_{n\to\infty} \sum\limits_{j=0}^{l-1} \sum\limits_{i=0}^{k-1-j} \binom{n}{i,j} F^{n-i-j}(x)(F(y)-F(x))^i(1-F(y))^j$. Throughout the article, dfs are specified for the values of the argument for which the df is in $(0,1)$ and are assumed to be absolutely continuous, to simplify proofs.

In the next section, we derive the joint distribution of the limit of normalized $k$-th and $l$-th upper order statistics from a random sample of size $n$, for fixed positive integers $k$ and $l$ with $k > l$, without loss of generality. In Section 3, assuming the sample size $n$ to be a rv $N_n$ with probability mass function (pmf) $P(N_n = u)$, $u = m, m+1, m+2, \ldots, m+n, \ldots,$ and $N_n$ to be independent of the iid rvs $X_1, X_2, \ldots,$ $m \geq 1$, a fixed integer, we derive the joint distribution of the limit of normalized $k$-th and $l$-th upper order statistics $(X_{N_n-k+1:N_n}, X_{N_n-l+1:N_n})$, when $N_n$ is shifted binomial, shifted Poisson, shifted geometric, shifted negative binomial and discrete uniform. Note that $(X_{N_n-k+1:N_n}, X_{N_n-l+1:N_n})$, is well defined for

$1 \le l < k \le m$. Vasantha (2010) discusses the case $k = 2, l = 1$ only, only for the case when the df $F \in D_l(\Phi_\alpha)$ and the sample size is shifted geometric, where $\Phi_\alpha(x) = \exp(-x^{-\alpha}), x > 0, \alpha > 0$, is the Freĉhet law. In Section 4, some new bivariate dfs are defined and their bivariate max domains of attraction are studied. In Section 5, some bivariate stochastic orderings are discussed. Examples are given in Section 6 and proofs of all results are given in Section 7. The last section gives concluding remarks.

## 2. Joint limit distribution of the normalized *k*-th and the *l*-th upper order statistics for fixed sample size *n*

In this section, we derive the joint limit distribution of the normalized $k$-th and the $l$-th upper order statistics from a random sample of size $n$, where $k > l$ are fixed positive integers.

**Theorem 2.1.** *If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(G)$ for some $l$-max stable df $G$ so that $F$ satisfies* (1) *for some norming sequence $g_n(x)$, then the limiting joint df* (3) *is given by*

$$G_{k,l}(x,y) = \begin{cases} G(y)\sum\limits_{j=0}^{l-1} \frac{(-\ln G(y))^j}{j!} & \text{if } x \ge y, \\ G(x)\sum\limits_{j=0}^{l-1} \frac{(-\ln G(y))^j}{j!} \sum\limits_{i=0}^{k-1-j} \frac{(\ln G(y) - \ln G(x))^i}{i!} & \text{if } x \le y. \end{cases} \tag{4}$$

**Corollary 2.2.** *The marginal dfs of normalized $X_{n-k+1:n}$ and $X_{n-l+1:n}$ with joint df as in* (3), *are respectively given by*

$$\begin{aligned} G_k(x) &= G(x)\sum_{i=0}^{k-1} \frac{(-\ln G(x))^i}{i!}, \quad x \in \{z : G(z) > 0\}, \\ G_l(y) &= G(y)\sum_{j=0}^{l-1} \frac{(-\ln G(y))^j}{j!}, \quad y \in \{z : G(z) > 0\}. \end{aligned}$$

## 3. Joint limit distribution of the normalized *k*-th and the *l*-th upper order statistics when the sample size *n* is random

In this section, we derive the joint limit distribution of the normalized $k$-th and the $l$-th upper order statistics when the sample size $n$ is random and $k > l$ are fixed positive integers. Let $N_n$ have pmf $P(N_n = u),\ u = m, m+1, \ldots, m+n, \ldots$ . From (2), we have

$$\begin{aligned} F_{k:N_n,l:N_n}(x,y) &= \begin{cases} \sum\limits_{u=m}^{\infty} P(X_{l:N_n} \le y, N_n = u) & \text{if } x \ge y, \\ \sum\limits_{u=m}^{\infty} P(X_{k:N_n} \le x, X_{l:N_n} \le y, N_n = u) & \text{if } x \le y, \end{cases} \\ &= \begin{cases} \sum\limits_{u=m}^{\infty}\sum\limits_{j=0}^{l-1} \binom{u}{j} F^{u-j}(y)(1-F(y))^j P(N_n = u) & \text{if } x \ge y, \\ \sum\limits_{u=m}^{\infty}\sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j} \binom{u}{i,\ j} F^{u-i-j}(x)(F(y)-F(x))^i(1-F(y))^j P(N_n = u) & \text{if } x \le y; \end{cases} \end{aligned}$$

so that

$$F_{k:N_n,l:N_n}(x,y) = \begin{cases} \sum\limits_{j=0}^{l-1}(1-F(y))^j \sum\limits_{u=m}^{\infty} \binom{u}{j} F^{u-j}(y) P(N_n = u), & x \ge y, \\ \sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j} \binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j \sum\limits_{u=m}^{\infty} \binom{u}{i+j} F^{u-i-j}(x) P(N_n = u), & x \le y. \end{cases} \tag{5}$$

**Theorem 3.1.** *If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(G)$ for some $l$-max stable df $G$ so that $F$ satisfies* (1) *for some norming sequence $g_n(x)$, then the limiting joint df of* (5), $\lim_{n\to\infty} F_{k:N_n,l:N_n}(g_n(x), g_n(y))$ *is*

(a) *the same as in* (4) *when*
   (i) *$N_n$ is a shifted binomial rv with $P(N_n = u) = \binom{n}{u-m} p_n^{u-m} q_n^{m+n-u}$, $u = m, m+1, m+2, \ldots, m+n$, for some integer $m \geq 1$, with $\lim_{n\to\infty} p_n = 1$; and*
   (ii) *$N_n$ is a shifted Poisson rv with $P(N_n = u) = \dfrac{e^{-\lambda_n}\lambda_n^{u-m}}{(u-m)!}$, $u = m, m+1, m+2, \ldots,$ for some integer $m \geq 1$, with $\lim_{n\to\infty} \dfrac{\lambda_n}{n} = 1$;*

(b) *equal to*

$$T_{k,l,G}(x,y) = \begin{cases} \dfrac{1}{(1-\ln G(y))^a} \sum\limits_{j=0}^{l-1} \binom{j+a-1}{j} \left(\frac{-\ln G(y)}{1-\ln G(y)}\right)^j & \text{if } x \geq y, \\ \dfrac{1}{(1-\ln G(x))^a} \sum\limits_{j=0}^{l-1} \left(\frac{-\ln G(y)}{1-\ln G(x)}\right)^j \sum\limits_{i=0}^{k-1-j} \binom{i+j+a-1}{i,j} \left(\frac{\ln G(y)-\ln G(x)}{1-\ln G(x)}\right)^i & \text{if } x \leq y, \end{cases} \tag{6}$$

*when $N_n$ is a shifted negative binomial rv with pmf $P(N_n = u) = \binom{u-m+a-1}{u-m} p_n^a q_n^{u-m}$, $u = m,\ m+1,\ m+2,\ \ldots,$ where $0 < p_n < 1, q_n = 1 - p_n$ and $\lim_{n\to\infty} np_n = 1$;*

(c) *equal to*

$$R_{k,l,G}(x,y) = \begin{cases} \sum\limits_{j=0}^{l-1} \dfrac{(-\ln G(y))^j}{(1-\ln G(y))^{1+j}} & \text{if } x \geq y, \\ \dfrac{1}{1-\ln G(x)} \sum\limits_{j=0}^{l-1} \left(\frac{-\ln G(y)}{1-\ln G(x)}\right)^j \sum\limits_{i=0}^{k-1-j} \binom{i+j}{i} \left(\frac{\ln G(y)-\ln G(x)}{1-\ln G(x)}\right)^i & \text{if } x \leq y, \end{cases} \tag{7}$$

*when $N_n$ is a shifted geometric rv with pmf $P(N_n = u) = p_n q_n^{u-m}, u = m, m+1, m+2, \ldots,$ where $0 < p_n < 1, q_n = 1 - p_n$ and $\lim_{n\to\infty} np_n = 1$;*

(d) *equal to*

$$U_{k,l,G}(x,y) = \begin{cases} l\left(\frac{1-G(y)}{-\ln G(y)}\right) - G(y)\sum_{v=1}^{l-1}(l-v)\frac{(-\ln G(y))^{v-1}}{v!} & \text{if } x \geq y, \\ \sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j} \binom{i+j}{i} \frac{(\ln G(y)-\ln G(x))^i(-\ln G(y))^j}{(-\ln G(x))^{i+j+1}} \left\{1 - G_{i+j+1}(x)\right\} & \text{if } x \leq y, \end{cases} \tag{8}$$

*when $N_n$ is a shifted discrete Uniform rv with pmf $P(N_n = u) = \frac{1}{n}$, $u = m+1, m+2, \ldots, m+n$ for some integer $m \geq 1$.*

**Corollary 3.2.** *The limiting marginal dfs of normalized $X_{N_n-k+1:N_n}$ and $X_{N_n-l+1:N_n}$ with joint df as in* (5)*, are*

(a) *as given in Corollary* 2.2 *when $N_n$ is the shifted binomial and the shifted Poisson rvs;*

(b) *respectively given by*

$$\begin{aligned} T_{k,G}(x) &= \sum_{i=0}^{k-1} \binom{i+a-1}{i} \frac{(-\ln G(x))^i}{(1-\ln G(x))^{a+i}}, \ x \in \{z : G(z) > 0\}, \\ T_{l,G}(y) &= \sum_{j=0}^{l-1} \binom{j+a-1}{j} \frac{(-\ln G(y))^j}{(1-\ln G(y))^{a+j}}, \ y \in \{z : G(z) > 0\}, \end{aligned}$$

*when $N_n$ is the shifted negative binomial rv,*

(c) *respectively given by*

$$R_{k,G}(x) = \sum_{i=0}^{k-1} \frac{(-\ln G(x))^i}{(1-\ln G(x))^{1+i}}\,,\ x \in \{z : G(z) > 0\}\,,$$

$$R_{l,G}(y) = \sum_{j=0}^{l-1} \frac{(-\ln G(y))^j}{(1-\ln G(y))^{1+j}}\,,\ x \in \{z : G(z) > 0\}\,,$$

*when $N_n$ is the shifted geometric rv,*

(d) *respectively given by*

$$U_{k,G}(x) = k\left(\frac{1-G(x)}{-\ln G(x)}\right) - G(x)\sum_{v=1}^{k-1}(k-v)\frac{(-\ln G(x))^{v-1}}{v!}\,,\ x \in \{z : G(z) > 0\}\,,$$

$$U_{l,G}(y) = l\left(\frac{1-G(y)}{-\ln G(y)}\right) - G(y)\sum_{v=1}^{l-1}(l-v)\frac{(-\ln G(y))^{v-1}}{v!}\,,\quad y \in \{z : G(z) > 0\}\,,$$

*when $N_n$ is the shifted discrete uniform rv.*

**Remark 3.3.** *The results in Sections* 2 *and* 3 *hold under power norming also.*

## 4. Bivariate max-domains of attraction for fixed and random sample sizes

In this section, we define new functions similar to those of the joint dfs $G_{k,l}$, $T_{k,l,G}$, $R_{k,l,G}$ and $U_{k,l,G}$ of the previous section, by replacing the df $G$ by a df $F$ and prove that the resulting function is a bivariate df and look at their bivariate max-domains. We discuss the case when $F \in D_l(\Phi_\alpha)$ only. The case when $F \in D_l(\Psi_\alpha)$ can be discussed on similar lines and the case when $F \in D_l(\Lambda)$ is not discussed in this article, where $\Psi_\alpha(x) = \exp(-|x|^\alpha), x < 0, \alpha > 0$, is the Weibull law and $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, is the Gumbel law.

**Definition 4.1.** *Similar to* (4), (6), (7), *and* (8), *with a df $F$ in place of the df $G$, for $x \in \mathbb{R}, y \in \mathbb{R}$, define*

$$F_{k,l}(x,y) = \begin{cases} F(y)\sum\limits_{j=0}^{l-1}\dfrac{(-\ln F(y))^j}{j!} & \text{if } x \geq y, \\ F(x)\sum\limits_{j=0}^{l-1}\dfrac{(-\ln F(y))^j}{j!}\sum\limits_{i=0}^{k-1-j}\frac{(\ln F(y)-\ln F(x))^i}{i!} & \text{if } x \leq y; \end{cases} \tag{9}$$

$$T_{k,l,F}(x,y) = \begin{cases} \sum\limits_{j=0}^{l-1}\binom{j+a-1}{j}\dfrac{(-\ln F(y))^j}{(1-\ln F(y))^{a+j}}, & \text{if } x \geq y, \\ \sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j}\binom{i+j+a-1}{i,j}\dfrac{(-\ln F(y))^j(\ln F(y)-\ln F(x))^i}{(1-\ln F(x))^{i+j+a}} & \text{if } x \leq y; \end{cases} \tag{10}$$

$$R_{k,l,F}(x,y) = \begin{cases} \sum\limits_{j=0}^{l-1}\dfrac{(-\ln F(y))^j}{(1-\ln F(y))^{1+j}} & \text{if } x \geq y, \\ \sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j}\binom{i+j}{i,j}\dfrac{(-\ln F(y))^j(\ln F(y)-\ln F(x))^i}{(1-\ln F(x))^{i+j+1}} & \text{if } x \leq y; \end{cases} \tag{11}$$

$$U_{k,l,F}(x,y) = \begin{cases} l\left(\frac{1-F(y)}{-\ln F(y)}\right) - F(y)\sum\limits_{v=1}^{l-1}(l-v)\dfrac{(-\ln F(y))^{v-1}}{v!} & \text{if } x \geq y, \\ \sum\limits_{j=0}^{l-1}\sum\limits_{i=0}^{k-1-j}\binom{i+j}{i}\dfrac{(\ln F(y)-\ln F(x))^i(-\ln F(y))^j\,\{1-F_{i+j+1}(x)\}}{(-\ln F(x))^{i+j+1}} & \text{if } x \leq y. \end{cases} \tag{12}$$

**Theorem 4.2.** *For the function $F_{k,l}$ defined in* (9)*, for $x, y \in \mathbb{R}$, with df $F$ absolutely continuous with pdf $f$, the following results hold:*

*(a) $F_{k,l}(x,y)$ is a joint df with the joint pdf*

$$f_{k,l}(x,y) = \frac{f(x)f(y)}{F(y)} \frac{(-\ln F(y))^{l-1}}{(l-1)!} \frac{(\ln F(y) - \ln F(x))^{k-l-1}}{(k-l-1)!}, \; x < y, \tag{13}$$

*and* 0 *otherwise, with the marginal pdfs given by*

$$\begin{aligned} f_k(x) &= \frac{f(x)}{(k-1)!}(-\ln F(x))^{k-1}, \; x \in \{z \in \mathbb{R} : F(z) > 0\}; \\ f_l(y) &= \frac{f(y)}{(l-1)!}(-\ln F(y))^{l-1}, \; y \in \{z \in \mathbb{R} : F(z) > 0\}. \end{aligned} \tag{14}$$

*(b) If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(\Phi_\alpha)$ with norming $g_n(x) = a_n x + b_n$ so that* (1) *holds with $G = \Phi_\alpha$, then $\lim_{n\to\infty} F^n_{k,l}(a^k_n x, a^l_n y) = \exp\left(-y^{-l\alpha}\right)$, $x > 0, y > 0$, where $a^i_n = F^-(1 - (i!/n)^{1/i})$, $i = k, l$.*

**Theorem 4.3.** *For the function $T_{k,l,F}$ defined in* (10) *with df $F$ absolutely continuous having pdf $f$, for $x \in \mathbb{R}, y \in \mathbb{R}$, the following results hold:*

*(a) $T_{k,l,F}(x,y)$ is a joint df with the joint pdf*

$$t_{k,l,F}(x,y) = \frac{\Gamma(k+a)}{\Gamma(k-l)\;\Gamma(l)\;\Gamma(a)} \frac{f(x)f(y)}{F(x)F(y)} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(1-\ln F(x))^{k+a}}, \; x < y, \tag{15}$$

*and* 0 *otherwise, with the marginal pdfs given by*

$$\begin{aligned} t_{k,F}(x) &= \frac{1}{B(a,k)} \frac{f(x)}{F(x)} \frac{(-\ln F(x))^{k-1}}{(1-\ln F(x))^{a+k}}, \; x \in \{z \in \mathbb{R} : F(z) > 0\}, \\ t_{l,F}(y) &= \frac{1}{B(a,l)} \frac{f(y)}{F(y)} \frac{(-\ln F(y))^{l-1}}{(1-\ln F(y))^{a+l}}, \; y \in \{z \in \mathbb{R} : F(z) > 0\}. \end{aligned} \tag{16}$$

*(b) If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(\Phi_\alpha)$ with norming $g_n(x) = a_n x + b_n$ so that* (1) *holds with $G = \Phi_\alpha$, then $\lim_{n\to\infty} T^n_{k,l,F}(a^k_n x, a^l_n y) = \exp\left(-y^{-l\alpha}\right)$, $x > 0, y > 0$, where $a^i_n = F^-(1 - (1/n)^{1/i})$, $i = k, l$.*

**Theorem 4.4.** *For the function $R_{k,l,F}$ defined in* (11) *with df $F$ absolutely continuous having pdf $f$, for $x \in \mathbb{R}, y \in \mathbb{R}$, the following results hold:*

*(a) $R_{k,l,F}(x,y)$ is a joint df with the joint pdf*

$$R'_{k,l,F}(x,y) = \frac{\Gamma(k+1)}{\Gamma(k-l)\;\Gamma(l)} \frac{f(x)f(y)}{F(x)F(y)} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(1-\ln F(x))^{k+1}}, \; x < y,$$

*and* 0 *otherwise, with the marginal pdfs given by*

$$\begin{aligned} R'_{k,F}(x) &= \frac{1}{B(1,k)} \frac{f(x)}{F(x)} \frac{(-\ln F(x))^{k-1}}{(1-\ln F(x))^{1+k}}, \; x \in \{z \in \mathbb{R} : F(z) > 0\}, \\ R'_{l,F}(y) &= \frac{1}{B(1,l)} \frac{f(y)}{F(y)} \frac{(-\ln F(y))^{l-1}}{(1-\ln F(y))^{1+l}}, \; y \in \{z \in \mathbb{R} : F(z) > 0\}. \end{aligned}$$

*(b) If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(\Phi_\alpha)$ with norming $g_n(x) = a_n x + b_n$ so that* (1) *holds with $G = \Phi_\alpha$, then $\lim_{n\to\infty} R^n_{k,l,F}(a^k_n x, a^l_n y) = \exp\left(-y^{-l\alpha}\right)$, $x > 0, y > 0$, where $a^i_n = F^-(1 - (1/n)^{1/i})$, $i = k, l$.*

The proof of Theorem 4.4 is immediate from that of Theorem 4.3 by putting $a = 1$.

**Theorem 4.5.** *For the function $U_{k,l,F}$ defined in* (12) *with df $F$ absolutely continuous having pdf $f$, for $x, y \in \mathbb{R}$, the following results hold:*

*(a) $U_{k,l,F}(x,y)$ is a joint df with the joint pdf*

$$u_{k,l,F}(x,y) = \frac{\Gamma(k+1)}{\Gamma(l)\ \Gamma(k-l)} \frac{f(x)f(y)}{F(x)F(y)} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(-\ln F(x))^{k+1}} \{1 - F_{k+1}(x)\}, x < y, \quad (17)$$

*and* 0 *otherwise, with the marginal pdfs given by*

$$\begin{aligned} u_{k,F}(x) &= \frac{kf(x)}{(-\ln F(x))^2} \left( \frac{1}{F(x)} - \sum_{l=0}^{k} \frac{(-\ln F(x))^l}{l!} \right), \ x \in \{z \in \mathbb{R} : F(z) > 0\}, \quad (18) \\ u_{l,F}(y) &= \frac{lf(y)}{(-\ln F(y))^2} \left( \frac{1}{F(y)} - \sum_{l=0}^{l} \frac{(-\ln F(y))^l}{l!} \right), \ y \in \{z \in \mathbb{R} : F(z) > 0\}. \end{aligned}$$

*(b) If df $F$ is absolutely continuous with pdf $f$, $F \in D_l(\Phi_\alpha)$ with norming $g_n(x) = a_n x + b_n$ so that* (1) *holds with $G = \Phi_\alpha$, then $\lim_{n\to\infty} U^n_{k,l,F}(a^k_n x, a^l_n y) = \exp\left(-y^{-l\alpha}\right)$, $x > 0, y > 0$, where $a^i_n = F^-(1 - ((i+1)!/n)^{1/i})$, $i = k, l$.*

**Remark 4.6.** *Observe that, in (b) of Theorems* 4.2*,* 4.3*,* 4.4 *and* 4.5*, the limit distribution depends only on $l$ and not on $k$.*

## 5. Some bivariate stochastic orderings

We refer to Shaked and Shanthikumar (1994) for definitions. Random vector $\mathbf{X}$ is said to be stochastically smaller than random vector $\mathbf{Y}$ in $\mathbb{R}^d$, denoted by $\mathbf{X} \leq_{\mathbf{st}} \mathbf{Y}$, if $P\{\mathbf{X} \in \mathbf{U}\} \leq P\{\mathbf{Y} \in \mathbf{U}\}$ for all upper sets $\mathbf{U} \in \mathbb{R}^d$, where a set $\mathbf{U} \in \mathbb{R}^d$ is an upper set if $\mathbf{x} \in \mathbf{U}$ and $\mathbf{y} \geq \mathbf{x}$ imply that $\mathbf{y} \in \mathbf{U}$, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^d$, and $\geq$ is component-wise, We repeat below the two dimensional version of Theorem 6.B.3 in page 268 in Shaked and Shanthikumar (1994) for later use.

**Theorem 5.1.** *Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two $2-$dimensional random vectors. If (a) $X_1 \leq_{st} Y_1$ and (b) $[X_2|X_1 = x_1] \leq_{st} [Y_2|Y_1 = y_1]$* whenever $-\infty < x_1 \leq y_1 < \infty$, *then $\mathbf{X} \leq_{\mathbf{st}} \mathbf{Y}$.*

**Theorem 5.2.** *With $V_{k,l}$ as equal to $F_{k,l}$ or $T_{k,l,F}$ or $R_{k,l,F}$ or $U_{k,l,F}$ as in Definition* 4.1*, if $(X_k, Y_l)$ has the joint df $V_{k,l}$, then $(X_{k+1}, Y_{l+1}) \leq_{st} (X_k, Y_l)$.*

## 6. Examples

In this section, we illustrate the forms of the limit laws. In Example (a), the forms of $G_{k,l}$, $T_{k,l,G}$, and $U_{k,l,G}$, are illustrated when $F$ is the standard Pareto law. In Example (b), the forms of $G_{k,l}$, $T_{k,l,G}$, and $U_{k,l,G}$, are illustrated when $F$ is the standard exponential distribution. In Example (c), Theorem 4.2 (b) is illustrated for the case when $k = 3$ and $l = 2$ for the standard Pareto law. Example (d) illustrates some results for the Uniform rv over (0,1), which belongs to the max domain $D_l(\Psi_\alpha)$ of attraction of the Weibull extreme value law under linear normalization.

(a) If $F(x) = 1 - \dfrac{1}{x}, x > 1$, then $\lim_{n\to\infty} F^n(nx) = e^{-1/x}$, $x > 0$. We have

$$G_{k,l}(x,y) = \begin{cases} e^{-1/y} \sum\limits_{j=0}^{l-1} \frac{(1/y)^j}{j!}, & x \geq y, \\ e^{-1/x} \sum\limits_{j=0}^{l-1} \frac{(1/y)^j}{j!} \sum\limits_{i=0}^{k-1-j} \frac{(1/x-1/y)^i}{i!}, & x \leq y; \end{cases}$$

$$T_{k,l,G}(x,y)=\begin{cases}\dfrac{1}{(1+1/y)^a}\sum_{j=0}^{l-1}\binom{j+a-1}{j}\left(\frac{1/y}{1+1/y}\right)^j, & x\geq y,\\ \dfrac{1}{(1+1/x)^a}\sum_{j=0}^{l-1}\left(\frac{1/y}{1+1/x}\right)^j\sum_{i=0}^{k-1-j}\binom{i+j+a-1}{i,j}\left(\frac{1/x-1/y}{1+1/x}\right)^i, & x\leq y;\end{cases}$$

when $N_n$ is the shifted negative binomial rv, and

$$U_{k,l,G}(x,y)=\begin{cases}l\left(\frac{1-e^{-1/y}}{1/y}\right)-G(y)\sum_{v=1}^{l-1}(l-v)\frac{(1/y)^{v-1}}{v!}, & x\geq y,\\ \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}\frac{(1/y-1/x)^i(1/y)^j}{(1/x)^{i+j+1}}\left\{1-e^{-1/x}\sum_{j=0}^{i+j}\frac{(1/x)^j}{j!}\right\}, & x\leq y,\end{cases}$$

when $N_n$ is the shifted discrete Uniform rv.

(b) If $F(x)=1-e^{-x}, x>0$, then $\lim_{n\to\infty}F^n(x+\ln n)=\exp\left(-e^{-x}\right), x\in\mathbb{R}$, and we have

$$G_{k,l}(x,y)=\begin{cases}\exp\left(-e^{-y}\right)\sum_{j=0}^{l-1}\frac{e^{-jy}}{j!}, & x\geq y,\\ \exp\left(-e^{-x}\right)\sum_{j=0}^{l-1}\frac{(e^{-y})^j}{j!}\sum_{i=0}^{k-1-j}\frac{(e^{-x}-e^{-y})^i}{i!}, & x\leq y;\end{cases}$$

$$T_{k,l,G}(x,y)=\begin{cases}\dfrac{1}{(1+e^{-y})^a}\sum_{j=0}^{l-1}\binom{j+a-1}{j}\left(\frac{e^{-y}}{1+e^{-y}}\right)^j, & x\geq y,\\ \dfrac{1}{(1+e^{-x})^a}\sum_{j=0}^{l-1}\left(\frac{e^{-y}}{1+e^{-x}}\right)^j\sum_{i=0}^{k-1-j}\binom{i+j+a-1}{i,j}\left(\frac{e^{-x}-e^{-y}}{1+e^{-x}}\right)^i, & x\leq y;\end{cases}$$

when $N_n$ is the shifted negative binomial rv, and

$$U_{k,l,G}(x,y)=\begin{cases}l\left(\frac{1-\exp\left(-e^{-y}\right)}{e^{-y}}\right)-G(y)\sum_{v=1}^{l-1}(l-v)\frac{(e^{-y})^{v-1}}{v!}, & x\geq y,\\ \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}\frac{(e^{-y}-e^{-x})^i(\exp(-y))^j}{(e^{-x})^{i+j+1}}\left\{1-\exp\left(-e^{-x}\right)\sum_{j=0}^{i+j}\frac{(e^{-x})^j}{j!}\right\}, & x\leq y,\end{cases}$$

when $N_n$ is the shifted discrete Uniform rv.

(c) If $F(x)=1-\dfrac{1}{x},\ x>1$, then $\lim_{t\to\infty}\dfrac{1-F(tx)}{1-F(t)}=\lim_{t\to\infty}\dfrac{(tx)^{-1}}{t^{-1}}=x^{-1}$. With $k=3$ and $l=2$ in Theorem 4.2 (b), we have

$$F_{3,2}(x,y)\quad=\quad\begin{cases}F(y)\sum_{j=0}^{1}\dfrac{(-\ln F(y))^j}{j!}, & x\geq y,\\ F(x)\sum_{j=0}^{1}\dfrac{(-\ln F(y))^j}{j!}\sum_{i=0}^{2-j}\frac{(\ln F(y)-\ln F(x))^i}{i!}, & x\leq y.\end{cases}$$

When $x\geq y$, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1-F_{3,2}(tx,ty)}{1-F_{3,2}(t,t)} &= \lim_{t\to\infty}\frac{1-F(ty)(1-\ln F(ty))}{1-F(t)(1-\ln F(t))}=\lim_{t\to\infty}\frac{1-(1-\frac{1}{ty})(1-\ln(1-\frac{1}{ty}))}{1-(1-\frac{1}{t})(1-\ln(1-\frac{1}{t}))}\\ &= \frac{1}{y}\lim_{t\to\infty}\frac{1-(1-\ln(1-\frac{1}{ty})}{1-(1-\ln(1-\frac{1}{t}))}=\frac{1}{y}\lim_{t\to\infty}\frac{\ln(1-\frac{1}{ty})}{\ln(1-\frac{1}{t})}\\ &= \frac{1}{y}\lim_{t\to\infty}\frac{(1-\frac{1}{ty})^{-1}(\frac{1}{t^2}\frac{1}{y})}{(1-\frac{1}{t})^{-1}(\frac{1}{t^2})}=\frac{1}{y^2}.\end{aligned}$$

When $x \leq y$, we compute $\lim_{t\to\infty} \dfrac{1-F_{3,2}(tx,ty)}{1-F_{3,2}(t,t)}$ now. Differentiating $F_{3,2}(tx,ty)$ with respect to $t$, we get $\frac{dF_{3,2}(tx,ty)}{dt}$ equal to

$$
\begin{aligned}
&\frac{1}{t^2x}\left(1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)+\frac{1}{2}\left\{\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}^2\right)\\
&+\left(1-\frac{1}{tx}\right)\left(\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}-\left(1-\frac{1}{tx}\right)^{-1}\frac{1}{t^2x}\right)\left\{1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}\\
&+\left\{-\ln\left(1-\frac{1}{ty}\right)\frac{1}{t^2x}-\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}\right\}\left(1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right)\\
&+\left(1-\frac{1}{tx}\right)\left\{-\ln\left(1-\frac{1}{ty}\right)\right\}\left(\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}-\left(1-\frac{1}{tx}\right)^{-1}\frac{1}{t^2x}\right)\\
=\ &\frac{1}{t^2x}\left(1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)+\frac{1}{2}\left\{\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}^2\right)\\
&+\frac{1}{t^2x}\left(-\ln\left(1-\frac{1}{ty}\right)-1\right)\left\{1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}\\
&+\left\{-\ln\left(1-\frac{1}{ty}\right)\right\}\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}-\frac{1}{t^2x}\right)\\
=\ &\frac{1}{t^2x}\left(\frac{1}{2}\left\{\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}^2\right)\\
&+\frac{1}{t^2x}\left(-\ln\left(1-\frac{1}{ty}\right)\right)\left\{1+\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}\\
&+\left\{-\ln\left(1-\frac{1}{ty}\right)\right\}\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}-\frac{1}{t^2x}\right)\\
=\ &\frac{1}{t^2x}\left(\frac{1}{2}\left\{\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}^2\right)+\frac{1}{t^2x}\left(-\ln\left(1-\frac{1}{ty}\right)\right)\left\{\ln\left(1-\frac{1}{ty}\right)-\ln\left(1-\frac{1}{tx}\right)\right\}\\
&+\left\{-\ln\left(1-\frac{1}{ty}\right)\right\}\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{t^2y}\right)\\
=\ &\frac{-1}{2xt^2}\left\{\left(\ln\left(1-\frac{1}{ty}\right)\right)^2-\left(\ln\left(1-\frac{1}{tx}\right)\right)^2\right\}-\ln\left(1-\frac{1}{ty}\right)\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{yt^2}\right),
\end{aligned}
$$

so that $\lim\limits_{t\to\infty} \dfrac{1-F_{3,2}(tx,ty)}{1-F_{3,2}(t,t)}$ is equal to

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{\frac{-1}{2xt^2}\left\{\left(\ln\left(1-\frac{1}{ty}\right)\right)^2-\left(\ln\left(1-\frac{1}{tx}\right)\right)^2\right\}-\ln\left(1-\frac{1}{ty}\right)\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{yt^2}\right)}{\frac{1}{t^2}\ln\left(1-\frac{1}{t}\right)}\\
=\ &\lim_{t\to\infty}\frac{\frac{-1}{2x}\left\{\left(\ln\left(1-\frac{1}{ty}\right)\right)^2-\left(\ln\left(1-\frac{1}{tx}\right)\right)^2\right\}-\ln\left(1-\frac{1}{ty}\right)\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{y}\right)}{\ln\left(1-\frac{1}{t}\right)}
\end{aligned}
$$

$$= \lim_{t\to\infty} \frac{1}{(1-\frac{1}{t})^{-1}\frac{1}{t^2}} \left\{ \frac{-1}{2x} \left\{ 2\left(\ln\left(1-\frac{1}{ty}\right)\right)(1-\frac{1}{ty})^{-1}\frac{1}{t^2y} - 2\left(\ln\left(1-\frac{1}{tx}\right)\right)\left(1-\frac{1}{tx}\right)^{-1}\frac{1}{t^2x} \right\} \right.$$
$$- \left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-2}\frac{1}{t^2y^2} - \ln\left(1-\frac{1}{ty}\right)\left(\left(-\frac{1}{t^2x}\right)\left(1-\frac{1}{ty}\right)^{-1}\frac{1}{y}\right)$$
$$\left. + \ln\left(1-\frac{1}{ty}\right)\left(\left(1-\frac{1}{tx}\right)\left(1-\frac{1}{ty}\right)^{-2}\frac{1}{t^2y}\right)\right\},$$

which is equal to $\frac{1}{y^2}$. Thus we get $w(x,y) = y^{-2}$ which satisfies $w(cx,cy) = c^{-2}w(x,y)$, and hence $\lim_{n\to\infty} F^n_{3,2}(nx,ny) = \exp(-y^{-2}),\ x>0, y>0$.

(d) Let $F(x) = x,\ 0<x<1$. Then $\lim_{t\to\infty} \frac{1-F(1-\frac{1}{tx})}{1-F(1-\frac{1}{t})} = \frac{1}{x}$ and the df $F \in D_l(\Psi_\alpha)$.

We have

$$T_{k,1,F}(x,y) = \begin{cases} \dfrac{1}{(1-\ln y)^a} & \text{if } x \geq y, \\ \sum\limits_{i=0}^{k-1} \binom{i+a-1}{i} \dfrac{(\ln y - \ln x)^i}{(1-\ln x)^{i+a}} & \text{if } x \leq y. \end{cases}$$

When $x \geq y$, we have

$$\lim_{t\to\infty} \frac{1-T_{k,1,F}(1-\frac{1}{tx}, 1-\frac{1}{ty})}{1-T_{k,1,F}(1-\frac{1}{t}, 1-\frac{1}{t})} = \lim_{t\to\infty} \frac{1-\left(1-\ln(1-\frac{1}{ty})\right)^{-a}}{1-\left(1-\ln(1-\frac{1}{t})\right)^{-a}},$$
$$= \lim_{t\to\infty} \frac{-a\left(1-\ln(1-\frac{1}{ty})\right)^{-a-1}(1-\frac{1}{ty})^{-1}\frac{1}{t^2y}}{-a\left(1-\ln(1-\frac{1}{t})\right)^{-a-1}(1-\frac{1}{t})^{-1}\frac{1}{t^2}} = \frac{1}{y}.$$

When $x \leq y$, we have

$$\lim_{t\to\infty} \frac{1-T_{k,1,F}(1-\frac{1}{tx}, 1-\frac{1}{ty})}{1-T_{k,1,F}(1-\frac{1}{t}, 1-\frac{1}{t})} = \lim_{t\to\infty} \frac{1-\sum\limits_{i=0}^{k-1}\binom{i+a-1}{i}\dfrac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^i}{(1-\ln(1-\frac{1}{tx}))^{i+a}}}{1-\left(1-\ln(1-\frac{1}{t})\right)^{-a}}.$$

We have

$$\begin{aligned}
\frac{dT_{k,1,F}(1-\frac{1}{tx}, 1-\frac{1}{ty})}{dt} &= \frac{d}{dt}\sum_{i=0}^{k-1}\frac{(i+a-1)!}{i!(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^i}{(1-\ln(1-\frac{1}{tx}))^{i+a}} \\
&= \frac{1}{t^2}\sum_{i=1}^{k-1}\frac{(i+a-1)!}{(i-1)!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^{i-1}}{(1-\ln(1-\frac{1}{tx}))^{i+a}} \times \\
&\quad \times\left((1-\frac{1}{ty})^{-1}\frac{1}{y} - (1-\frac{1}{tx})^{-1}\frac{1}{x}\right) \\
&\quad + \frac{1}{t^2}\sum_{i=0}^{k-1}\frac{(i+a)!}{i!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^i}{(1-\ln(1-\frac{1}{tx}))^{i+a+1}}\left(1-\frac{1}{tx}\right)^{-1}\frac{1}{x} \\
&= \frac{1}{t^2y}\sum_{i=0}^{k-2}\frac{(i+a)!}{i!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^i}{(1-\ln(1-\frac{1}{tx}))^{i+a+1}}\left(1-\frac{1}{ty}\right)^{-1} \\
&\quad + \frac{1}{t^2x}\frac{(k-1+a)!}{(k-1)!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^{k-1}}{(1-\ln(1-\frac{1}{tx}))^{i+a+1}}\left(1-\frac{1}{tx}\right)^{-1}
\end{aligned}$$

Then

$$\lim_{t\to\infty} \frac{1-T_{k,1,F}(1-\frac{1}{tx},1-\frac{1}{ty})}{1-T_{k,1,F}(1-\frac{1}{t},1-\frac{1}{t})}$$

$$= \lim_{t\to\infty} \frac{1}{a\left(1-\ln(1-\frac{1}{t})\right)^{-a-1}(1-\frac{1}{t})^{-1}\frac{1}{t^2}} \left\{ \frac{1}{t^2}\sum_{i=0}^{k-2}\frac{(i+a)!}{i!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^i}{(1-\ln(1-\frac{1}{tx}))^{i+a+1}}(1-\frac{1}{ty})^{-1}\frac{1}{y}\right.$$

$$\left. +\frac{1}{t^2}\frac{(k-1+a)!}{(k-1)!\,(a-1)!}\frac{(\ln(1-\frac{1}{ty})-\ln(1-\frac{1}{tx}))^{k-1}}{(1-\ln(1-\frac{1}{tx}))^{i+a+1}}\left(1-\frac{1}{tx}\right)^{-1}\frac{1}{x}\right\} = \frac{1}{y}.$$

Thus, we get, $w(x,y)=y^{-1}$, which satisfies $w(cx,cy)=c^{-1}w(x,y)$, and hence $\lim_{n\to\infty} T^n_{k,1}\left(1+\frac{x}{n},1+\frac{y}{n}\right)=\exp(-y^{-1}),\ x>0,y>0.$

## 7. Proofs

**Proof of Theorem 2.1.** Regrouping terms, (2) is rewritten as

$$\begin{aligned}
F_{k:n,l:n}(x,y) &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{n!}{(n-i-j)!\,i!\,j!}F^{n-i-j}(x)(F(y)-F(x))^i(1-F(y))^j \\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{1}{i!\,j!}\prod_{t=0}^{i+j-1}(n-t)F^{n-i-j}(x)(F(y)-F(x))^i(1-F(y))^j \\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{1}{i!\,j!}\prod_{t=0}^{i+j-1}\left(1-\frac{t}{n}\right)F^{n-i-j}(x)(n(F(y)-F(x)))^i(n(1-F(y)))^j.
\end{aligned}$$

Replacing $x$ by $g_n(x)$ and $y$ by $g_n(y)$ above and using (1), we get $\lim_{n\to\infty} n(1-F(g_n(x))=-\ln G(x)$ and $\lim_{n\to\infty} n\{F(g_n(y))-F(g_n(x))\}=\lim_{n\to\infty} n\{1-F(g_n(x))-(1-F(g_n(y)))\}=\ln G(y)-\ln G(x)$, so that

$$G_{k,l}(x,y)=\lim_{n\to\infty}F_{k:n,l:n}(x,y)=\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{1}{i!\,j!}G(x)(\ln G(y)-\ln G(x))^i(-\ln G(y))^j,$$

and (4) holds, completing the proof. □

**Proof of Corollary 2.2.** From (4), for $k>l>1$, we have, $G_{k,l}(x,y)$ equal to

$$G(x)\left(1+\sum_{i=1}^{k-1}\frac{(\ln G(y)-\ln G(x))^i}{i!}+\sum_{j=1}^{l-1}\frac{(-\ln G(y))^j}{j!}+\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\frac{(\ln G(y)-\ln G(x))^i(-\ln G(y))^j}{i!\,j!}\right).$$

Taking limit as $x\to y$ in the above, we get the marginal $G_l(.)$ and taking limit as $y\to r(F)$ in the above, we get the marginal $G_k(.)$. When $l=1$, $G_{k,1}(x,y)=G(x)+\sum_{i=1}^{k-1}\frac{G(x)(\ln G(y)-\ln G(x))^i}{i!}$, and taking limit as $x\to y$, we get the marginal $G_1(.)=G(.)$ and taking limit as $y\to r(F)$, we get the marginal $G_k(.)$, completing the proof. □

**Proof of Theorem 3.1.** (a) When $x\ge y$, (i) and (ii) follow as in the proof in pages 15 and 16 respectively, of Theorem 4.1 (a), Ravi and Manohar (2017). Let $x\le y$.

(i) Substituting $P(N_n=u)$ in (5), we get $F_{k:N_n,l:N_n}(x,y)$ equal to

$$\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\left\{\sum_{u=m}^{m+n}\binom{u}{i+j}F^{u-i-j}(x)\binom{n}{r-m}p_n^{r-m}q_n^{m+n-r}\right\}.$$

Using the summation results

$$\sum_{r=m}^{m+n} F^r(x)\binom{n}{r-m}p_n^{r-m}q_n^{m+n-r} = F^m(x)(p_nF(x)+q_n)^n$$

and

$$\sum_{r=m}^{m+n}\binom{r}{i}F^{r-i}(x)\binom{n}{r-m}p_n^{r-m}q_n^{m+n-r} = F^{m-i}(x)\sum_{l=0}^{i}\binom{m}{l}\binom{n}{i-l}(p_nF(x))^{i-l}(p_nF(x)+q_n)^{n-i+l},$$

as in page 15 of the proof of Theorem 4.1 (a), Ravi and Manohar (2017), we get $F_{k:N_n,l:N_n}(x,y)$ equal to

$$\begin{aligned}
&\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}n^{i+j}(F(y)-F(x))^i(1-F(y))^jF^{m-i-j}(x)\times\\
&\qquad\times\left\{\sum_{v=0}^{i+j}\binom{m}{v}\binom{n}{i+j-v}(p_nF(x))^{i+j-v}(p_nF(x)+q_n)^{n-i-j+v}\right\},\\
=&\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}n^{i+j}(F(y)-F(x))^i(1-F(y))^jF^{m-i-j}(x)\times\\
&\times\left\{\sum_{v=0}^{i+j}\binom{m}{v}\frac{1}{(i+j-v)!}\prod_{t=0}^{i+j-v-1}\left(1-\frac{t}{n}\right)(p_nF(x))^{i+j-v}(p_nF(x)+q_n)^{n-i-j+v}\frac{1}{n^v}\right\},\\
=&F^m(x)(p_nF(x)+q_n)^n+\sum_{i=1}^{k-1}\left\{n^i(F(y)-F(x))^iF^{m-i}(x)\sum_{v=0}^{i}\binom{m}{v}\frac{1}{(i-v)!}\times\right.\\
&\left.\times\prod_{t=0}^{i-v-1}\left(1-\frac{t}{n}\right)(p_nF(x))^{i-v}(p_nF(x)+q_n)^{n-i+v}\frac{1}{n^v}\right\}\\
&+\sum_{j=1}^{l-1}\left\{n^j(1-F(y))^jF^{m-j}(x)\sum_{v=0}^{j}\binom{m}{v}\frac{1}{(j-v)!}\times\right.\\
&\left.\times\prod_{t=0}^{j-v-1}\left(1-\frac{t}{n}\right)(p_nF(x))^{j-v}(p_nF(x)+q_n)^{n-j+v}\frac{1}{n^v}\right\}\\
&+\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\left\{\binom{i+j}{i}n^{i+j}(F(y)-F(x))^i(1-F(y))^jF^{m-i-j}(x)\sum_{v=0}^{i+j}\binom{m}{v}\times\right.\\
&\left.\times\frac{1}{(i+j-v)!}\prod_{t=0}^{i+j-v-1}\left(1-\frac{t}{n}\right)(p_nF(x))^{i+j-v}(p_nF(x)+q_n)^{n-i-j+v}\frac{1}{n^v}\right\}.
\end{aligned}$$

Replacing $x$ by $g_n(x)$, $y$ by $g_n(y)$, above and using (1) and

$$\lim_{n\to\infty}(q_n+p_nF(g_n(x)))^n = \lim_{n\to\infty}\left(1-\frac{np_n(1-F(g_n(x)))}{n}\right)^n = e^{-(-\ln G(x))} = G(x),$$

we get

$$G_{k,l}(x,y) = \lim_{n\to\infty}F_{k:N_n,l:N_n}(g_n(x),g_n(y))$$

$$\begin{aligned}
&= G(x)+\sum_{i=1}^{k-1}(\ln G(y)-\ln G(x))^i\left\{\frac{1}{i!}G(x)+\sum_{v=1}^{i}0\right\}+\sum_{j=1}^{l-1}(-\ln G(y))^j\left\{\frac{1}{j!}G(x)+\sum_{v=1}^{j}0\right\}\\
&\quad+\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}(\ln G(y)-\ln G(x))^i(-\ln G(y))^j\left\{\frac{1}{(i+j)!}G(x)+\sum_{v=1}^{i+j}0\right\}\\
&= \sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\frac{G(x)(\ln G(y)-\ln G(x))^i(-\ln G(y))^j}{i!\,j!},
\end{aligned}$$

which is the same as (4), proving (i).

(ii) Substituting $P(N_n=u)$ in (5), we have

$$F_{k:N_n,l:N_n}(x,y)=\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\left\{\sum_{u=m}^{\infty}\binom{u}{i+j}F^{u-i-j}(x)\frac{e^{-\lambda_n}\lambda_n^{r-m}}{(r-m)!}\right\}.$$

Using the summation results

$$\sum_{r=m}^{\infty}F^r(x)\frac{e^{-\lambda_n}\lambda_n^{r-m}}{(r-m)!}=e^{-\lambda_n}F^m(x)\sum_{r=m}^{\infty}\frac{(\lambda_nF(x))^{r-m}}{(r-m)!}=e^{-\lambda_n(1-F(x))}F^m(x)$$

and

$$\sum_{r=m}^{\infty}\binom{r}{i}F^{r-i}(x)\frac{e^{-\lambda_n}\lambda_n^{r-m}}{(r-m)!}=\frac{e^{-\lambda_n(1-F(x))}F^{m-i}(x)}{i!}\sum_{l=0}^{i}\binom{i}{l}\frac{m!}{(m-l)!}(\lambda_nF(x))^{i-l},$$

as in the proof of Theorem 4.1 (b) in page 16 in Ravi and Manohar (2017), one gets $F_{k:N_n,l:N_n}(x,y)$ equals to

$$\begin{aligned}
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\left\{\sum_{u=m}^{\infty}\binom{u}{i+j}F^{u-i-j}(x)\frac{e^{-\lambda_n}\lambda_n^{r-m}}{(r-m)!}\right\},\\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(\bar{F}(y))^j\frac{e^{-\lambda_n(1-F(x))}F^{m-i-j}(x)}{(i+j)!}\sum_{v=0}^{i+j}\binom{i+j}{v}\frac{m!}{(m-v)!}(\lambda_nF(x))^{i+j-v},\\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}n^{i+j}(F(y)-F(x))^i(1-F(y))^j\frac{e^{-\frac{\lambda_n}{n}n(1-F(x))}F^{m-i-j}(x)}{(i+j)!}\times\\
&\quad\times\sum_{v=0}^{i+j}\left\{\binom{i+j}{v}\frac{m!}{(m-v)!}\left(\frac{\lambda_n}{n}F(x)\right)^{i+j-v}\frac{1}{n^v}\right\}\\
&= e^{-\dfrac{\lambda_n}{n}n(1-F(x))}F^m(x)\\
&\quad+\sum_{i=1}^{k-1}\left\{n^i(F(y)-F(x))^i\frac{e^{-\frac{\lambda_n}{n}n(1-F(x)}F^{m-i}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}\frac{m!}{(m-v)!}\left(\frac{\lambda_n}{n}F(x)\right)^{i-v}\frac{1}{n^v}\right\}
\end{aligned}$$

$$+\sum_{j=1}^{l-1}\left\{n^j(1-F(x))^j\frac{e^{-\frac{\lambda_n}{n}n(1-F(x))}F^{m-j}(x)}{j!}\sum_{v=0}^{j}\binom{j}{v}\frac{m!}{(m-v)!}\left(\frac{\lambda_n}{n}F(x)\right)^{j-v}\frac{1}{n^v}\right\}$$
$$+\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\left\{\binom{i+j}{i}n^{i+j}(F(y)-F(x))^i(1-F(y))^j\times\right.$$
$$\left.\times\frac{e^{-\frac{\lambda_n}{n}n(1-F(x))}F^{m-i-j}(x)}{(i+j)!}\sum_{v=0}^{i+j}\binom{i+j}{v}\frac{m!}{(m-v)!}\left(\frac{\lambda_n}{n}F(x)\right)^{i+j-v}\frac{1}{n^v}\right\}.$$

Replacing $x$ by $g_n(x)$, $y$ by $g_n(y)$ above, and using (1) and $\lim_{n\to\infty}\frac{\lambda_n}{n}=1$, which is equivalent to

$$\lim_{n\to\infty}e^{-\lambda_n(1-F(g_n(x)))}=G(x),$$

we get

$$\begin{aligned}G_{k,l}(x,y) &= \lim_{n\to\infty}F_{k:N_n,l:N_n}(g_n(x),g_n(y))\\ &= G(x)+\sum_{i=1}^{k-1}\frac{(\ln G(y)-\ln G(x))^i}{i!}G(x)+\sum_{i=1}^{k-1}0=G(x)\sum_{i=0}^{k-1}\frac{(\ln G(y)-\ln G(x))^i}{i!},\end{aligned}$$

which is the same as (4), proving (ii).

(b) When $x\geq y$, the proof is the same as the proof in page 18 of Theorem 4.5, Ravi and Manohar (2017). When $x\leq y$, the proof is as follows. Substituting $P(N_n=u)$ in (5), we get

$$\begin{aligned}F_{k:N_n,l:N_n}(x,y) &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\times\\ &\quad\times\left\{\sum_{u=m}^{\infty}\binom{u}{i+j}F^{u-i-j}(x)\binom{u-m+a-1}{u-m}p_n^rq_n^{u-m}\right\}.\end{aligned}$$

Using the summation results

$$\sum_{l=m}^{\infty}F^l(x)\binom{l-m+r-1}{l-m}p_n^rq_n^{l-m}=\frac{p_n^rF^m(x)}{(1-q_nF(x))^r}$$

and

$$\sum_{l=m}^{\infty}\binom{l}{i}F^{l-i}(x)\binom{l-m+r-1}{l-m}p_n^rq_n^{l-m}=p_n^r\sum_{l=0}^{i}\binom{m}{i-l}\binom{r-1+l}{l}\frac{q_n^lF^{m-i+l}(x)}{(1-q_nF(x))^{r+l}},$$

as in the proof of Theorem 4.5 in page 18 in Ravi and Manohar (2017), one gets

$$\begin{aligned}&F_{k:N_n,l:N_n}(x,y)\\ &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\left\{\sum_{u=m}^{\infty}\binom{u}{i+j}F^{u-i-j}(x)\binom{u-m+a-1}{u-m}p_n^rq_n^{u-m}\right\},\\ &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^jp_n^a\sum_{v=0}^{i+j}\binom{m}{i+j-v}\binom{a-1+v}{v}\frac{q_n^vF^{m-i-j+v}(x)}{(1-q_nF(x))^{a+v}},\\ &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(n(F(y)-F(x)))^i(n\bar{F}(y))^j(np_n)^a\times\\ &\quad\times\sum_{v=0}^{i+j}\binom{m}{i+j-v}\binom{a-1+v}{v}\frac{q_n^vF^{m-i-j+v}(x)}{(n(1-q_nF(x)))^{a+v}}\frac{1}{n^{i+j-v}},\end{aligned}$$

$$
\begin{aligned}
&= \frac{(np_n)^a F^m(x)}{n^a(1-q_nF(x))^a} \\
&\quad + \sum_{i=1}^{k-1}(n(F(y)-F(x)))^i (np_n)^a \sum_{v=0}^{i}\binom{m}{i-v}\binom{a-1+v}{v}\frac{q_n^v F^{m-i+v}(x)}{(n(1-q_nF(x)))^{a+v}}\frac{1}{n^{i-v}} \\
&\quad + \sum_{j=1}^{l-1}(n(1-F(y)))^j (np_n)^a \sum_{v=0}^{j}\binom{m}{j-v}\binom{a-1+v}{v}\frac{q_n^v F^{m-j+v}(x)}{(n(1-q_nF(x)))^{a+v}}\frac{1}{n^{j-v}} \\
&\quad + \sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}(n(F(y)-F(x)))^i (n(1-F(y)))^j (np_n)^a \times \\
&\quad \times \sum_{v=0}^{i+j}\binom{m}{i+j-v}\binom{a-1+v}{v}\frac{q_n^v F^{m-i-j+v}(x)}{(n(1-q_nF(x)))^{a+v}}\frac{1}{n^{i+j-v}}.
\end{aligned}
$$

Replacing $x$ by $g_n(x)$, $y$ by $g_n(y)$ above and using (1) and

$$
\lim_{n\to\infty} q_n^k = \lim_{n\to\infty}\Big(1-\frac{np_n}{n}\Big)^k = 1,\ \lim_{n\to\infty} n(1-q_nF(g_n(x))) = 1-\ln G(x),
$$

we get

$$
\begin{aligned}
G_{k,l}(x,y) &= \lim_{n\to\infty} F_{k:N_n,l:N_n}(g_n(x),g_n(y)) \\
&= \frac{1}{1-\ln G(x)} + \sum_{i=1}^{k-1}(\ln G(y)-\ln G(x))^i\left\{\sum_{v=0}^{i-1}0+\binom{a-1+i}{i}\frac{1}{(1-\ln G(x))^{a+i}}\right\} \\
&\quad + \sum_{j=1}^{l-1}(-\ln G(y))^j\left\{\sum_{v=0}^{j-1}0+\binom{a-1+j}{j}\frac{1}{(1-\ln G(x))^{a+j}}\right\} \\
&\quad + \sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}(\ln G(y)-\ln G(x))^i(-\ln G(y))^j \times \\
&\quad \times \left\{\sum_{v=0}^{i+j-1}0+\binom{a-1+i+j}{i+j}\frac{1}{(1-\ln G(x))^{a+i+j}}\right\}, \\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j+a-1}{i,j}\left(\frac{\ln G(y)-\ln G(x)}{1-\ln G(x)}\right)^i\left(\frac{-\ln G(y)}{1-\ln G(x)}\right)^j\frac{1}{(1-\ln G(x))^a},
\end{aligned}
$$

and (6) holds.

(c) The proof follows from the proof of (b) above by putting $a=1$.

(d) When $x \geq y$, the proof follows as in the proof of Theorem 3.1 in page 13 in Ravi and Manohar (2017). When $x \leq y$, the proof is as follows. Substituting $P(N_n = u)$ in (5), we have

$$
F_{k:N_n,l:N_n}(x,y) = \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j\left\{\frac{1}{n}\sum_{u=m}^{\infty}\binom{u}{i+j}F^{u-i-j}(x)\right\}.
$$

Using the summation results $\frac{1}{n}\sum_{r=m+1}^{m+n}F^r(x) = \frac{F^{m+1}(x)-F^{m+n+1}(x)}{n(1-F(x))}$ and

$$
\frac{1}{n}\sum_{r=m+1}^{m+n}\binom{r}{i}F^{r-i}(x) = \frac{1}{n}\sum_{l=0}^{i}\left\{\binom{m+1}{l}\frac{F^{m+1-l}(x)}{(1-F(x))^{i+1-l}} - \binom{m+n+1}{l}\frac{F^{m+n+1-l}(x)}{(1-F(x))^{i+1-l}}\right\},
$$

as in the proof of Theorem 3.1 in page 13 in Ravi and Manohar (2017), one gets

$$\begin{aligned}
F_{k:N_n,l:N_n}(x,y) &= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}(F(y)-F(x))^i(1-F(y))^j \times \\
&\quad \times\frac{1}{n}\sum_{v=0}^{i+j}\left\{\binom{m+1}{v}\frac{F^{m+1-v}(x)}{(1-F(x))^{i+j+1-v}}-\binom{m+n+1}{v}\frac{F^{m+n+1-v}(x)}{(1-F(x))^{i+j+1-v}}\right\} \\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}\left(\frac{n(F(y)-F(x))}{n(1-F(x))}\right)^i\left(\frac{n(1-F(y))}{n(1-F(x))}\right)^j \times \\
&\quad \times\sum_{v=0}^{i+j}\left\{\binom{m+1}{v}\frac{F^{m+1-v}(x)}{(n(1-F(x)))^{1-v}}\frac{1}{n^v}-\prod_{t=-1}^{v-2}\left(1+\frac{m-t}{n}\right)\frac{F^{m+n+1-v}(x)}{v!(n(1-F(x)))^{1-v}}\right\} \\
&= \frac{F^{m+1}(x)-F^{m+n+1}(x)}{n(1-F(x))}+\sum_{i=1}^{k-1}\left(\frac{n(F(y)-F(x))}{n(1-F(x))}\right)^i \times \\
&\quad \times\sum_{v=0}^{i}\left\{\binom{m+1}{v}\frac{F^{m+1-v}(x)}{(n(1-F(x)))^{1-v}}\frac{1}{n^v}-\prod_{t=-1}^{v-2}\left(1+\frac{m-t}{n}\right)\frac{F^{m+n+1-v}(x)}{v!(n(1-F(x)))^{1-v}}\right\} \\
&\quad +\sum_{j=1}^{l-1}\left(\frac{n(1-F(y))}{n(1-F(x))}\right)^j\sum_{v=0}^{j}\left\{\binom{m+1}{v}\frac{F^{m+1-v}(x)}{(n(1-F(x)))^{1-v}}\frac{1}{n^v}\right. \\
&\quad \left.-\prod_{t=-1}^{v-2}\left(1+\frac{m-t}{n}\right)\frac{F^{m+n+1-v}(x)}{v!(n(1-F(x)))^{1-v}}\right\} \\
&\quad +\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}\left(\frac{n(F(y)-F(x))}{n(1-F(x))}\right)^i\left(\frac{n(1-F(y))}{n(1-F(x))}\right)^j \times \\
&\quad \times\sum_{v=0}^{i+j}\left\{\binom{m+1}{v}\frac{F^{m+1-v}(x)}{(n(1-F(x)))^{1-v}}\frac{1}{n^v}-\prod_{t=-1}^{v-2}\left(1+\frac{m-t}{n}\right)\frac{F^{m+n+1-v}(x)}{v!(n(1-F(x)))^{1-v}}\right\}.
\end{aligned}$$

Replacing $x$ by $g_n(x)$, $y$ by $g_n(y)$ above, and using (1) and $\lim_{n\to\infty}F^n(g_n(x))=G(x)$, we get

$$\begin{aligned}
U_{k,l}(x,y) &= \lim_{n\to\infty}F_{k:N_n,l:N_n}(x,y) \\
&= \frac{1-G(x)}{-\ln G(x)}+\sum_{i=1}^{k-1}\left(\frac{\ln G(y)-\ln G(x)}{-\ln G(x)}\right)^i\left\{\frac{1}{-\ln G(x)}+\sum_{v=1}^{i}0-G(x)\sum_{v=0}^{i}\frac{(-\ln G(x))^{v-1}}{v!}\right\} \\
&\quad +\sum_{j=1}^{l-1}\left(\frac{-\ln G(y)}{-\ln G(x)}\right)^j\left\{\frac{1}{-\ln G(x)}+\sum_{v=1}^{j}0-G(x)\sum_{v=0}^{j}\frac{(-\ln G(x))^{v-1}}{v!}\right\} \\
&\quad +\sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}\left(\frac{\ln G(y)-\ln G(x)}{-\ln G(x)}\right)^i\left(\frac{-\ln G(y)}{-\ln G(x)}\right)^j \times \\
&\quad \times\left\{\frac{1}{-\ln G(x)}+\sum_{v=1}^{i+j}0-G(x)\sum_{v=0}^{i+j}\frac{(-\ln G(x))^{v-1}}{v!}\right\} \\
&= \sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\binom{i+j}{i}\left(\frac{\ln G(y)-\ln G(x)}{-\ln G(x)}\right)^i\left(\frac{-\ln G(y)}{-\ln G(x)}\right)^j\left\{\frac{1}{-\ln G(x)}-G(x)\sum_{v=0}^{i+j}\frac{(-\ln G(x))^{v-1}}{v!}\right\},
\end{aligned}$$

so that (8) holds. □

**Proof of Corollary 3.2.** (a) The proof is the same as the proof of Corollary 2.2.

(b) From (6), for $k > l > 1$, we have

$$\begin{aligned} T_{k,l,G}(x,y) &= \frac{1}{1-\ln G(x)} + \sum_{i=1}^{k-1}\binom{a-1+i}{i}\frac{(\ln G(y)-\ln G(x))^i}{(1-\ln G(x))^{a+i}} + \sum_{j=1}^{l-1}\binom{a-1+j}{j}\frac{(-\ln G(y))^j}{(1-\ln G(x))^{a+j}} \\ &\quad + \sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}\binom{a-1+i+j}{i+j}\frac{(\ln G(y)-\ln G(x))^i(-\ln G(y))^j}{(1-\ln G(x))^{a+i+j}}. \end{aligned}$$

Taking limit as $x \to y$ in the above expression, we get the marginal $T_{l,G}(.)$ and taking limit as $y \to r(G)$, we get the marginal $T_{k,G}(.)$.
For the case $l = 1$, $T_{k,1,G}(x,y) = \frac{1}{1-\ln G(x)} + \sum_{i=1}^{k-1}\binom{a-1+i}{i}\frac{(\ln G(y)-\ln G(x))^i}{(1-\ln G(x))^{a+i}}$. Taking limit as $x \to y$, we get the marginal $T_{1,G}(y) = \dfrac{1}{1-\ln G(y)}$, and taking limit as $y \to r(G)$, we get the marginal $T_{k,G}(.)$, completing the proof.

(c) The proof follows from the proof of (b) by putting $a = 1$.

(d) From (8), for $k > l > 1$, we have

$$\begin{aligned} U_{k,l,G}(x,y) &= \frac{1-G(x)}{-\ln G(x)} + \sum_{i=1}^{k-1}\left(\frac{\ln G(y)-\ln G(x))}{-\ln G(x)}\right)^i\left\{\frac{1}{-\ln G(x)} - G(x)\sum_{v=0}^{i}\frac{(-\ln G(x))^{v-1}}{v!}\right\} \\ &\quad + \sum_{j=1}^{l-1}\left(\frac{-\ln G(y))}{-\ln G(x)}\right)^j\left\{\frac{1}{-\ln G(x)} - G(x)\sum_{v=0}^{j}\frac{(-\ln G(x))^{v-1}}{v!}\right\} \\ &\quad + \sum_{j=1}^{l-1}\sum_{i=1}^{k-1-j}\binom{i+j}{i}\left(\frac{\ln G(y)-\ln G(x))}{-\ln G(x)}\right)^i\left(\frac{-\ln G(y))}{-\ln G(x)}\right)^j \times \\ &\quad \times\left\{\frac{1}{-\ln G(x)} - G(x)\sum_{v=0}^{i+j}\frac{(-\ln G(x))^{v-1}}{v!}\right\}. \end{aligned}$$

Taking limit as $x \to y$, we get the marginal $U_{l,G}(.)$ and taking limit as $y \to r(G)$, we get the marginal $U_{k,G}(.)$.
For the case $l = 1$, $G_{k,1}(x,y) = \frac{1-G(x)}{-\ln G(x)} + \sum_{i=1}^{k-1}\left(\frac{\ln G(y)-\ln G(x))}{-\ln G(x)}\right)^i\left\{\frac{1}{-\ln G(x)} - G(x)\sum_{v=0}^{i}\frac{(-\ln G(x))^{v-1}}{v!}\right\}$.
Taking limit as $x \to y$, we get the marginal $U_{1,G}(y) = \dfrac{1-G(y)}{-\ln G(y)}$ and taking limit as $y \to r(G)$, we get the marginal $U_{k,G}(.)$. □

**Proof of Theorem 4.2.** (a) We have

$$\begin{aligned} \frac{\partial F_{k,l}(x,y)}{\partial x} &= \frac{-f(x)}{F(x)}\sum_{j=0}^{l-1}\frac{(-\ln F(y))^j}{j!}\left\{f(x)\sum_{i=0}^{k-1-j}\frac{(\ln F(y)-\ln F(x))^i}{i!} + F(x)\sum_{i=1}^{k-1-j}\frac{(\ln F(y)-\ln F(x))^{i-1}}{(i-1)!}\right\} \\ &= f(x)\sum_{j=0}^{l-1}\frac{(-\ln F(y))^j}{j!}\left\{\frac{(\ln F(y)-\ln F(x))^{k-1-j}}{(k-1-j)!}\right\}. \end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial^2 F_{k,l}(x,y)}{\partial x \partial y} &= f(x)\sum_{j=0}^{l-1}\frac{(-\ln F(y))^j}{j!}\left\{\frac{(\ln F(y)-\ln F(x))^{k-2-j}}{(k-2-j)!}\right\}\left\{\frac{f(y)}{F(y)}\right\} \\ &+f(x)\sum_{j=1}^{l-1}\frac{(-\ln F(y))^{j-1}}{(j-1)!}\left\{\frac{(\ln F(y)-\ln F(x))^{k-1-j}}{(k-1-j)!}\right\}\left\{\frac{-f(y)}{F(y)}\right\} \\ &= \frac{f(x)f(y)}{F(y)(l-1)!(k-l-1)!}(-\ln F(y))^{l-1}(\ln F(y)-\ln F(x))^{k-l-1}.\end{aligned}$$

To prove that the total integral is 1, we consider

$$\int_{-\infty}^{\infty}\int_{x}^{\infty} f_{k,l}(x,y)\ dy\ dx = \int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{f(x)f(y)}{F(y)(l-1)!(k-l-1)!}(-\ln F(y))^{l-1}(\ln F(y)-\ln F(x))^{k-l-1}\ dy\ dx.$$

Putting $t=\dfrac{-\ln F(y)}{-\ln F(x)}$, we get

$$\begin{aligned}\int_{x}^{\infty}\frac{f(y)}{F(y)}(-\ln F(y))^{l-1}(\ln F(y)-\ln F(x))^{k-l-1}\ dy &= (-\ln F(x))^{k-1}\int_0^1 t^{l-1}(1-t)^{k-l-1}\ dt \\ &= B(l,k-l)(-\ln F(x))^{k-1},\end{aligned}$$

so that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{x}^{\infty} f_{k,l}(x,y)\ dy\ dx &= \int_{-\infty}^{\infty}\frac{f(x)}{(l-1)!(k-l-1)!}B(l,k-l)(-\ln F(x))^{k-1}\ dx \\ &= \frac{1}{(k-1)!}\int_{-\infty}^{\infty} f(x)(-\ln F(x))^{k-1}\ dx = \frac{1}{(k-1)!}\int_0^{\infty} e^{-u}u^{k-1} = 1.\end{aligned}$$

The marginals can be obtained by differentiating the marginal dfs as in the proof Corollary 3.2 with the df $F$ in place of the df $G$.

(b) Let $F \in D_l(\Phi_\alpha)$. Then by Proposition 1.11 in page 54 in Resnick (1987), $1-F$ is regularly varying with index $\alpha$ so that $\lim_{t\to\infty}\dfrac{1-F(tx)}{1-F(t)} = x^{-\alpha}$, $x>0$. Equivalently,

$$\lim_{t\to\infty}\frac{-xf(tx)}{-f(t)} = x^{-\alpha}, \quad \text{and} \quad \lim_{t\to\infty}\frac{-\ln F(tx)}{1-F(t)} = \lim_{t\to\infty}\frac{1}{F(tx)}\frac{-xf(tx)}{-f(t)} = x^{-\alpha}. \tag{19}$$

We consider two cases, $l=1$ and $l>1$.

If $l=1$, then, $F_{k,1}(t,t)=F(t)$ and hence

$$\begin{aligned}\lim_{t\to\infty}\frac{1-F_{k,1}(tx,ty)}{(1-F(t))} &= \lim_{t\to\infty}\frac{1}{(1-F(t))}\left\{1-F(tx)\sum_{i=0}^{k-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\right\} \\ &= \lim_{t\to\infty}\left\{\frac{1-F(tx)}{1-F(t)} - F(tx)\frac{(\ln F(ty)-\ln F(tx))}{1-F(t)} - F(tx)\sum_{i=2}^{k-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!(1-F(t))}\right\},\end{aligned}$$

so that using (19),

$$\lim_{t\to\infty}\frac{1-F_{k,1}(tx,ty)}{(1-F_{k,1}(t,t))} = x^{-\alpha} - 1(-y^{-\alpha}+x^{-\alpha}) - 1\sum_{i=2}^{k-1}\frac{(-y^{-\alpha}+x^{-\alpha})}{i!}(0) = y^{-\alpha} = W(x,y). \tag{20}$$

If $l > 1$, we have $F_{k,l}(t,t) = F(t)\sum_{j=0}^{l-1}\frac{(-\ln F(t))^j}{j!} = F_l(t)$, as in Definition 2.1 in page 3 in Ravi and Manohar (2017). Since $\lim_{t\to\infty}\frac{1-F_l(t)}{(1-F(t))^l} = \frac{1}{l!}$, it follows that

$$\lim_{t\to\infty}\frac{1-F_{k,l}(tx,ty)}{1-F_{k,l}(t,t)} = \lim_{t\to\infty}\frac{(1-F(t))^l}{1-F_l(t)}\lim_{t\to\infty}\frac{1-F_{k,l}(tx,ty)}{(1-F(t))^l} = l!\lim_{t\to\infty}\frac{1-F_{k,l}(tx,ty)}{(1-F(t))^l}.$$

To apply L'Hospital's rule, we differentiate $1-F_{k,l}(tx,ty)$ with respect to $t$. We have

$$\begin{aligned}
\frac{d}{dt}\{1-F_{k,l}(tx,ty)\} &= -xf(tx)\sum_{j=0}^{l-1}\frac{(-\ln F(ty))^j}{j!}\sum_{i=0}^{k-1-j}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\\
&\quad -F(tx)\sum_{j=1}^{l-1}\frac{(-\ln F(ty))^{j-1}}{(j-1)!}\sum_{i=0}^{k-1-j}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\left(-\frac{yf(ty)}{F(ty)}\right)\\
&\quad -F(tx)\sum_{j=0}^{l-1}\frac{(-\ln F(ty))^j}{j!}\sum_{i=1}^{k-1-j}\frac{(\ln F(ty)-\ln F(tx))^{i-1}}{(i-1)!}\left(\frac{yf(ty)}{F(ty)}-\frac{xf(tx)}{F(tx)}\right)\\
&= -xf(tx)\sum_{j=0}^{l-1}\frac{(-\ln F(ty))^j}{j!}\left\{\sum_{i=0}^{k-1-j}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\right.\\
&\quad \left.-\sum_{i=0}^{k-2-j}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\right\}\\
&\quad -\frac{yf(ty)F(tx)}{F(ty)}\frac{(-\ln F(ty))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\\
&= -xf(tx)\sum_{j=0}^{l-1}\frac{(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^{k-1-j}}{j!\,(k-1-j)!}\\
&\quad -\frac{yf(ty)F(tx)(-\ln F(ty))^{l-1}}{F(ty)(l-1)!}\sum_{i=0}^{k-l-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}.
\end{aligned}$$

Thus one gets

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F_{k,l}(tx,ty)}{(1-F(t))^l} &= \lim_{t\to\infty}\frac{1}{-lf(t)(1-F(t))^{l-1}}\left\{-xf(tx)\sum_{j=0}^{l-1}\frac{(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^{k-1-j}}{j!\,(k-1-j)!}\right.\\
&\quad\left.-\frac{yf(ty)F(tx)}{F(ty)}\frac{(-\ln F(ty))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\right\}\\
&= \lim_{t\to\infty}\frac{1}{l}\left\{\frac{xf(tx)}{f(t)}\sum_{j=0}^{l-1}\frac{(1-F(t))^{k-l}}{j!\,(k-1-j)!}\left(\frac{-\ln F(ty)}{1-F(t)}\right)^j\left(\frac{\ln F(ty)-\ln F(tx)}{1-F(t)}\right)^{k-1-j}\right.\\
&\quad\left.+\frac{yf(ty)F(tx)}{F(ty)}\frac{1}{(l-1)!}\left(\frac{-\ln F(ty)}{1-F(t)}\right)^{l-1}\left(1+\sum_{i=1}^{k-l-1}\frac{(\ln F(ty)-\ln F(tx))^i}{i!}\right)\right\},
\end{aligned}$$

so that, using (19), $\lim_{t\to\infty}\frac{1-F_{k,l}(tx,ty)}{(1-F(t))^l} = \frac{1}{l}\left\{0+\frac{1}{(l-1)!}y^{-l\alpha}\right\}$, since $\lim_{t\to\infty}(1-F(t))^{k-l} = 0$ and for

$i \geq 1$, $\lim_{t\to\infty}(\ln F(ty) - \ln F(tx))^i = 0$. Then

$$\lim_{t\to\infty} \frac{1 - F_{k,l}(tx, ty)}{(1 - F_{k,l}(t,t))} = y^{-l\alpha} = W(x,y). \tag{21}$$

From (20) and (21), it follows that $W(cx, cy) = (cy)^{-l\alpha} = c^{-l\alpha}W(x,y)$. Then from Corollary 5.18, in page 281, Resnick (1987), the proof is complete. The norming constants $a_n^k$ and $a_n^l$ in (b) are as in (b) of Theorem 2.3, Ravi and Manohar (2017). □

**Proof of Theorem 4.3.** (a) We have

$$\begin{aligned}
\frac{\partial T_{k,l,F}(x,y)}{\partial y} &= \sum_{j=0}^{l-1}\sum_{i=1}^{k-1-j} \frac{(i+j+a-1)!}{(i-1)!\ j!\ (a-1)!} \frac{(-\ln F(y))^j(\ln F(y) - \ln F(x))^{i-1}}{(1-\ln F(x))^{i+j+a}} \left(\frac{f(y)}{F(y)}\right) \\
&\quad + \sum_{j=1}^{l-1}\sum_{i=0}^{k-1-j} \frac{(i+j+a-1)!}{i!\ (j-1)!\ (a-1)!} \frac{(-\ln F(y))^{j-1}(\ln F(y) - \ln F(x))^{i}}{(1-\ln F(x))^{i+j+a}} \left(\frac{-f(y)}{F(y)}\right) \\
&= \frac{f(y)}{F(y)} \sum_{i=0}^{k-l-1} \frac{(i+l+a-1)!}{(i)!\ (l-1)!\ (a-1)!} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{i}}{(1-\ln F(x))^{i+l+a}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial^2 T_{k,l,F}(x,y)}{\partial x \partial y} &= \frac{-f(x)f(y)(-\ln F(y))^{l-1}}{F(x)F(y)} \sum_{i=0}^{k-l-1} \frac{(i+l+a-1)!}{(i)!\ (l-1)!\ (a-1)!} \frac{i(\ln F(y) - \ln F(x))^{i-1}}{(1-\ln F(x))^{i+l+a}} \\
&\quad - \frac{-f(x)f(y)(-\ln F(y))^{l-1}}{F(x)F(y)} \sum_{i=0}^{k-l-1} \frac{(i+l+a-1)!}{(i)!\ (l-1)!\ (a-1)!} \frac{(i+l+a)(\ln F(y) - \ln F(x))^{i}}{(1-\ln F(x))^{i+l+a+1}} \\
&= \frac{-f(x)f(y)}{F(x)F(y)}(-\ln F(y))^{l-1} \left\{ \sum_{i=0}^{k-l-2} \frac{(i+l+a)!}{i!\ (l-1)!\ (a-1)!} \frac{(\ln F(y) - \ln F(x))^{i}}{D^{i+l+a+1}} \right. \\
&\quad \left. - \sum_{i=0}^{k-l-1} \frac{(i+l+a)!}{i!\ (l-1)!\ (a-1)!} \frac{(\ln F(y) - \ln F(x))^{i}}{(1-\ln F(x))^{i+l+a+1}} \right\} \\
&= \frac{-f(x)f(y)}{F(x)F(y)}(-\ln F(y))^{l-1} \left\{ - \frac{(k+a-1)!}{(k-l-1)!\ (l-1)!\ (a-1)!} \frac{(\ln F(y) - \ln F(x))^{k-l-1}}{(1-\ln F(x))^{k+a}} \right\} \\
&= \frac{\Gamma(k+a)}{\Gamma(k-l)\ \Gamma(l)\ \Gamma(a)} \frac{f(x)f(y)}{F(x)F(y)} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(1-\ln F(x))^{k+a}} > 0.
\end{aligned}$$

To prove that total integral is 1, we consider

$$\int_{-\infty}^{\infty}\int_{x}^{\infty} t_{k,l}(x,y)\, dy\, dx = \int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\Gamma(k+a)}{\Gamma(k-l)\ \Gamma(l)\ \Gamma(a)} \frac{f(x)f(y)}{F(x)F(y)} \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(1-\ln F(x))^{k+a}}\, dy\, dx.$$

Putting $t = \dfrac{-\ln F(y)}{-\ln F(x)}$, we get

$$\begin{aligned}
\int_{x}^{\infty} \frac{f(y)}{F(y)}(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}\, dy &= (-\ln F(x))^{k-1}\int_0^1 t^{l-1}(1-t)^{k-l-1}\, dt \\
&= B(l, k-l)(-\ln F(x))^{k-1},
\end{aligned}$$

so that

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{x}^{\infty} t_{k,l}(x,y)\, dy\, dx &= \int_{-\infty}^{\infty} \frac{\Gamma(k+a)}{\Gamma(k-l)\ \Gamma(l)\ \Gamma(a)} \frac{f(x)}{F(x)(1-\ln F(x))^{k+a}} B(l,k-l)(-\ln F(x))^{k-1}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{B(a,k)} \frac{f(x)(-\ln F(x))^{k-1}}{F(x)(1-\ln F(x))^{k+a}}\, dx = \int_0^{\infty} \frac{1}{B(a,k)} \frac{u^{k-1}}{(1+u)^{k+a}}\, du = 1.
\end{aligned}$$

The marginals can be obtained by differentiating the marginal dfs as in the proof Corollary 3.2 with the df $F$ in place of the df $G$.

(b) Let $F \in D_l(\Phi_\alpha)$. Then by Proposition 1.11 in page 54 in Resnick (1987), $1-F$ is regularly varying with index $-\alpha$ and hence $\lim_{t\to\infty} \dfrac{1-F(tx)}{1-F(t)} = x^{-\alpha}$.

First, $T_{k,l,F}(t,t) = \dfrac{1}{(1-\ln F(t))^a}\displaystyle\sum_{j=0}^{l-1}\binom{j+a-1}{j}\left(\frac{-\ln F(t)}{1-\ln F(t)}\right)^j = T_{l,F}(t)$, as defined in page 7 in Ravi and Manohar (2017). Since $\lim_{t\to\infty}\dfrac{1-T_{l,F}(t)}{(1-F(t))^l} = \dfrac{1}{lB(a,l)}$, it follows that

$$\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{1-T_{k,l,F}(t,t)} = \lim_{t\to\infty}\frac{(1-F(t))^l}{1-T_{l,F}(t)}\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{(1-F(t))^l} = lB(a,l)\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{(1-F(t))^l}.$$

We consider two cases. For $l=1$, we have

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-T_{k,1,F}(tx,ty)}{(1-F(t))} &= \lim_{t\to\infty}\frac{1}{(1-F(t))}\left\{1-\frac{1}{(1-\ln F(tx))^a}\sum_{i=0}^{k-1}\binom{i+a-1}{i}\left(\frac{\ln F(ty)-\ln F(tx)}{1-\ln F(tx)}\right)^i\right\}\\
&= \lim_{t\to\infty}\frac{1}{(1-F(t))}\left\{1-\frac{1}{(1-\ln F(tx))^a}\right\} - \lim_{t\to\infty}\frac{a}{(1-\ln F(tx))^{a+1}}\frac{\ln F(ty)-\ln F(tx)}{1-F(t)}\\
&\quad -\lim_{t\to\infty}\frac{1}{(1-\ln F(tx))^a}\sum_{i=2}^{k-1}\binom{i+a-1}{i}\left(\frac{(\ln F(ty)-\ln F(tx))^{i-1}}{(1-\ln F(tx))^i}\right)\frac{\ln F(ty)-\ln F(tx)}{(1-F(t))}\\
&= \lim_{t\to\infty}\frac{1}{(-f(t))}\left\{-a(1-\ln F(tx))^{-a-1}\left(-\frac{f(tx)x}{F(tx)}\right)\right\} - a(-y^{-\alpha}+x^{-\alpha}) - 0,
\end{aligned}$$

so that, using ( 19),

$$\lim_{t\to\infty}\frac{1-T_{k,1,F}(tx,ty)}{(1-F(t))^l} = ax^{-\alpha} - a(-y^{-\alpha}+x^{-\alpha}) - 0 = ay^{-\alpha}.$$

Then

$$\lim_{t\to\infty}\frac{1-T_{k,1,F}(tx,ty)}{1-T_{k,1,F}(t,t)} = B(a,1)ay^{-\alpha} = y^{-\alpha} = W(x,y).$$

If $l>1$, we consider $\lim_{t\to\infty}\dfrac{1-T_{k,l,F}(tx,ty)}{(1-F(t))^l}$. To apply L'Hospital's rule, differentiating $T_{k,l,F}(x,y)$ with respect to $t$, we get

$$\begin{aligned}
\frac{dT_{k,l,F}(x,y)}{dt} &= \sum_{j=1}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j+a-1)!}{i!(j-1)!(a-1)!}\frac{(-\ln F(ty))^{j-1}(\ln F(ty)-\ln F(tx))^i}{(1-\ln F(tx))^{i+j+a}}\left(-\frac{yf(ty)}{F(ty)}\right)\\
&\quad +\sum_{j=0}^{l-1}\sum_{i=1}^{k-1-j}\frac{(i+j+a-1)!}{(i-1)!j!(a-1)!}\frac{(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^{i-1}}{(1-\ln F(tx))^{i+j+a}}\left(\frac{yf(ty)}{F(ty)}-\frac{xf(tx)}{F(tx)}\right)\\
&\quad -\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j+a-1)!}{i!j!(a-1)!}\frac{(i+j+a)(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^i}{(1-\ln F(tx))^{i+j+a+1}}\left(-\frac{xf(tx)}{F(tx)}\right)\\
&= \frac{yf(ty)}{F(ty)}\sum_{i=0}^{k-l-1}\frac{(l+i+a-1)!}{(l-1)!i!(a-1)!}\frac{(-\ln F(ty))^{l-1}(\ln F(ty)-\ln F(tx))^i}{(1-\ln F(tx))^{l+i+a}}\\
&\quad +\frac{xf(tx)}{F(tx)}\sum_{j=0}^{l-1}\frac{(k-1+a)!}{(k-1-j)!j!(a-1)!}\frac{(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^{k-1-j}}{(1-\ln F(tx))^{k+a}}.
\end{aligned}$$

Thus one gets

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{(1-F(t))^l}\\
=\;&\lim_{t\to\infty}\frac{1}{-lf(t)(1-F(t))^{l-1}}\left\{-\frac{xf(tx)}{F(tx)}\sum_{j=0}^{l-1}\frac{(k-1+a)!}{(k-1-j)!j!(a-1)!}\frac{(-\ln F(ty))^j(\ln F(ty)-\ln F(tx))^{k-1-j}}{(1-\ln F(tx))^{k+a}}\right.\\
&\left.-\frac{yf(ty)}{F(ty)}\sum_{i=0}^{k-l-1}\frac{(l+i+a-1)!}{(l-1)!i!(a-1)!}\frac{(-\ln F(ty))^{l-1}(\ln F(ty)-\ln F(tx))^i}{(1-\ln F(tx))^{l+i+a}}\right\}\\
=\;&\lim_{t\to\infty}\frac{1}{l}\left\{\frac{xf(tx)}{f(t)F(tx)}\sum_{j=0}^{l-1}\frac{(k-1+a)!}{j!\,(k-1-j)!}\frac{(1-F(t))^{k-l}}{(1-\ln F(tx))^{k+a}}\left(\frac{-\ln F(ty)}{1-F(t)}\right)^j\left(\frac{\ln F(ty)-\ln F(tx)}{1-F(t)}\right)^{k-1-j}\right.\\
&\left.+\frac{yf(ty)}{f(t)F(ty)}\left(\frac{-\ln F(ty)}{1-F(t)}\right)^{l-1}\left(\sum_{i=0}^{k-l-1}\frac{(l+i+a-1)!}{(l-1)!i!(a-1)!}\frac{(\ln F(ty)-\ln F(tx))^i}{(1-\ln F(tx))^{l+i+a}}\right)\right\},
\end{aligned}$$

so that, using (19),

$$\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{(1-F(t))^l}=\frac{(l+a-1)!}{l!\,(a-1)!}\,y^{-l\alpha},$$

since $\lim_{t\to\infty}(1-F(t))^{k-l}=0$ and for $i\geq 1,\ \lim_{t\to\infty}(\ln F(ty)-\ln F(tx))^i=0$. Then

$$\lim_{t\to\infty}\frac{1-T_{k,l,F}(tx,ty)}{1-T_{k,l,F}(t,t)}=lB(a,l)\frac{(l+a-1)!}{l!\,(a-1)!}y^{-l\alpha}=y^{-l\alpha}=W(x,y).$$

From (20) and (21), it follows that that $W(cx,cy)=(cy)^{-l\alpha}=c^{-l\alpha}W(x,y)$. Then using Corollary 5.18, in page 281, Resnick (1987), the proof is complete. The norming constants $a_n^k$ and $a_n^l$ in (b) are as in (b) of Theorem 4.7, Ravi and Manohar (2017). □

**Proof of Theorem 4.5.** (a) We have

$$\begin{aligned}
\frac{\partial U_{k,l,F}(x,y)}{\partial y}&=\frac{f(y)}{F(y)}\sum_{j=0}^{l-1}\sum_{i=1}^{k-1-j}\frac{(i+j)!}{(i-1)!j!}\frac{(\ln F(y)-\ln F(x))^{i-1}(-\ln F(y))^j}{(-\ln F(x))^{i+j+1}}\left\{1-F_{i+j+1}(x)\right\}\\
&\quad-\frac{f(y)}{F(y)}\sum_{j=1}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j)!}{(j-1)!i!}\frac{(\ln F(y)-\ln F(x))^{i}(-\ln F(y))^{j-1}}{(-\ln F(x))^{i+j+1}}\left\{1-F_{i+j+1}(x)\right\}\\
&=\frac{f(y)}{F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(i+l)!}{i!}\frac{(\ln F(y)-\ln F(x))^{i}}{(-\ln F(x))^{i+l+1}}\left\{1-F_{i+l+1}(x)\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial^2 U_{k,l,F}(x,y)}{\partial x\partial y}&=\frac{-f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=1}^{k-l-1}\frac{(i+l)!}{(i-1)!}\frac{(\ln F(y)-\ln F(x))^{i-1}}{(-\ln F(x))^{i+l+1}}\left\{1-F_{i+l+1}(x)\right\}\\
&\quad+\frac{f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(i+l)!}{i!}\frac{(i+l+1)(\ln F(y)-\ln F(x))^{i}}{(-\ln F(x))^{i+l+2}}\left\{1-F_{i+l+1}(x)\right\}\\
&\quad+\frac{f(y)}{F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(i+l)!}{i!}\frac{(\ln F(y)-\ln F(x))^{i}}{(-\ln F(x))^{i+l+1}}\left\{-f(x)\frac{(-\ln F(x))^{i+l}}{(i+l)!}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-2}\frac{(i+l+1)!}{i!}\frac{(\ln F(y)-\ln F(x))^i}{(-\ln F(x))^{i+l+2}}\{1-F_{i+l+2}(x)\}\\
&\quad+\frac{f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{(i+l+1)!}{i!}\frac{(\ln F(y)-\ln F(x))^i}{(-\ln F(x))^{i+l+2}}\{1-F_{i+l+1}(x)\}\\
&\quad-\frac{f(x)f(y)}{F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{1}{i!}\frac{(\ln F(y)-\ln F(x))^i}{(-\ln F(x))}\\
&= \frac{f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-2}\frac{(i+l+1)!}{i!}\frac{(\ln F(y)-\ln F(x))^i}{(-\ln F(x))^{i+l+2}}\left\{F(x)\frac{(-\ln F(x))^{i+l+1}}{(i+l+1)!}\right\}\\
&\quad+\frac{f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\frac{k!}{(k-l-1)!}\frac{(\ln F(y)-\ln F(x))^{k-l-1}}{(-\ln F(x))^{k-1+2}}\{1-F_k(x)\}\\
&\quad-\frac{f(x)f(y)}{F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\sum_{i=0}^{k-l-1}\frac{1}{i!}\frac{(\ln F(y)-\ln F(x))^i}{(-\ln F(x))}\\
&= \frac{f(x)f(y)}{F(x)F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\frac{k!}{(k-l-1)!}\frac{(\ln F(y)-\ln F(x))^{k-l-1}}{(-\ln F(x))^{k+1}}\{1-F_k(x)\}\\
&\quad-\frac{f(x)f(y)}{F(y)}\frac{(-\ln F(y))^{l-1}}{(l-1)!}\left\{\frac{1}{(k-l-1)!}\frac{(\ln F(y)-\ln F(x))^{k-l-1}}{(-\ln F(x))}\right\}\\
&= \frac{f(x)f(y)}{F(x)F(y)}\frac{k!}{(l-1)!(k-l-1)!}\frac{(-\ln F(y))^{l-1}(\ln F(y)-\ln F(x))^{k-l-1}}{(-\ln F(x))^{k+1}}\{1-F_{k+1}(x)\}.
\end{aligned}
$$

To prove that the total integral is 1, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_x^{\infty}u_{k,l}(x,y)\,dy\,dx &= \int_{-\infty}^{\infty}\frac{\Gamma(k+1)}{\Gamma(l)\Gamma(k-l)}\frac{f(x)}{F(x)(-\ln F(x))^{k+1}}I(l,k-l)\{1-F_{k+1}(x)\}\,dx\\
&= \int_{-\infty}^{\infty}\frac{k\Gamma(k)}{\Gamma(l)\Gamma(k-l)}\frac{f(x)(-\ln F(x))^{k-1}}{F(x)(-\ln F(x))^{k+1}}B(l,k-l)\{1-F_{k+1}(x)\}\,dx\\
&= k\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\frac{1-F_{k+1}(x)}{(-\ln F(x))^2}\,dx.
\end{aligned}
$$

Integrating by parts, we get,

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\frac{1-F_{k+1}(x)}{(-\ln F(x))^2}\,dx &= \frac{1-F_{k+1}(x)}{(-\ln F(x))^2}\{\ln F(x)\}\Big|_{-\infty}^{\infty}-\\
&\quad\int_{-\infty}^{\infty}\{\ln F(x)\}\frac{(-\ln F(x))^2(-f_{k+1}(x))-2(1-F_{k+1}(x))(-\ln F(x))\left(-\frac{f(x)}{F(x)}\right)}{(-\ln F(x))^4}\,dx\\
&= 0+\int_{-\infty}^{\infty}\frac{(-f_{k+1}(x))}{(-\ln F(x))}\,dx+2\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\frac{(1-F_{k+1}(x))}{(-\ln F(x))^2}\,dx,\\
&= -\int_{-\infty}^{\infty}\frac{1}{k!}f(x)(-\ln F(x))^{k-1}\,dx+2\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\frac{1-F_{k+1}(x)}{(-\ln F(x))^2}\,dx.
\end{aligned}
$$

Hence $\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\frac{1-F_{k+1}(x)}{(-\ln F(x))^2}\,dx=\int_0^{\infty}\frac{1}{\Gamma(k+1)}e^{-u}u^{k-1}=\frac{1}{k}$ and the result follows.

The marginals can be obtained by differentiating the marginal dfs as in the proof of Corollary 3.2 with the df $F$ in place of the df $G$.

(b) Let $F\in D_l(\Phi_\alpha)$. Then by Proposition 1.11 in page 54 in Resnick (1987), $1-F$ is regularly varying with index $-\alpha$ and hence $\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)}=x^{-\alpha}$.

First, $U_{k,l,F}(t,t) = U_{l,F}(t)$ as in Definition 3.1 in page 5 in Ravi and Manohar (2017). Since

$$\lim_{t\to\infty} \frac{1-U_{l,F}(t)}{(1-F(t))^l} = \frac{1}{(l+1)!},$$

it follows that

$$\lim_{t\to\infty} \frac{1-U_{k,l,F}(tx,ty)}{1-U_{k,l,F}(t,t)} = \lim_{t\to\infty} \frac{(1-F(t))^l}{1-U_{l,F}(t)} \lim_{t\to\infty} \frac{1-U_{k,l,F}(tx,ty)}{(1-F(t))^l} = (l+1)! \lim_{t\to\infty} \frac{1-U_{k,l,F}(tx,ty)}{(1-F(t))^l}.$$

We have $U_{k,1,F}(tx,ty) = \sum_{i=0}^{k-1} \frac{(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+1}} \{1 - F_{i+1}(tx)\}$. Note that

$$\begin{aligned}
F_{i+1}(tx) &= F(tx) \sum_{v=0}^{i} \frac{(-\ln F(tx))^v}{(v)!}, \text{ and} \\
\frac{dF_{i+1}(tx)}{dt} &= xf(tx) \sum_{v=0}^{i} \frac{(-\ln F(tx))^v}{(v)!} + F(tx) \sum_{v=1}^{i} \frac{(-\ln F(tx))^{v-1}}{(v-1)!} \left( \frac{-xf(tx)}{F(tx)} \right), \\
&= xf(tx) \sum_{v=0}^{i} \frac{(-\ln F(tx))^v}{(v)!} - xf(tx) \sum_{v=0}^{i-1} \frac{(-\ln F(tx))^v}{(v)!} = xf(tx) \frac{(-\ln F(tx))^i}{i!}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\frac{dU_{k,l,F}(tx,ty)}{dt} &= \sum_{i=1}^{k-1} \frac{i(\ln F(ty) - \ln F(tx))^{i-1}}{(-\ln F(tx))^{i+1}} \{1 - F_{i+1}(tx)\} \left( \frac{yf(ty)}{F(ty)} - \frac{xf(tx)}{F(tx)} \right) \\
&\quad - \sum_{i=0}^{k-1} \frac{(i+1)(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+2}} \{1 - F_{i+1}(tx)\} \left( -\frac{xf(tx)}{F(tx)} \right) \\
&\quad + \sum_{i=0}^{k-1} \frac{(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+1}} \left\{ -xf(tx) \frac{(-\ln F(tx))^i}{i!} \right\}, \\
&= \frac{yf(ty)}{F(ty)} \sum_{i=0}^{k-2} \frac{(i+1)(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+2}} \{1 - F_{i+2}(tx)\} \\
&\quad + \frac{xf(tx)}{F(tx)} \frac{k(\ln F(ty) - \ln F(tx))^{k-1}}{(-\ln F(tx))^{k+1}} \{1 - F_k(tx)\} \\
&\quad + \frac{xf(tx)}{F(tx)} \sum_{i=0}^{k-2} \frac{(i+1)(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+2}} \left\{ F(tx) \frac{(-\ln F(tx))^{i+1}}{(i+1)!} \right\} \\
&\quad - xf(tx) \sum_{i=0}^{k-1} \frac{(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))i!} \\
&= \frac{yf(ty)}{F(ty)} \sum_{i=0}^{k-2} \frac{(i+1)(\ln F(ty) - \ln F(tx))^i}{(-\ln F(tx))^{i+2}} \{1 - F_{i+2}(tx)\} \\
&\quad + \frac{xf(tx)}{F(tx)} \frac{k(\ln F(ty) - \ln F(tx))^{k-1}}{(-\ln F(tx))^{k+1}} \{1 - F_k(tx)\} - xf(tx) \frac{(\ln F(ty) - \ln F(tx))^{k-1}}{(-\ln F(tx))(k-1)!}.
\end{aligned}$$

Then

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1-U_{k,1,F}(tx,ty)}{(1-F(t))}\\
=\ &\lim_{t\to\infty}\left\{\frac{yf(ty)}{F(ty)}\sum_{i=0}^{k-2}(i+1)\left(\frac{\ln F(ty)-\ln F(tx)}{-\ln F(tx)}\right)^i\left\{\frac{1-F_{i+2}(tx)}{(-\ln F(tx))^2}\right\}\right.\\
&\left.+\frac{kxf(tx)}{F(tx)f(t)}\left(\frac{\ln F(ty)-\ln F(tx)}{-\ln F(tx)}\right)^{k-1}\left\{\frac{1-F_k(tx)}{(-\ln F(tx))^2}\right\}-\frac{xf(tx)}{f(t)}\frac{(\ln F(ty)-\ln F(tx))^{k-1}}{(-\ln F(tx))(k-1)!}\right\}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\ln F(ty)-\ln F(tx)}{-\ln F(tx)} &= \lim_{t\to\infty}\frac{(\ln F(ty)-\ln F(tx))/(1-F(t))}{(-\ln F(tx))/(1-F(t))}=\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}},\\
\lim_{t\to\infty}\frac{1-F_{i+2}(tx)}{(-\ln F(tx))^2} &= \lim_{t\to\infty}\frac{-xf(tx)(-\ln F(tx))^{i+1}/(i+1)!}{(2)(-\ln F(tx))(-xf(tx)/F(tx))}=\lim_{t\to\infty}\frac{F(tx)(-\ln F(tx))^i}{2(i+1)!}\\
&= \begin{cases}1/2, & i=0\\ 0, & i\geq 1,\end{cases}\\
\lim_{t\to\infty}\frac{1-F_k(tx)}{(-\ln F(tx))^2} &= \lim_{t\to\infty}\frac{-xf(tx)(-\ln F(tx))^{k-1}/(k-1)!}{(2)(-\ln F(tx))(-xf(tx)/F(tx))}=\lim_{t\to\infty}\frac{F(tx)(-\ln F(tx))^{k-2}}{2(k-1)!}\\
&= \begin{cases}1/2, & k=2\\ 0, & k\geq 3.\end{cases}
\end{aligned}
$$

Then, for $k=2$, we get

$$
\lim_{t\to\infty}\frac{1-U_{k,1,F}(tx,ty)}{(1-F(t))}=y^{-\alpha}(\frac{1}{2})+2(x^{-\alpha})\left(\frac{x^{-\alpha}-x^{-\alpha}}{x^{-\alpha}}\right)\frac{1}{2}-(x^{-\alpha})\left(\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}}\right)(1)=\frac{1}{2}y^{-\alpha},
$$

and for $k>2$, we get

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1-U_{k,1,F}(tx,ty)}{(1-F(t))} &= y^{-\alpha}\left(\frac{1}{2}+\sum_{i=1}^{k-2}0\right)+2(x^{-\alpha})\left(\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}}\right)(0)-(x^{-\alpha})\left(\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}}\right)(0)\\
&= \frac{1}{2}y^{-\alpha}.
\end{aligned}
$$

Thus

$$
\lim_{t\to\infty}\frac{1-U_{k,1,F}(tx,ty)}{1-U_{k,1,F}(t,t)}=2\left\{\frac{1}{2}y^{-\alpha}\right\}=y^{-\alpha}=W(x,y).
$$

Let $l>1$. Then we have

$$
\begin{aligned}
&\frac{dU_{k,l,F}(tx,ty)}{dt}\\
=\ &\sum_{j=1}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j)!}{i!\,(j-1)!}\frac{(\ln F(ty)-\ln F(tx))^i(-\ln F(ty))^{j-1}}{(-\ln F(tx))^{i+j+1}}\{1-F_{i+j+1}(tx)\}\left(-\frac{yf(ty)}{F(ty)}\right)\\
&+\sum_{j=0}^{l-1}\sum_{i=1}^{k-1-j}\frac{(i+j)!}{(i-1)!\,j!}\frac{(\ln F(ty)-\ln F(tx))^{i-1}(-\ln F(ty))^j}{(-\ln F(tx))^{i+j+1}}\{1-F_{i+j+1}(tx)\}\left(\frac{yf(ty)}{F(ty)}-\frac{xf(tx)}{F(tx)}\right)\\
&-\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j+1)!}{i!\,j!}\frac{(\ln F(ty)-\ln F(tx))^i(-\ln F(ty))^j}{(-\ln F(tx))^{i+j+2}}\{1-F_{i+j+1}(tx)\}\left(-\frac{xf(tx)}{F(tx)}\right)\\
&+\sum_{j=0}^{l-1}\sum_{i=0}^{k-1-j}\frac{(i+j)!}{i!\,j!}\frac{(\ln F(ty)-\ln F(tx))^i(-\ln F(ty))^j}{(-\ln F(tx))^{i+j+1}}\left\{-xf(tx)\frac{(-\ln F(tx))^{i+j}}{(i+j)!}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{yf(ty)}{F(ty)} \sum_{i=0}^{k-1-l} \frac{(i+l)!}{i!\,(l-1)!} \frac{(\ln F(ty) - \ln F(tx))^i (-\ln F(ty))^{l-1}}{(-\ln F(tx))^{i+l+1}} \{1 - F_{i+l+1}(tx)\} \\
&\quad + \frac{xf(tx)}{F(ty)} \sum_{j=0}^{l-1} \sum_{i=0}^{k-2-j} \frac{(i+j+1)!}{i!\,j!} \frac{(\ln F(ty) - \ln F(tx))^i (-\ln F(ty))^j}{(-\ln F(tx))^{i+j+2}} \left\{ \frac{F(tx)(-\ln F(tx))^{i+j+1}}{(i+j+1)!} \right\} \\
&\quad + \frac{xf(tx)}{F(tx)} \sum_{j=0}^{l-1} \frac{k!}{(k-1-j)!\,j!} \frac{(\ln F(ty) - \ln F(tx))^{k-1-j} (-\ln F(ty))^j}{(-\ln F(tx))^{k+1}} \{1 - F_k(tx)\} \\
&\quad - xf(tx) \sum_{j=0}^{l-1} \sum_{i=0}^{k-1-j} \frac{1}{i!\,j!} \frac{(\ln F(ty) - \ln F(tx))^i (-\ln F(ty))^j}{(-\ln F(tx))} \\
&= \frac{yf(ty)}{F(ty)} \sum_{i=0}^{k-1-l} \frac{(i+l)!}{i!\,(l-1)!} \frac{(\ln F(ty) - \ln F(tx))^i (-\ln F(ty))^{l-1}}{(-\ln F(tx))^{i+l+1}} \{1 - F_{i+l+1}(tx)\} \\
&\quad + \frac{xf(tx)}{F(tx)} \sum_{j=0}^{l-1} \frac{k!}{(k-1-j)!\,j!} \frac{(\ln F(ty) - \ln F(tx))^{k-1-j} (-\ln F(ty))^j}{(-\ln F(tx))^{k+1}} \{1 - F_k(tx)\} \\
&\quad - xf(tx) \sum_{j=0}^{l-1} \frac{1}{(k-1-j)!\,j!} \frac{(\ln F(ty) - \ln F(tx))^{k-1-j} (-\ln F(ty))^j}{(-\ln F(tx))}.
\end{aligned}
$$

Thus one gets

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1 - U_{k,l,F}(tx,ty)}{(1-F(t))^l} \\
&= \lim_{t\to\infty} \frac{1}{lf(t)(1-F(t))^{l-1}} \left\{ \frac{yf(ty)}{F(ty)} \sum_{i=0}^{k-1-l} \frac{(i+l)!}{i!\,(l-1)!} \frac{(\ln F(ty) - \ln F(tx))^i (-\ln F(ty))^{l-1}}{(-\ln F(tx))^{i+l+1}} \{1 - F_{i+l+1}(tx)\} \right. \\
&\quad + \frac{xf(tx)}{F(tx)} \sum_{j=0}^{l-1} \frac{k!}{(k-1-j)!\,j!} \frac{(\ln F(ty) - \ln F(tx))^{k-1-j} (-\ln F(ty))^j}{(-\ln F(tx))^{k+1}} \{1 - F_k(tx)\} \\
&\quad \left. - xf(tx) \sum_{j=0}^{l-1} \frac{1}{(k-1-j)!\,j!} \frac{(\ln F(ty) - \ln F(tx))^{k-1-j} (-\ln F(ty))^j}{(-\ln F(tx))} \right\} \\
&= \lim_{t\to\infty} \frac{1}{l} \left\{ \frac{yf(ty)F(tx)}{F(ty)f(t)} \left( \frac{-\ln F(ty)}{1-F(t)} \right)^{l-1} \sum_{i=0}^{k-1-l} \frac{(i+l)!}{i!\,(l-1)!} \left( \frac{\ln F(ty) - \ln F(tx)}{-\ln F(tx)} \right)^i \frac{1 - F_{i+l+1}(tx)}{(-\ln F(tx))^{l+1}} \right. \\
&\quad + \frac{xf(tx)}{F(tx)f(t)} \sum_{j=0}^{l-1} \frac{k!(1-F(t))^{k-l-1}}{(k-1-j)!\,j!} \left( \frac{\ln F(ty) - \ln F(tx)}{1-F(t)} \right)^{k-1-j} \left( \frac{-\ln F(ty)}{1-F(t)} \right)^j \frac{1-F_k(tx)}{(-\ln F(tx))^k} \frac{1-F(t)}{-\ln F(tx)} \\
&\quad \left. - \frac{xf(tx)}{f(t)} \sum_{j=0}^{l-1} \frac{1}{(k-1-j)!\,j!} \left( \frac{\ln F(ty) - \ln F(tx)}{1-F(t)} \right)^{k-1-j} \left( \frac{-\ln F(ty)}{1-F(t)} \right)^j \frac{(1-F(t))}{(-\ln F(tx))} (1-F(t))^{k-l-1} \right\}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1 - F_{i+l+1}(tx)}{(-\ln F(tx))^{l+1}} &= \lim_{t\to\infty} \frac{-xf(tx)(-\ln F(tx))^{i+l}/(i+l)!}{(l+1)(-\ln F(tx))^i(-xf(tx)/F(tx))} = \lim_{t\to\infty} \frac{F(tx)(-\ln F(tx))^i}{(l+1)(i+1)!} \\
&= \begin{cases} \frac{1}{(l+1)!}, & i = 0 \\ 0, & i \geq 1, \end{cases} \\
\lim_{t\to\infty} \frac{1 - F_k(tx)}{(-\ln F(tx))^k} &= \lim_{t\to\infty} \frac{-xf(tx)(-\ln F(tx))^{k-1}/(k-1)!}{k(-\ln F(tx))^{k-1}(-xf(tx)/F(tx)} = \frac{1}{k!}.
\end{aligned}
$$

Then for $k = l+1$, we get

$$\lim_{t\to\infty} \frac{1-U_{k,1,F}(tx,ty)}{(1-F(t))} = y^{-\alpha}(y^{-\alpha})^{l-1}\left(\frac{1}{(l+1)!}\right) + x^{-\alpha}\sum_{j=0}^{l-1}\frac{k!}{(k-1-j)!}\left(\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}}\right)^{k-1-j}\left(y^{-\alpha}\right)^j\frac{1}{k!}x^{\alpha}$$
$$-x^{-\alpha}\sum_{j=0}^{l-1}\frac{1}{(k-1-j)!}\left(\frac{x^{-\alpha}-y^{-\alpha}}{x^{-\alpha}}\right)^{k-1-j}\left(y^{-\alpha}\right)^j x^{\alpha} = \frac{1}{(l+1)!}y^{-l\alpha},$$

and for $k > l+1$, we get

$$\lim_{t\to\infty} \frac{1-U_{k,l,F}(tx,ty)}{(1-F(t))^l} = y^{-\alpha}(y^{-\alpha})^{l-1}\left(\frac{1}{(l+1)!}+\sum_{i=1}^{k-l-1}0\right) + (x^{-\alpha})(0) - (x^{-\alpha})(0) = \frac{1}{(l+1)!}y^{-l\alpha}.$$

Thus

$$\lim_{t\to\infty} \frac{1-U_{k,1,F}(tx,ty)}{1-U_{k,1,F}(t,t)} = (l+1)!\left\{\frac{1}{(l+1)!}y^{-l\alpha}\right\} = y^{-l\alpha}.$$

From (20) and (21), it follows that $W(cx,cy) = (cy)^{-l\alpha} = c^{-l\alpha}W(x,y)$. Then using Corollary 5.18, in page 281, Resnick (1987), the proof is complete. The norming constants $a_n^k$ and $a_n^l$ in (b) are as in (c) of Theorem 3.3, Ravi and Manohar (2017). □

**Proof of Theorem 5.2.** $X_1 \le_{st} X_2$ holds true in all the cases since we have

(a) $F_{k+1}(x) - F_k(x) = \frac{F(x)}{k!}(-\ln F(x))^k \ge 0$, so that $(1-F_k(x)) - (1-F_{k+1}(x)) = \bar{F}_k(x) - \bar{F}_{k+1}(x) \ge 0$, which implies that $F_{k+1}(x) \le_{\text{st}} F_k(x)$,

(b) $T_{k+1,F}(x) - T_{k,F}(x) = \binom{k+r-1}{k}\frac{(-\ln F(x))^k}{(1-\ln F(x))^{k+r}} \ge 0$, so that $(1-T_{k,F}(x)) - (1-T_{k+1,F}(x)) = \bar{T}_{k,F}(x) - \bar{T}_{k+1,F}(x) \ge 0$, which implies that $T_{k+1,F}(x) \le_{\text{st}} T_{k,F}(x)$,

(c) $(1-R_{k+1,F}(x))/(1-R_{k,F}(x)) < 1$, so that $(1-R_{k+1,F}(x)) < (1-R_{k+1,F}(x))$, which implies that $R_{k+1,F}(x) \le_{\text{st}} R_{k,F}(x)$,

(d) $U_{k+1,F}(x) - U_{k,F}(x) = \frac{1-F(x)}{-\ln F(x)} - F(x)\sum_{l=1}^{k}\frac{(-\ln F(x))^{l-1}}{l!} = 1 - F_{k+1}(x) \ge 0$, so that $(1-U_{k,F}(x)) - (1-U_{k+1,F}(x)) = \bar{U}_{k,F}(x) - \bar{U}_{k+1,F}(x) \ge 0$, which implies that $U_{k+1,F}(x) \le_{\text{st}} U_{k,F}(x)$.

Note that the above proofs are similar to those of Theorem 6.1, Ravi and Manohar (2017) albeit the inequalities there should have been $\ge_{st}$ instead of $\le_{st}$, an oversight.

It remains to establish that

$$[Y_1|X_1 = x_1] \le_{st} [Y_2|X_2 = x_2] \text{ whenever } x_1 \le x_2.$$

(a) From equations (13) and (14), one gets

$$f_{l|k}(y|x) = \frac{f_{k.l}(x,y)}{f_k(x)} = \frac{f(y)}{F(y)}\left\{\frac{(k-1)!}{(l-1)!(k-l-1)!}\right\}\left\{\frac{(-\ln F(y))^{l-1}(\ln F(y)-\ln F(x))^{k-l-1}}{(-\ln F(x))^{k-1}}\right\},$$

and

$$\begin{aligned} F_{l|k}(y|x) &= \int_x^y f_{l|k}(w|x)\,dw \\ &= \int_x^y \frac{f(w)}{F(w)}\left\{\frac{(k-1)!}{(l-1)!(k-l-1)!}\right\}\left\{\frac{(-\ln F(w))^{l-1}(\ln F(w)-\ln F(x))^{k-l-1}}{(-\ln F(x))^{k-1}}\right\}dw, \\ &= \int_x^y \frac{1}{B(l,k-l)}\left(\frac{-\ln F(w)}{-\ln F(x)}\right)^{l-1}\left(\frac{\ln F(w)-\ln F(x)}{-\ln F(x)}\right)^{k-l-1}\left(\frac{f(w)}{-F(w)\ln F(w)}\right)dw. \end{aligned}$$

Putting $t = \dfrac{-\ln F(w)}{-\ln F(x)}$, $dt = \dfrac{-f(w)}{-F(w)\ln F(x)}\,dw$ and changing the range from $(x, y)$ to $(1, l(y))$ where $l(y) = \dfrac{-\ln F(y)}{-\ln F(x)}$, we get

$$F_{l|k}(y|x) = \int_1^{l(y)} \frac{t^{l-1}(1-t)^{k-l-1}}{B(l,k-l)}\,(-dt) = \int_{l(y)}^1 \frac{t^{l-1}(1-t)^{k-l-1}}{B(l,k-l)}dt.$$

Note that $l(x) = 1$ and $l(\infty) = 0$. Further $\dfrac{d}{dy}(l(y)) = \dfrac{-f(y)}{-F(y)\ln F(x)} < 0$, so that $l(y)$ is a decreasing function of $y$. Hence $F_{l|k}(y|x)$ is a nondecreasing function of $y$. When $x_1 \le x_2$, we have

$$F(x_1) \le F(x_2) \Rightarrow -\ln F(x_1) \ge -\ln F(x_2) \Rightarrow \frac{1}{-\ln F(x_1)} \le \frac{1}{-\ln F(x_2)} \Rightarrow \frac{-\ln F(y)}{-\ln F(x_1)} \le \frac{-\ln F(y)}{-\ln F(x_2)}.$$

Setting $l_1(y) = \dfrac{-\ln F(y)}{-\ln F(x_1)}$, $l_2(y) = \dfrac{-\ln F(y)}{-\ln F(x_2)}$, so that $l_1(y) < l_2(y)$, we have

$$F_{l+1|k+1}(y|x_1) = P_{l+1|k+1}(Y_1 \le y|X_1 = x_1) = \int_{l_1(y)}^1 \frac{t^l(1-t)^{k-l-1}}{B(l+1,k-l)}dt,$$

$$F_{l|k}(y|x_2) = P_{l|k}(Y_2 \le y|X_2 = x_2) = \int_{l_2(y)}^1 \frac{t^{l-1}(1-t)^{k-l-1}}{B(l,k-l)}dt.$$

The incomplete beta function is defined as $\mathrm{B}(x;\, a, b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt$, and the regularized incomplete beta function is defined as $I_x(a, b) = \dfrac{\mathrm{B}(x;\, a, b)}{\mathrm{B}(a, b)}$. The function $I_x(a, b)$ is the df of the Beta distribution, and is related to the df of a rv $X$ with a binomial distribution with success probability $p$ and sample size $n$:

$$F(k;\, n, p) = \Pr(X \le k) = I_{1-p}(n-k, k+1) = 1 - I_p(k+1, n-k).$$

We have $I_x(a+1, b) = I_x(a, b) - \dfrac{x^a(1-x)^b}{aB(a,b)}$. Let $\bar{I}_x(a, b) = 1 - I_x(a, b)$. Then

$$\bar{I}_x(a+1, b) = 1 - I_x(a+1, b) = 1 - \left\{ I_x(a, b) - \frac{x^a(1-x)^b}{aB(a,b)} \right\} = \bar{I}_x(a, b) + \frac{x^a(1-x)^b}{aB(a,b)}.$$

Applying this result, we have

$$\begin{aligned} F_{l+1|k+1}(y|x_1) - F_{l|k}(y|x_2) &= \int_{l_1(y)}^{l_2(y)} \frac{t^l(1-t)^{k-l-1}}{B(l+1,k-l)}dt + \int_{l_2(y)}^1 \left\{ \frac{t^l(1-t)^{k-l-1}}{B(l+1,k-l)} - \frac{t^{l-1}(1-t)^{k-l-1}}{B(l,k-l)} \right\} dt \\ &= \int_{l_1(y)}^{l_2(y)} \frac{t^l(1-t)^{k-l-1}}{B(l+1,k-l)}dt + \bar{I}_{l_2(y)}(l+1, k-l) - \bar{I}_{l_2(y)}(l, k-l) \\ &= \int_{l_1(y)}^{l_2(y)} \frac{t^l(1-t)^{k-l-1}}{B(l+1,k-l)}dt + \frac{(l_2(y))^l\,(1-l_2(y))^{k-l}}{lB(l,k-l)} > 0. \end{aligned}$$

Hence

$$F_{l+1|k+1}(y|x_1) - F_{l|k}(y|x_2) > 0 \Rightarrow \bar{F}_{l+1|k+1}(y|x_1) < \bar{F}_{l|k}(y|x_2) \Rightarrow F_{l+1|k+1}(y|x_1) <_{st} F_{l|k}(y|x_2).$$

So we conclude that $(X_1, Y_1) \le_{st} (X_2, Y_2)$.

(b) From equations (15) and (16), one gets

$$t_{l|k}(y|x) = \frac{t_{k.l,F}(x,y)}{t_{k,F}(x)} = \frac{f(y)}{F(y)} \left\{ \frac{(k-1)!}{(l-1)!(k-l-1)!} \right\} \left\{ \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(-\ln F(x))^{k-1}} \right\}.$$

Rest of the proof follows from (a).

(c) Follows by putting $a = 1$ in (b) above.

(d) From equations (17) and (18), one gets

$$u_{l|k}(y|x) = \frac{u_{k.l,F}(x,y)}{u_{k,F}(x)} = \frac{f(y)}{F(y)} \left\{ \frac{(k-1)!}{(l-1)!(k-l-1)!} \right\} \left\{ \frac{(-\ln F(y))^{l-1}(\ln F(y) - \ln F(x))^{k-l-1}}{(-\ln F(x))^{k-1}} \right\}.$$

Rest of the proof follows from (a). □

## 8. Concluding remarks

In this article, the limit laws of joint distribution of two normalized upper order statistics are obtained under fixed sample size and when the sample size is shifted binomial, shifted Poisson, shifted geometric, shifted negative binomial and discrete Uniform. The results are illustrated using some examples. Some interesting bivariate stochastic orderings have been studied. The results obtained in this article are a sequel to those obtained in Ravi and Manohar (2017).

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