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A new lifetime model by mixing gamma and geometric distributions useful in hydrology data

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Abstract. In this paper, we propose a new distribution obtained by mixing gamma and geometric distributions. We discuss different shapes of the probability density function and the hazard rate functions. We study several statistical properties. The maximum likelihood estimation method is performed for estimating the parameters. We determine the observed information matrix and discuss inference. Illustrative three hydrology data sets are given to show the flexibility and potentiality of the proposed distribution.

1. Introduction

An important aspect of statistics is the determination of flexible distributions to elaborate useful models for lifetime data. Among the existing approaches, new distributions can be obtained by mixing discrete and continuous distributions. Those using geometric distributions include the exponential geometric distribution [1], the exponential-power series distribution [9], the extended exponential geometric distribution [2], the complementary exponential geometric distribution [14], the Weibull-geometric distribution [6], the Weibull-power series distribution [18], the generalized exponential-power series distribution [16], the complementary exponentiated exponential geometric distribution [15], the extended Weibull-power series distribution [21], the complementary extended Weibull-power series distribution [11], the exponentiated extended Weibull-power series distribution [23], the G-geometric distribution [3], the alternative G-geometric distribution [8], the generalized linear failure rate-geometric distribution [12] and the linear failure rate-power series distribution [17]. We also refer to the review of [22], and the references therein.

On the other side, among the continuous distributions, the gamma distribution is one of the most commonly used in modeling life-time data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions. To the best of our knowledge, the mixing of the geometric distribution with the gamma distribution (not reduced to the exponential one) has not ever been considered in the literature. Based on such a mixing, this paper offers a new distribution with two parameters, called the gamma-geometric (GG) distribution. The formulation and motivations of such distribution are as follows. Let $\lambda > 0$, $\theta \in (0,1)$ and $\bar{\theta} = 1 - \theta$. We say that a random variable X follows the GG distribution with

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parameters (θ,λ), denoted by $\mathrm{GG}(\theta,\lambda)$, if it has the probability density function (pdf) given by

$$f(x)=\theta\lambda^2 xe^{-\lambda x}\frac{1+\bar{\theta}e^{-\lambda x}}{\left(1-\bar{\theta}e^{-\lambda x}\right)^3},\qquad x>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x)=1-\theta e^{-\lambda x}\frac{1+\lambda x-\bar{\theta}e^{-\lambda x}}{(1-\bar{\theta}e^{-\lambda x})^2},\qquad x>0. \tag{2}$$

The GG distribution arises from the following stochastic representation. Let X be a random variable having the following stochastic representation:

$$X\mid\{N=n\}\sim\mathcal{G}_{am}(2,\lambda n),\qquad N\mid\theta\sim G_{trunc}(\theta), \tag{3}$$

that is N is a random variable having the truncated geometric G_{trunc} distribution with parameter θ: $P(N=n)=\theta\bar{\theta}^{n-1}$, $n=1,2,\ldots$ and the distribution of X conditionally to $\{N=n\}$ is the gamma distribution $\mathcal{G}_{am}(2,\lambda n)$, with a conditional pdf given by $f_{X\mid\{N=n\}}(x)=\lambda^2n^2xe^{-\lambda nx}$, $x>0$. Then X follows the $\mathrm{GG}(\theta,\lambda)$ distribution; using the geometric series expansion: $\sum_{n=1}^{+\infty}n^2x^n=\frac{x(1+x)}{(1-x)^3}$, $|x|<1$, the pdf of X is given by

$$f_X(x)=\sum_{n=1}^{+\infty}f_{X\mid\{N=n\}}(x)P(N=n)=\frac{\theta}{\bar{\theta}}\lambda^2x\sum_{n=1}^{+\infty}n^2(\bar{\theta}e^{-\lambda x})^n=\theta\lambda^2xe^{-\lambda x}\frac{1+\bar{\theta}e^{-\lambda x}}{\left(1-\bar{\theta}e^{-\lambda x}\right)^3}. \tag{4}$$

The stochastic representation (3) can be viewed as a natural extension of the stochastic representation $X\mid\{N=n\}\sim\mathcal{E}_{xp}(\lambda n)=\mathcal{G}_{am}(1,\lambda n)$, with pdf corresponding to the one of the G-geometric class proposed by [3] (applied with the exponential distribution).

Ratio of two independent variables. An example of simple model using the GG distribution is given by the ratio of two independent variables as described as follows. Let $Y\sim\mathcal{G}_{am}(2,\lambda)$ and $N\sim G_{trunc}(\theta)$. Suppose that Y and N are independent. Then the ratio of Y and N given by

$$X=\frac{Y}{N},$$

follows the $\mathrm{GG}(\theta,\lambda)$ distribution. It is enough to note that $X\mid\{N=n\}=Y/n\sim\mathcal{G}_{am}(2,\lambda n)$. This ratio representation will be useful to determine statistical properties of the GG distribution.

Note that, from the ratio representation, the cdf of X given by (2) can be expressed directly: using the cdf of Y: $F_Y(x)=1-e^{-\lambda x}-\lambda xe^{-\lambda x}$, $x>0$, and the geometric series expansions: $\sum_{n=0}^{+\infty}x^n=\frac{1}{1-x}$ and $\sum_{n=1}^{+\infty}nx^{n-1}=\frac{1}{(1-x)^2}$, $|x|<1$, the cdf of X is given by

$$\begin{aligned}F_X(x)&=P(Y\le xN)=\sum_{n=1}^{\infty}F_Y(xn)P(N=n)\\&=1-\theta e^{-\lambda x}\sum_{n=0}^{\infty}(\bar{\theta}e^{-\lambda x})^n-\lambda x\theta e^{-\lambda x}\sum_{n=1}^{\infty}n(\bar{\theta}e^{-\lambda x})^{n-1}\\&=1-\theta e^{-\lambda x}\frac{1+\lambda x-\bar{\theta}e^{-\lambda x}}{(1-\bar{\theta}e^{-\lambda x})^2}.\end{aligned}$$

Some limit properties for $f(x)$ are given as:

$$f(x)\sim\lambda^2\frac{2-\theta}{\theta^2}x\to 0,\quad x\to 0,\qquad\qquad f(x)\sim\theta\lambda^2xe^{-\lambda x}\to 0,\quad x\to+\infty.$$

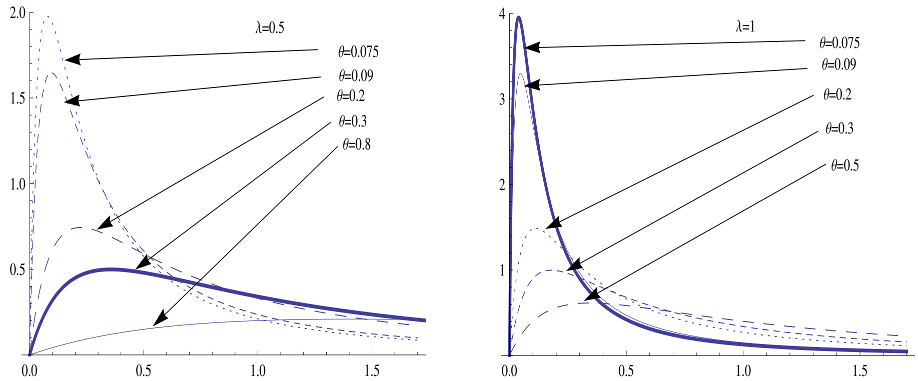

Figure 1: Plots of the GG density function

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Moreover, one can show that $f(x)$ has a unique maximum on $(0, +\infty)$ given by $f(x_*)$ where x_* satisfies the equation: $4\bar{\theta}\lambda x_* e^{\lambda x_*} + (\lambda x_* - 1)e^{2\lambda x_*} + \bar{\theta}^2(\lambda x_* + 1) = 0$. Some plots of $f(x)$ are given in Figure 1 for several values of (θ, λ). The rest of the paper is organized as follows. In Section 2, we give some properties of the GG distribution. The estimation by maximum likelihood is discussed in Section 3. Three illustrative real-life data examples are provided in Section 4.

2. Properties of the GG distribution

In this section, we propose many features and statistical properties of the GG distribution.

2.1. The survival and hazard rate functions

The survival function (sf) of X is given by

$$S(x) = 1 - F(x) = \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2}, \qquad x > 0, \tag{5}$$

and the associated hazard rate function (hrf) of X is

$$h(x) = \frac{f(x)}{S(x)} = \lambda^2 x \frac{1 + \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})(1 + \lambda x - \bar{\theta} e^{-\lambda x})}, \qquad x > 0. \tag{6}$$

Observe that

$$h(x) \sim \lambda^2 \frac{2 - \theta}{\theta^2} x \to 0, \quad x \to 0, \qquad\qquad h(x) \sim \lambda, \quad x \to +\infty.$$

Some plots of $h(x)$ are given in Figure 2 for several values of (θ, λ).

2.2. Quantile function

The quantile function of X is determined by inverting the cdf $F(x)$. The p-th quantile x_p of X is the solution of the nonlinear equation:

$$F(x_p) = p \quad \Leftrightarrow \quad \theta e^{-\lambda x_p}(1 + \lambda x_p - \bar{\theta} e^{-\lambda x_p}) = (1 - p)(1 - \bar{\theta} e^{-\lambda x_p})^2.$$

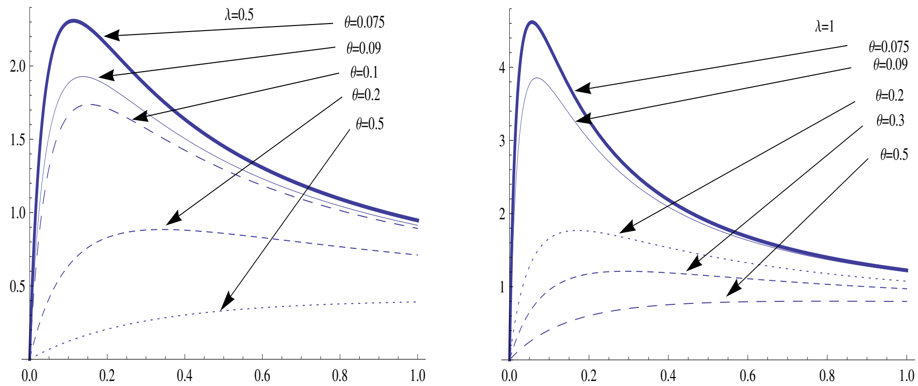

Figure 2: Plots of the GG hazard rate function

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2.3. Moments

Some key features of a distribution, like mean and variance, can be investigated through its r-th moments $E(X^r)$. For finding $E(X^r)$, we can use the ratio representation of X: X has the same distribution of the ratio of 2 random variables: Y/N with $Y \sim \mathcal{G}_{am}(2,\lambda)$ and $N \sim G_{trunc}(\theta)$, Y and N independent. Therefore, considering the Gamma function: $\Gamma(\nu) = \int_0^{+\infty} x^{\nu-1}e^{-x}dx$, $\nu > 0$, $E\left(Y^r\right) = \frac{\Gamma(2+r)}{\lambda^r}$ and the polylogarithm function: $\mathrm{Li}_r(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^r}$, $r > 0$, $|x| < 1$, we have

$$E(X^r) = E\left(\frac{Y^r}{N^r}\right) = E\left(Y^r\right) E\left(\frac{1}{N^r}\right) = \frac{\Gamma(2+r)}{\lambda^r}\sum_{n=1}^{+\infty}\frac{1}{n^r}P(N=n) = \frac{\Gamma(2+r)}{\lambda^r}\frac{\theta}{\bar{\theta}}\,\mathrm{Li}_r(\bar{\theta}).$$

In particular, by taking $r = 1$, since $\mathrm{Li}_1(x) = -\log(1-x)$, we obtain

$$E(X) = -\frac{2\theta}{\lambda\bar{\theta}}\log(\theta).$$

The variance of X can be explicit in some cases. For instance, if $\theta = \frac{1}{2}$, since $\mathrm{Li}_1(\frac{1}{2}) = \log(2)$ and $\mathrm{Li}_2(\frac{1}{2}) = \frac{1}{12}[\pi^2 - 6(\log(2))^2]$, we have

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{\Gamma(4)}{\lambda^2}\mathrm{Li}_2\left(\frac{1}{2}\right) - \left[\frac{\Gamma(3)}{\lambda}\mathrm{Li}_1\left(\frac{1}{2}\right)\right]^2 = \frac{1}{\lambda^2}\left(\frac{\pi^2}{2} - 7(\log(2))^2\right).$$

2.4. Moment generating function

Also the moment generating function of X can be obtain via the ratio representation Y/N. Using $E(e^{tY}) = \frac{\lambda^2}{(\lambda-t)^2}$, $t < \lambda$, and the conditional expectation, we get

$$\begin{aligned} M(t) &= E(e^{tX}) = E(e^{t\frac{Y}{N}}) = E\left(E\left(e^{t\frac{Y}{N}} \mid N\right)\right) = E\left(\frac{\lambda^2}{\left(\lambda - \frac{t}{N}\right)^2}\right) = \lambda^2 E\left(\frac{N^2}{\left(\lambda N - t\right)^2}\right) \\ &= \lambda^2\sum_{n=1}^{+\infty}\frac{n^2}{\left(\lambda n - t\right)^2}P(N=n) = \lambda^2\theta\sum_{n=1}^{+\infty}\frac{n^2}{\left(\lambda n - t\right)^2}\bar{\theta}^{n-1}. \end{aligned}$$

2.5. Conditional and reversed moments

The r-th conditional moments of X is given by

$$E(X^r \mid X > t) = \frac{1}{S(t)} \left(E(X^r) - \int_0^t x^r f(x) dx \right), \qquad t > 0,$$

and the r-th reversed moments of X is given by

$$E(X^r \mid X \leq t) = \frac{1}{F(t)} \int_0^t x^r f(x) dx, \qquad t > 0.$$

The integral term can be expressed using the expansion (4). Indeed, introducing the lower incomplete gamma function $\Gamma(t, \nu) = \int_0^t x^{\nu-1} e^{-x} dx$, $\nu > 0$, $t > 0$, we have

$$\int_0^t x^r f(x) dx = \frac{\theta}{\bar{\theta}} \lambda^2 \sum_{n=1}^{+\infty} n^2 \bar{\theta}^n \int_0^t x^{r+1} e^{-n\lambda x} dx = \frac{\theta}{\bar{\theta}} \frac{1}{\lambda^r} \sum_{n=1}^{+\infty} \frac{\bar{\theta}^n}{n^r} \Gamma(n\lambda t, r+2). \tag{7}$$

2.6. Rényi entropy

An entropy plays a central role in information theory. It provides a suitable measure of randomness or uncertainty of X. For continuous distributions, Rényi entropy (see [20]) can be determined as follows:

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left(\int_{-\infty}^{+\infty} [f(x)]^\gamma dx \right), \qquad \gamma > 0, \, \gamma \neq 1.$$

We have

$$[f(x)]^\gamma = \theta^\gamma \lambda^{2\gamma} x^\gamma e^{-\lambda\gamma x} \frac{(1 + \bar{\theta} e^{-\lambda x})^\gamma}{\left(1 - \bar{\theta} e^{-\lambda x}\right)^{3\gamma}}.$$

Using the generalized binomial series: $(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k$, $\alpha \in \mathbb{C}$, $|x| < 1$, $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$, we have

$$(1 + \bar{\theta} e^{-\lambda x})^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} \bar{\theta}^k e^{-\lambda k x}, \qquad \frac{1}{\left(1 - \bar{\theta} e^{-\lambda x}\right)^{3\gamma}} = \sum_{\ell=0}^{+\infty} \binom{-3\gamma}{\ell} (-1)^\ell \bar{\theta}^\ell e^{-\lambda \ell x}.$$

Hence,

$$[f(x)]^\gamma = \theta^\gamma \lambda^{2\gamma} \sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} x^\gamma e^{-\lambda(\ell+k+\gamma)x}.$$

Therefore,

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma} \left[\gamma \log(\theta) + 2\gamma \log(\lambda) + \log \left(\sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} \int_0^{+\infty} x^\gamma e^{-\lambda(\ell+k+\gamma)x} dx \right) \right] \\ &= \frac{1}{1-\gamma} \left[\gamma \log(\theta) + 2\gamma \log(\lambda) + \log \left(\sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} \frac{\Gamma(\gamma+1)}{\lambda^{\gamma+1}(\ell+k+\gamma)^{\gamma+1}} \right) \right]. \end{aligned}$$

2.7. Order statistics distributions

The order statistics are central tools in non-parametric statistics and inference. Let us now present the distributions of some fundamental order statistics related to the GG(θ,λ) distribution. Let a sample $X_1, X_2, \ldots, X_n$ is randomly chosen from the GG(θ,λ) distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ are its corresponding order statistics. A pdf of $X_{i:n}$ is given by

$$\begin{aligned} f_{X_{i:n}}(x) &= \frac{n!}{(i-1)!\,(n-i)!} f(x) \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l \left[F(x)\right]^{i-1+l} \\ &= \frac{n!}{(i-1)!\,(n-i)!} \theta\lambda^2 x e^{-\lambda x} \frac{1+\bar{\theta}e^{-\lambda x}}{\left(1-\bar{\theta}e^{-\lambda x}\right)^3} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l \left[1-\theta e^{-\lambda x}\frac{1+\lambda x-\bar{\theta}e^{-\lambda x}}{(1-\bar{\theta}e^{-\lambda x})^2}\right]^{i-1+l}, \\ & \quad x>0. \end{aligned}$$

The cdf of $X_{i:n}$ is given by

$$\begin{aligned} F_{X_{i:n}}(x) &= \frac{n!}{(i-1)!\,(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} \frac{(-1)^l}{i+l} \left[F(x)\right]^{i+l} \\ &= \frac{n!}{(i-1)!\,(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} \frac{(-1)^l}{i+l} \left[1-\theta e^{-\lambda x}\frac{1+\lambda x-\bar{\theta}e^{-\lambda x}}{(1-\bar{\theta}e^{-\lambda x})^2}\right]^{i+l}, \quad x>0. \end{aligned}$$

A joint pdf of $(X_{1:n}, \ldots, X_{n:n})$ is given by

$$\begin{aligned} f_{(X_{1:n},\ldots,X_{n:n})}(x_1,\ldots,x_n) &= n! \prod_{k=1}^{n} f(x_k) = n!\theta^n\lambda^{2n}\left(\prod_{k=1}^{n} x_k\right) e^{-\lambda\sum_{k=1}^{n} x_k} \frac{\prod_{k=1}^{n}(1+\bar{\theta}e^{-\lambda x_k})}{\prod_{k=1}^{n}\left(1-\bar{\theta}e^{-\lambda x_k}\right)^3}, \\ & \quad 0<x_1<\ldots<x_n. \end{aligned}$$

A joint pdf of $(X_{i:n}, X_{j:n})$, $i<j$, is given by

$$\begin{aligned} f_{(X_{i:n},X_{j:n})}(x_i,x_j) &= \frac{n!}{(i-1)!\,(n-j)!\,(j-i-1)} [F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1}[S(x_j)]^{n-j} f(x_i)f(x_j) \\ &= \frac{n!}{(i-1)!\,(n-j)!\,(j-i-1)} \left[1-\theta e^{-\lambda x_i}\frac{1+\lambda x_i-\bar{\theta}e^{-\lambda x_i}}{(1-\bar{\theta}e^{-\lambda x_i})^2}\right]^{i-1} \\ &\times \left[\theta e^{-\lambda x_i}\frac{1+\lambda x_i-\bar{\theta}e^{-\lambda x_i}}{(1-\bar{\theta}e^{-\lambda x_i})^2} - \theta e^{-\lambda x_j}\frac{1+\lambda x_j-\bar{\theta}e^{-\lambda x_j}}{(1-\bar{\theta}e^{-\lambda x_j})^2}\right]^{j-i-1} \\ &\times \left[\theta e^{-\lambda x_j}\frac{1+\lambda x_j-\bar{\theta}e^{-\lambda x_j}}{(1-\bar{\theta}e^{-\lambda x_j})^2}\right]^{n-j} \theta^2\lambda^4 x_i x_j e^{-\lambda(x_i+x_j)} \frac{1+\bar{\theta}e^{-\lambda x_i}}{\left(1-\bar{\theta}e^{-\lambda x_i}\right)^3} \frac{1+\bar{\theta}e^{-\lambda x_j}}{\left(1-\bar{\theta}e^{-\lambda x_j}\right)^3}, \\ & \quad 0<x_i<x_j. \end{aligned}$$

2.8. Record values distributions

Record values arise in a wide varity of real-life applications as hydrology, industry, lifetesting, economics, among the others. See, for instance, [4], [5] and [10]. We now present important distributions related to record values using the GG(θ,λ) distribution as baseline. Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. random variables having the GG(θ,λ) distribution. We define a sequence of record times $U(n)$ as follows: $U(1)=1$, $U(n)=\min\{j;\ j>U(n-1),\ X_j>X_{U(n-1)}\}$ for $n\geq 2$. We define the i-th upper record value by $R_i = X_{U(i)}$, with $R_1 = X_1$. A pdf of R_i is given by

$$f_{R_i}(x) = \frac{1}{(i-1)!}[-\log(S(x))]^{i-1} f(x)$$
$$= \frac{1}{(i-1)!}\left[-\log\left(\theta e^{-\lambda x}\frac{1+\lambda x-\bar{\theta}e^{-\lambda x}}{(1-\bar{\theta}e^{-\lambda x})^2}\right)\right]^{i-1} \theta\lambda^2 x e^{-\lambda x}\frac{1+\bar{\theta}e^{-\lambda x}}{\left(1-\bar{\theta}e^{-\lambda x}\right)^3}, \quad x>0.$$

A joint pdf of $(R_1, \ldots, R_n)$ is given by

$$f_{(R_1,\ldots,R_n)}(x_1,\ldots,x_n) = f(x_n)\prod_{k=1}^{n-1} h(x_k) = \theta\lambda^{2n} x_n e^{-\lambda x_n}\frac{1+\bar{\theta}e^{-\lambda x_n}}{\left(1-\bar{\theta}e^{-\lambda x_n}\right)^3}$$
$$\times\left(\prod_{k=1}^{n-1} x_k\right)\frac{\prod_{k=1}^{n-1}(1+\bar{\theta}e^{-\lambda x_k})}{\prod_{k=1}^{n-1}(1-\bar{\theta}e^{-\lambda x_k})(1+\lambda x_k-\bar{\theta}e^{-\lambda x_k})}, \quad 0<x_1<\ldots<x_n.$$

A joint pdf of (R_i, R_j), $i<j$, is given by

$$f_{(R_i,R_j)}(x_i,x_j) = \frac{1}{(i-1)!(j-i-1)!}[-\log(S(x_i))]^{i-1}\left[\log(S(x_i))-\log(S(x_j))\right]^{j-i-1} h(x_i) f(x_j)$$
$$= \frac{1}{(i-1)!(j-i-1)!}\left[-\log\left(\theta e^{-\lambda x_i}\frac{1+\lambda x_i-\bar{\theta}e^{-\lambda x_i}}{(1-\bar{\theta}e^{-\lambda x_i})^2}\right)\right]^{i-1}$$
$$\times\left[\log\left(\theta e^{-\lambda x_i}\frac{1+\lambda x_i-\bar{\theta}e^{-\lambda x_i}}{(1-\bar{\theta}e^{-\lambda x_i})^2}\right)-\log\left(\theta e^{-\lambda x_j}\frac{1+\lambda x_j-\bar{\theta}e^{-\lambda x_j}}{(1-\bar{\theta}e^{-\lambda x_j})^2}\right)\right]^{j-i-1}$$
$$\times \lambda^2 x_i\frac{1+\bar{\theta}e^{-\lambda x_i}}{(1-\bar{\theta}e^{-\lambda x_i})(1+\lambda x_i-\bar{\theta}e^{-\lambda x_i})}\times\theta\lambda^2 x_j e^{-\lambda x_j}\frac{1+\bar{\theta}e^{-\lambda x_j}}{\left(1-\bar{\theta}e^{-\lambda x_j}\right)^3}, \quad 0<x_i<x_j.$$

2.9. Residuals life functions

The residual life functions play a fundamental role in survival or reliability studies. See, for instance, [7], [13] and [19]. We now present some related mathematical objects with a potential of interest in the context of the GG(θ,λ) distribution.

The residual life is described by the conditional random variable $R_{(t)} = X - t \mid \{X > t\}$, $t \geq 0$. The sf of the residual lifetime $R_{(t)}$ is given by

$$S_{R_{(t)}}(x) = \frac{S(x+t)}{S(t)} = e^{-\lambda x}\frac{(1+\lambda(x+t)-\bar{\theta}e^{-\lambda(x+t)})(1-\bar{\theta}e^{-\lambda t})^2}{(1-\bar{\theta}e^{-\lambda(x+t)})^2(1+\lambda t-\bar{\theta}e^{-\lambda t})}, \quad x>0.$$

The associated cdf is given by

$$F_{R_{(t)}}(x) = 1 - e^{-\lambda x}\frac{(1+\lambda(x+t)-\bar{\theta}e^{-\lambda(x+t)})(1-\bar{\theta}e^{-\lambda t})^2}{(1-\bar{\theta}e^{-\lambda(x+t)})^2(1+\lambda t-\bar{\theta}e^{-\lambda t})}, \quad x>0.$$

Then, the corresponding pdf is given by

$$f_{R_{(t)}}(x) = \lambda^2(x+t)e^{-\lambda x}\frac{(1+\bar{\theta}e^{-\lambda(x+t)})(1-\bar{\theta}e^{-\lambda t})^2}{\left(1-\bar{\theta}e^{-\lambda(x+t)}\right)^3(1+\lambda t-\bar{\theta}e^{-\lambda t})}, \quad x>0.$$

The associated hrf is given by

$$h_{R_{(t)}}(x) = \lambda^2(x+t)\frac{1+\bar{\theta}e^{-\lambda(x+t)}}{\left(1-\bar{\theta}e^{-\lambda(x+t)}\right)\left(1+\lambda(x+t)-\bar{\theta}e^{-\lambda(x+t)}\right)} \quad x>0,$$

and the mean residual life is defined as

$$K(t) = E(R_{(t)}) = E(X - t \mid \{X > t\}) = \frac{1}{S(t)}\left(E(X) - \int_0^t xf(x)dx\right) - t.$$

The integral term can be expressed as (7) with $r = 1$.

On the other side, the variance residual life is given by

$$V(t) = Var(R_{(t)}) = Var(X - t \mid \{X > t\}) = \frac{1}{S(t)}\left(E(X^2) - \int_0^t x^2 f(x)dx\right) - t^2 - 2tK(t) - [K(t)]^2.$$

Again, the integral term can be expressed as (7) with $r = 2$.

The reverse residual life is described by the conditional random variable $\overline{R}_{(t)} = t - X \mid \{X \le t\}$, $t \ge 0$. The sf of the reversed residual lifetime $\overline{R}_{(t)}$ is given by

$$S_{\overline{R}_{(t)}}(x) = \frac{F(t-x)}{F(t)} = \frac{(1-\bar{\theta}e^{-\lambda t})^2\left[(1-\bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1+\lambda(t-x)-\bar{\theta}e^{-\lambda(t-x)})\right]}{(1-\bar{\theta}e^{-\lambda(t-x)})^2\left[(1-\bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1+\lambda t-\bar{\theta}e^{-\lambda t})\right]}, \quad 0 < x \le t.$$

The associated cdf is given by

$$F_{\overline{R}_{(t)}}(x) = 1 - \frac{(1-\bar{\theta}e^{-\lambda t})^2\left[(1-\bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1+\lambda(t-x)-\bar{\theta}e^{-\lambda(t-x)})\right]}{(1-\bar{\theta}e^{-\lambda(t-x)})^2\left[(1-\bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1+\lambda t-\bar{\theta}e^{-\lambda t})\right]}, \quad 0 < x \le t.$$

Therefore, the corresponding pdf is given by

$$f_{\overline{R}_{(t)}}(x) = \frac{\theta\lambda^2(t-x)e^{-\lambda(t-x)}(1+\bar{\theta}e^{-\lambda(t-x)})(1-\bar{\theta}e^{-\lambda t})^2}{\left(1-\bar{\theta}e^{-\lambda(t-x)}\right)^3\left[(1-\bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1+\lambda t-\bar{\theta}e^{-\lambda t})\right]}, \quad 0 < x \le t,$$

and the associated hrf is given by

$$h_{\overline{R}_{(t)}}(x) = \frac{\theta\lambda^2(t-x)e^{-\lambda(t-x)}(1+\bar{\theta}e^{-\lambda(t-x)})(1-\bar{\theta}e^{-\lambda(t-x)})^2}{\left(1-\bar{\theta}e^{-\lambda(t-x)}\right)^3\left[(1-\bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1+\lambda(t-x)-\bar{\theta}e^{-\lambda(t-x)})\right]}, \quad 0 < x \le t.$$

The mean reversed residual life is defined as

$$L(t) = E(\overline{R}_{(t)}) = E(t - X \mid \{X \le t\}) = t - \frac{1}{F(t)}\int_0^t xf(x)dx.$$

The integral term can be expressed as (7) with $r = 1$.

The variance reversed residual life is given by

$$W(t) = Var(\overline{R}_{(t)}) = Var(t - X \mid \{X \le t\}) = 2tL(t) - [L(t)]^2 - t^2 + \frac{1}{F(t)}\int_0^t x^2 f(x)dx.$$

Again, the last integral can be expressed as (7) with $r = 2$.

3. Maximum likelihood estimation

In this section, we estimate the unknown parameters of the GG distribution using the method of maximum likelihood. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the GG(θ, λ) distribution with observed values $x_1, x_2, \ldots, x_n$. Set $\Theta = \{\theta, \lambda\}$. The likelihood function associated to $x_1, \ldots, x_n$ is given by

$$L(\boldsymbol{\Theta}) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n \left(\theta\lambda^2 x_i e^{-\lambda x_i}\frac{1+\bar{\theta}e^{-\lambda x_i}}{\left(1-\bar{\theta}e^{-\lambda x_i}\right)^3}\right) = \theta^n\lambda^{2n}\left(\prod_{i=1}^n x_i\right) e^{\lambda\sum_{i=1}^n x_i}\frac{\prod_{i=1}^n\left(e^{\lambda x_i}+1-\theta\right)}{\prod_{i=1}^n\left(e^{\lambda x_i}-1+\theta\right)^3}.$$

The maximum likelihood estimators (MLEs) of θ and λ are obtained by maximization of $L(\mathbf{\Theta})$, or alternatively, the log-likelihhod defined by

$$\ell(\mathbf{\Theta}) = \log(L(\Theta)) = n\log(\theta) + 2n\log(\lambda) + \sum_{i=1}^{n}\log(x_i) + \lambda\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(e^{\lambda x_i}+1-\theta) - 3\sum_{i=1}^{n}\log(e^{\lambda x_i}-1+\theta).$$

It follows that the MLEs are the simultaneous solutions of the equations according to (θ,λ):

$$\frac{\partial\ell(\Theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\frac{1}{e^{\lambda x_i}+1-\theta} - 3\sum_{i=1}^{n}\frac{1}{e^{\lambda x_i}-1+\theta} = 0$$

and

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = \frac{2n}{\lambda} + \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{e^{\lambda x_i}x_i}{e^{\lambda x_i}+1-\theta} - 3\sum_{i=1}^{n}\frac{e^{\lambda x_i}x_i}{e^{\lambda x_i}-1+\theta} = 0.$$

Since we have no analytic forms, numerical methods, as the quasi-Newton algorithm, can be applied to determine the estimators. The observed information matrix is given by

$$J(\Theta) = \begin{pmatrix} J_{\theta\,\theta}(\Theta) & J_{\theta\,\lambda}(\Theta) \\ J_{\lambda\,\theta}(\Theta) & J_{\lambda\,\lambda}(\Theta) \end{pmatrix},$$

where

$$J_{\theta\,\theta}(\Theta) = -\frac{\partial^2\ell(\Theta)}{\partial\theta^2} = \frac{n}{\theta^2} + \sum_{i=1}^{n}\frac{1}{(e^{\lambda x_i}+1-\theta)^2} - 3\sum_{i=1}^{n}\frac{1}{(e^{\lambda x_i}-1+\theta)^2},$$

$$J_{\theta\,\lambda}(\Theta) = -\frac{\partial^2\ell(\Theta)}{\partial\theta\partial\lambda} = -\sum_{i=1}^{n}\frac{e^{\lambda x_i}x_i}{(e^{\lambda x_i}+1-\theta)^2} - 3\sum_{i=1}^{n}\frac{e^{\lambda x_i}x_i}{(e^{\lambda x_i}-1+\theta)^2},$$

$$J_{\lambda\,\lambda}(\Theta) = -\frac{\partial^2\ell(\Theta)}{\partial\lambda^2} = \frac{2n}{\lambda^2} + \sum_{i=1}^{n}\left(\frac{e^{2\lambda x_i}x_i^2}{(e^{\lambda x_i}+1-\theta)^2} - \frac{e^{\lambda x_i}x_i^2}{e^{\lambda x_i}+1-\theta}\right) - 3\sum_{i=1}^{n}\left(\frac{e^{2\lambda x_i}x_i^2}{(e^{\lambda x_i}-1+\theta)^2} - \frac{e^{\lambda x_i}x_i^2}{(e^{\lambda x_i}-1+\theta)}\right).$$

This matrix is a key mathematical tool to obtain approximate confidence intervals or Wald tests for θ and λ in the case of a large sample.

4. Illustrative hydrology data examples

In this section, we take three hydrology data sets to show the flexibility and potentiality of the proposed distribution.

We fit the GG distribution to three hydrologic data sets and compare with the Weibull, Gumbel, Exponentiated Exponential, Generalized Gumbel, Kappa and Weibull Geometric distributions for three data sets. Most of those distributions have received great attention for fitting hydrology data, like rainfall data, precipitation data and flood data. More precisely, the densities of the compared distributions are given as follows:

- Weibull distribution with pdf:

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}, \quad \lambda>0,\, k>0,\, x>0.$$

- Gumbel distribution with pdf:

$$f(x) = \frac{e^{-e^{-\frac{x-\mu}{\sigma}}-\frac{x-\mu}{\sigma}}}{\sigma}, \quad \sigma > 0,\ x, \mu \in \mathbb{R}.$$

- Exponentiated Exponential (EE) distribution [24] with pdf:

$$f(x) = \alpha\lambda \left(1 - e^{-\lambda x}\right)^{(\alpha-1)} e^{-\lambda x}, \quad \alpha,\ \lambda,\ x > 0.$$

- Generalized Gumbel (GGu) distribution [25] with pdf:

$$f(x) = \frac{\alpha\left(1 - e^{-e^{-\frac{x-\mu}{\sigma}}}\right)^{\alpha-1}\left(e^{-\frac{x-\mu}{\sigma}}\right)e^{-e^{-\frac{x-\mu}{\sigma}}}}{\sigma}, \quad \alpha,\ \sigma > 0,\ \mu,\ x \in \mathbb{R}.$$

- Kappa distribution [26] with pdf:

$$f(x) = \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}\left(\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right)^{\frac{-(\alpha+1)}{\alpha}}, \quad \alpha,\ \theta,\ \beta,\ x > 0.$$

- Weibull geometric (WG) distribution [6] with pdf:

$$f(x) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\left(1 - p\,e^{-(\beta x)^{\alpha}}\right)^{-2}, \quad p \in (0,1),\ \alpha,\ \beta,\ x > 0.$$

For goodness-of-fit we have two main test statistics, i.e., information criterion and empirical distribution. The measures Akaike information criterion (AIC) [27], corrected Akaike information criterion (AICC) [28], Hannan–Quinn information criterion (HQIC) [29], and consistent Akaike information criterion (CAIC) [30] are widely used information criterion for selecting the appropriate model among other models. The Anderson-Darling (A^*) due to Anderson and Darling [31], the Cramér–von Mises (W^*) due to Cramér and Mises [32] and the Kolmogorov Smirnov (KS) statistics due to Kolmogorov [33] with their p-values to compare the fitted models. These statistics are used to evaluate how a particular distribution with cdf, for a given data set, fits the corresponding empirical distribution. The distribution with better fit than the others will be the one having the smallest statistics and largest p-value.

The descriptions of the data sets are as follows.
The first data set is taken from engineering department consists of a sample of 30 failure times of air-conditioned system of an aeroplane (in hours) and is presented by Linhart and Zucchini [34].
The data points are 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

The second data is taken from the U.S. Geological Survey (USGS) gaging station 08230500 (Carnero creek near La Garita, Colorado) which describes the annual maximum stream flow amount measured in cubic feet per second. The data is based on a time series data that consist of 58 periods from 1920 to 1981 (data for 1924, 1925, 1929 and 1931 were missing). The data is available in U.S. Geological Survey (USGS) website ($http://nwis.waterdata.usgs.gov$).
The data points are 144, 179, 105, 280, 81, 35, 320, 248, 159, 570, 278, 315, 327, 182, 186, 102, 31, 350, 435, 520, 715, 1600, 660, 173, 239, 667, 44, 82, 70, 68, 69, 42, 16, 450, 333, 114, 121, 175, 299, 102, 93, 287, 64, 36, 438, 63, 146, 48, 37, 214, 25, 161, 104, 115, 32, 109, 128, 30

The third data describes the maximum rainfall in mm of the whole year of Jiwani town located along the Gulf of Oman in the Gwadar district of the Balochistan province in Pakistan from 1981 to 2010. The data points are 21.7, 172.9, 69.5, 96.5, 12.6, 265.5, 154, 28, 142.8, 14.2, 74.8, 32.5, 25, 28.5, 113.8, 25.7,

Table 1: Estimates of the parameters (standard errors in parenthesis) for Aeroplane data

Distributions	Estimates		
GG(θ, λ)	0.155401	0.012459	
	(0.039411)	(0.002647)	
Weibul(λ, k)	54.6134	0.853587	
	(12.361500)	(0.119402)	
Gumbel(μ, σ)	31.360100	40.609000	
	(0.039411)	(6.566440)	
EE(α, λ)	0.809287	0.014543	
	(0.188512)	(0.003720)	
GGu(α, σ, μ)	1.611330	63.122100	58.019100
	(0.361676)	(13.339200)	(12.796100)
Kappa(α, θ, β)	1.035070	1.271930	29.742000
	(0.827705)	(0.695590)	(12.688400)
WG(λ, a, b)	0.793447	0.007379	1.137020
	(0.281986)	(0.005622)	(0.251924)

Table 2: Goodness of fit statistics for Aeroplane data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	**-151.102**	**306.205**	**306.649**	**307.101**	**306.649**
Weibul(λ, k)	-151.937	307.874	308.318	308.770	308.318
Gumbel(μ, σ)	-161.982	327.964	328.408	328.860	328.408
EE(α, λ)	-152.201	308.401	308.846	309.298	308.846
GGu(α, σ, μ)	-164.116	334.233	335.156	335.577	335.156
Kappa(α, θ, β)	-152.183	310.367	311.290	311.712	311.290
WG(λ, a, b)	-151.278	308.557	309.480	309.902	309.480
Distributions	A^*	W^*	KS	p	
GG(θ, λ)	**0.51583**	**0.0892352**	**0.114073**	**0.829838**	
Weibul(λ, k)	0.567419	0.0990543	0.153363	0.480628	
Gumbel(μ, σ)	5.90727	0.931143	0.355422	0.00102163	
EE(α, λ)	0.691402	0.123059	0.171971	0.337505	
GGu(α, σ, μ)	8.98938	1.39876	0.39446	0.000176396	
Kappa(α, θ, β)	0.477954	0.0885657	0.121557	0.767156	
WG(λ, a, b)	0.476917	0.0869025	0.125281	0.734085	

116.3, 28, 16.9, 6, 9, 17.6, 47.3, 55, 129, 72, 92, 28, 113, 194

For the first data set, Table 1 gives estimates of the parameters of the considered models with their corresponding standard errors. Table 2 presents their goodness-of-fit statistics. Concerning the GG model, the MLEs corresponding to the data are given by $\hat{\theta} = 0.155401$ and $\hat{\lambda} = 0.0124595$, and the following information criterion are obtained: AIC = 306.205, AICC = 306.649, HQIC = 307.101 and CAIC = 306.649. These values are the smallest in comparison to those obtained for the other models. On the other side, we have $A^* = 0.51583$, $W^* = 0.0892352$, KS = 0.114073 with $p = 0.829838$, which are also the best. The superiority of the GG model, in terms of goodness-of-fit statistics, in comparison to the others, is also observed for second data set (estimates are given in Table 3 and goodness of fit statistics in Table 4) and the third data set (estimates are given in Table 5 and goodness of fit statistics in Table 6).

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References

[1] Adamidis, K. and Loukas, S. (1998): A lifetime distribution with decreasing failure rate, Statistics and Probability Letters, 39, 35-42.

[2] Adamidis, K., Dimitrakopoulou, T. and Loukas, S. (2005): On a generalization of the exponential-geometric distribution, Statistics and Probability Letters, 73, 259-269.

Table 3: Estimates of the parameters (standard errors in parenthesis) for CARNERO data

Distributions	Estimates		
GG(θ, λ)	0.336657	0.00480789	
	(0.0567315)	(0.000630488)	
Weibul(λ, k)	229.722	1.05334	
	(30.3587)	(0.100504)	
Gumbel(μ, σ)	133.735	132.415	
	(18.0217)	(15.0402)	
EE(α, λ)	1.27283	0.00520083	
	(0.234449)	(0.000861595)	
GGu(α, σ, μ)	70.6639	1229.31	2083.55
	(5.77893)	(73.4427)	(55.7163)
Kappa(α, θ, β)	0.885474	1.88622	136.533
	(0.63983)	(0.937705)	(35.2732)
WG(λ, a, b)	0.952363	0.00108257	1.64806
	(0.06844)	(0.00071915)	(0.224096)

Table 4: Goodness of fit statistics for CARNERO data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	**-369.200**	**742.400**	**742.618**	**744.005**	**742.618**
Weibul(λ, k)	-371.839	747.678	747.896	749.283	747.896
Gumbel(μ, σ)	-381.104	766.207	766.426	767.813	766.426
EE(α, λ)	-371.153	746.305	746.523	747.910	746.523
GGu(α, σ, μ)	-410.482	826.963	827.408	829.371	827.408
Kappa(α, θ, β)	-369.537	745.075	745.519	747.483	745.519
WG(λ, a, b)	-369.040	744.080	744.524	746.488	744.524
Distributions	A^*	W^*	KS	p	
GG(θ, λ)	**0.26411**	**0.0282419**	**0.0586334**	**0.988461**	
Weibul(λ, k)	0.563524	0.0712539	0.0869865	0.772452	
Gumbel(μ, σ)	3.94994	0.727	0.197178	0.0219974	
EE(α, λ)	0.570699	0.0915234	0.0938397	0.686709	
GGu(α, σ, μ)	25.9378	4.87121	0.419101	2.8×10^{-9}	
Kappa(α, θ, β)	0.314206	0.0400859	0.0737592	0.910551	
WG(λ, a, b)	0.248297	0.0308173	0.067118	0.956358	

Table 5: Estimates of the parameters (standard errors in parenthesis) for Jiwani data

Distributions	Estimates		
GG(θ, λ)	0.453529	0.0182027	
	(0.105628)	(0.00303367)	
Weibul(λ, k)	77.5581	1.16238	
	(12.8831)	(0.165544)	
Gumbel(μ, σ)	45.4199	43.7452	
	(8.35456)	(6.80433)	
EE(α, λ)	1.34541	0.0163824	
	(0.340988)	(0.00365062)	
GGu(α, σ, μ)	10.4962	156.021	226.737
	(1.5249)	(17.0742)	(16.1174)
Kappa(α, θ, β)	0.881944	1.83423	46.7766
	(1.20898)	(1.74096)	(23.4308)
WG(λ, a, b)	0.648328	0.00815663	1.40982
	(0.454225)	(0.00451551)	(0.333492)

Table 6: Goodness of fit statistics for Jiwani data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	**-157.521**	**319.042**	**319.487**	**319.939**	**319.487**
Weibul(λ, k)	-158.359	320.717	321.162	321.614	321.162
Gumbel(μ, σ)	-162.542	329.084	329.529	329.981	329.529
EE(α, λ)	-158.223	320.446	320.891	321.343	320.891
GGu(α, σ, μ)	-166.937	339.874	340.797	341.219	340.797
Kappa(α, θ, β)	-159.344	324.688	325.611	326.033	325.611
WG(λ, a, b)	-158.038	322.077	323.000	323.422	323.000
Distributions	A^*	W^*	KS	p	
GG(θ, λ)	**0.4193**	**0.0798668**	**0.141087**	**0.589015**	
Weibul(λ, k)	0.517542	0.095075	0.165072	0.387045	
Gumbel(μ, σ)	2.10048	0.349666	0.265105	0.0294899	
EE(α, λ)	0.54145	0.102138	0.167958	0.365793	
GGu(α, σ, μ)	5.2192	0.858108	0.321454	0.00405898	
Kappa(α, θ, β)	0.567106	0.101475	0.141822	0.582326	
WG(λ, a, b)	0.503657	0.0945876	0.153904	0.476065	

[3] Alkarni, S. H. (2012): A compound class of geometric and lifetime distributions, Open Statist. Probab. J. 5, 1-5.

[4] Ahsanullah, M. (1995): Record Statistics. Nova Science Publishers, Commack, New Jersey.

[5] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998): Records. Wiley, New York.

[6] Barreto-Souza, W., Alice, L. M. and Cordeiro, G. M. (2011): The Weibull-Geometric Distribution, Statistical Computation and Simulation, 81, 645-657.

[7] Bryson, C. and Siddiqui, M. M. (1969): Some criteria for aging. Journal of the American Statistical Association, 64, 1472-1483.

[8] Castellares, F. and Lemonte, A. J. (2016): On the Marshall-Olkin extended distributions. Communications in Statistics - Theory and Methods, 45, 15, 4537-4555.

[9] Chahkandi, M. and Ganjali, M. (2009): On some lifetime distributions with decreasing failure rate, Computational Statistics and Data Analysis, 53, 12, 4433-4440.

[10] Chandler, K. N. (1952): The distribution and frequency of record values, Journal of the Royal Statistical Society: Series B, 14, 220-228.

[11] Cordeiro, G. M. and Silva, R. B. (2014): The complementary extended Weibull power series class of distributions, Ciência e Natura, 36, 1-13.

[12] Harandi, S. S. and Alamatsaz, M. A. (2017): A complementary generalized linear failure rate-geometric distribution, Communications in Statistics - Simulation and Computation, 46, 1, 679-703.

[13] Hollander, W. and Proschan, F. (1975): Tests for mean residual life, Biometrika, 62, 585-593.

[14] Louzada, F., Roman, M. and Cancho, V. G. (2011): The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart, Computational Statistics and Data Analysis 55, 8, 2516-2524.

[15] Louzada, F., Marchi, V. and Carpenter, J. (2013): The complementary exponentiated exponential geometric lifetime distribution, Journal of Probability and Statistics, Article ID 502159, 12 pages.

[16] Mahmoudi, E. and Jafari, A. A. (2012): Generalized exponential-power series distributions. Computational Statistics and Data Analysis, 56, 12, 4047-4066.

[17] Mahmoudi, E. and Jafari, A. A. (2017): The compound class of linear failure rate-power series distributions: Model, properties, and applications, Communications in Statistics - Simulation and Computation, 46, 2, 1414-1440.

[18] Morais, A. L. and Barreto-Souza, W. (2011): A compound class of Weibull and power series distributions, Computational Statistics and Data Analysis, 55(3):1410-1425.

[19] Muth, E. J. (1977): Reliability models with positive memory derived from the mean residual life function, in Theory and Applications of Reliability, (C. P. Tsokos and I. N. Shimi eds.), Academic Press, 401-434.

[20] Rényi, A. (1961): On measures of entropy and information, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Berkeley (CA):University of California Press, I, 547 - 561.

[21] Silva, R. B., Bourguignon, M., Dias, C. R. B., and Cordeiro, G. M. (2013): The compound class of extended Weibull power series distributions, Computational Statistics and Data Analysis, 58, 352-367.

[22] Tahir, M. H. and Cordeiro, G. M. (2016): Compounding of distributions: a survey and new generalized classes. Journal of Statistical Distributions and Applications, 3, 1, 1-35.

[23] Tahmasebi, S. and Jafari, A. A. (2015): Exponentiated extended Weibull-power series class of distributions, Ciência e Natura, 37, 183-193.

[24] Ahuja, J. C. and Nash, S. W. (1967): The generalized Gompertz-Verhulst family of distributions, Sankhya, Ser. A, 29, 141-156.

[25] Cooray, K (2010): Generalized Gumbel distribution, Journal of Applied Statistics, 37, 171-179.

[26] Mielke P. W (1973): Another Family of Distributions for Describing and Analyzing Precipitation Data, Journal of Applied Meteorology, 12 (2), 275-280.

[27] H. Akaike (1974): A new look at the statistical model identification, IEEE Transactions on Automatic Control, 19, 716–723.

[28] C. M. Hurvich, C. L. Tsai (1989): Regression and time series model selection in small samples, Biometrika, 76, 297–307.

[29] E. J. Hannan, B. G. Quinn (1979): The Determination of the order of an autoregression, Journal of the Royal Statistical Society, Series B, 41, 190-195.
[30] H. Bozdogan (1987): Model selection and Akaikes information criterion, The general theory and its analytical extensions, Psychometrika, 52, 345–370.
[31] T. W. Anderson, D. A. Darling (1952): Asymptotic theory of certain "goodness–of–fit" criteria based on stochastic processes, Annals of Mathematical Statistics, 23, 193-212.
[32] H. Cramer (1928): On the composition of elementary errors II, Skand, Aktuarietidskr, 1, 13–74.
[33] A. Kolmogorov (1993): Sulla determinazione empirica di una legge di distribuzione, G. Inst. Ital. Attuari, 4, 83–91.
[34] H. Linhart, W. Zucchini (1986): Model Selection, Wiley, New York.