ProbStat Forum, Volume 12, January 2019, Pages 15–35
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

Performance and economic study of heterogeneous $M/M/2/N$ feedback queue with working vacation and impatient customers

Amina Angelika Bouchentouf[a], Abdelhak Guendouzi[b], Abdeldjebbar Kandouci[b]

[a]*Department of Mathematics, Laboratory of Mathematics, Djillali Liabes University of Sidi Bel Abbes, B. P. 89, 22000 Sidi Bel Abbes, Algeria.*
[b]*Laboratory of Stochastic Models, Statistic and Applications, Dr. Tahar Moulay University of Saida, B. P. 138, En-nasr, 20000 Saida, Algeria.*

Abstract. This paper presents the analysis of a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers. We suppose that the impatience timer of a customer in the system depends on the server's states. The steady-state probabilities of the model are obtained. Various performance measures of the model have been discussed. Then, we develop a model for the costs incurred and carry out a sensitive analysis for this queueing system with respect to all system parameters. Further, numerical results have been presented.

1. Introduction

Recent decades have seen an increasing interest in queueing systems with customer's impatience because of their great advantage in many real life applications such as situations involving impatient telephone switchboard customers, inventory systems with storage of perishable goods, business and industry etc. The readers can be referred to Gupta et al. [11; 12], Boxma et al. [8], Choudhury and Medhi [9], Jose and Manoharan [13; 14], Kumar and Sharma [16; 17], Bouchentouf et al. [6] and references therein.

Queueing models with vacation and working vacation have gained the interest of many researchers in the last three decades, due to their wide range of applications, especially in the communication and the manufacturing systems. Altman and Yechiali [2] analyzed the infinite-server queues with system's additional tasks and impatient customers, both multiple and single U-task scenarios are studied considering both exponentially and generally distributed task and impatience times. Jain and Jain [20] considered a working vacation queueing model with multiple types of server breakdowns, via a matrix geometric approach, the stationary queue length distribution has been obtained. Laxmi et al. [19] presented the analysis of a finite buffer $M/M/1$ queue with multiple and single working vacations. Then, Goswami [10] dealt with a queueing system with Bernoulli schedule working vacations, vacation interruption and impatient customers. Abidini et al. [1] gave an analysis of vacation and polling models with retrials. Panda and Goswami [21] established an equilibrium balking strategies in renewal input queue with bernoulli-schedule controlled vacation and vacation interruption. Later, Bouchentouf and Yahiaoui [7] presented an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation.

2010 *Mathematics Subject Classification.* MSC 2010: 60K25, 68M20, 90B22.
Keywords. Queueing models, heterogeneity, working vacation, impatient customers, Bernoulli feedback.
Received: 28 August 2018; Revised: 13 January 2019, Accepted: 23 January 2019.

Email addresses: bouchentouf_amina@yahoo.fr (Amina Angelika Bouchentouf), a.guendouzi@yahoo.com (Abdelhak Guendouzi), kandouci1974@yahoo.fr (Abdeldjebbar Kandouci)

Recently, there has been growing interest in the study of multiserver queues with vacation. For instance, Yue and Yue [22] considered heterogeneous two-server network system with balking and a Bernoulli vacation schedule. An $M/M/2$ queueing system with heterogeneous servers including one with working vacation has been analyzed by Krishnamoorthy and Sreenivasan [15]. Ammar [3] investigated the transient analysis of a two-heterogeneous servers queue with impatient behavior, the explicit solution for the considered model has been obtained. Later, Laxmi and Jyothsna [18] presented the analysis of a renewal input multiple working vacations queue with balking, reneging and heterogeneous servers. Using supplementary variable and recursive techniques, the steady-state probabilities of the model are obtained. Recently, the cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers has been given by Bouchentouf and Guendouzi [5].

In this paper, we present a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, and impatient customers. In this work, we extend the analytical results of the model given in Laxmi and Jyothsna [18] to the case where the impatience timer of customers in the system depend on the server's states, moreover the concept of feedback and retention of reneged customers is incorporated.

The rest of the paper is organized as follows, in Section 2, we give a detailed description of the model. In Section 3, the steady-state probabilities of the model are obtained using supplementary variable and recursive techniques. In Section 4, various performance measures of the model are presented. In Section 5, we develop the cost model. Then, numerical results are presented in Section 6. Finally, conclusion and some future aspects of research done are stated in Section 7.

2. The model

Consider a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, server's states-dependent reneging and retention of reneged customers.

• The inter-arrival times are assumed to be independent and identically distributed random variables with cumulative distribution function $A(u)$, probability density function $a(u)$, $u \geq 0$, Laplace-Stieltjes transform (L.S.T.) $A^*(\varpi)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\varpi)$ evaluated at $\varpi = 0$.

• There exist two heterogeneous servers, server 1 and server 2. The service times are supposed to be exponentially distributed with parameters μ_1 and μ_2, respectively, with $\mu_2 \leq \mu_1$. Whenever server 2 becomes idle and there are no waiting customers in the queue, he leaves for an exponential working vacation 'WV' with parameter ϕ. During a WV, server 2 serves the waiting customers at a rate lower than the normal service rate which is assumed to be exponentially distributed with parameter ν. At the end of vacation period, if there are customers waiting in the queue, server 2 switches to normal working level, otherwise he continues the vacation. Moreover, it is supposed that server 1 is always available in the system.

• The capacity of the system is taken finite N, and the customers are served on a FCFS discipline.

• An arriving customer who finds i customers in the system decides either to join the queue with probability $b_i = 1-\frac{i}{N^2}$ or balk with probability $\overline{b}_i = 1-b_i = \frac{i}{N^2}$. Suppose that $b_0 = b_1 = 1$, $0 \leq b_{i+1} \leq b_i \leq 1$, $2 \leq i \leq N-1$, and $b_N = 0$.

• If there are i customers in the system, one of the $(i-2)$ waiting customers in the queue may renege. Whenever a customer arrives at the system and finds the server 2 on working vacation (resp. on normal busy period), he activates an impatience timer T_1 (respectively. T_2,) which is exponentially distributed with parameter ξ_1 (resp. ξ_2). If the customer's service has not begun before the customer's timer expires, the customer abandons the queue. Thus, customer's average reneging rate is given by $(i-2)\xi_1$ (resp. $(i-2)\xi_2$) when server 2 is on working vacation (resp. on normal busy period), $2 \leq i \leq N$. We assume that impatience timers are independent and identically distributed random variables and independent of the number of waiting customers.

• Using certain mechanism, each reneged customer may leave the queue definitively with probability α or may be retained in the system with complimentary probability α'.

• After getting incomplete or unsatisfactory service either from working vacation service or normal busy service, with probability β', a customer may rejoin the system as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively, i.e. with probability β, where $\beta'+\beta = 1$.

- The inter-arrival times, service times and vacation times are assumed to be independent.

3. Steady-State Solution

In this section, the distributions of the steady-state of the system will be obtained following the same method given in Laxmi and Jyothsna [18]. Thus, using the supplementary variable and recursive techniques the steady-state probabilities will be derived. To get the system length distributions at arbitrary epoch, the differential difference equations using the remaining inter-arrival time as the supplementary variable will be developed.

Let $N_s(t)$ be the number of customers in the system at time t. And let $I(t)$ be the remaining inter-arrival time at time t for the next arrival.

Let

$$S(t) = \begin{cases} 0, & \text{when server 2 is idle during working vacation (WV) period;} \\ 1, & \text{when server 2 is busy during working vacation (WV) period;} \\ 2, & \text{when server 2 is busy during normal busy period.} \end{cases}$$

Then, the joint probabilities are presented as

$$\pi_{i,0}(u,t)du = \mathbb{P}(N_s(t) = i, u \leq I(t) < u + du, S(t) = 0), u \geq 0, i = 0, 1,$$

$$\pi_{i,j}(u,t)du = \mathbb{P}(N_s(t) = i, u \leq I(t) < u + du, S(t) = j), u \geq 0, j = 1, 2,$$

$$1 \leq i \leq N.$$

Thus

$$\pi_{i,0}(u) = \lim_{t\to\infty} \pi_{i,0}(u,t), \;\; i = 0, 1, \;\; \pi_{i,j}(u) = \lim_{t\to\infty} \pi_{i,j}(u,t), \;\; j = 1, 2, \;\; 1 \leq i \leq N.$$

The L.S.T. of the steady-state probabilities are given as

$$\pi^*_{i,0}(\varpi) = \int_0^\infty e^{-\varpi u}\pi_{i,0}(u)du, \;\; i = 0, 1, \;\; \pi^*_{i,j}(\varpi) = \int_0^\infty e^{-\varpi u}\pi_{i,j}(u)du,$$

$$j = 1, 2, \;\; 1 \leq i \leq N.$$

Let $\pi_{i,j} = \pi^*_{i,j}(0)$ be the probability of i customers in the system when the server is in state j at an arbitrary epoch.

The system of differential difference equations at steady-state is given as follows:

$$-\pi^{(1)}_{0,0}(u) = \beta\mu_1\pi_{1,0}(u) + \beta\nu\pi_{1,1}(u) + \beta\mu_2\pi_{1,2}(u), \tag{1}$$

$$-\pi^{(1)}_{1,0}(u) = -\beta\mu_1\pi_{1,0}(u) + \beta\nu\pi_{2,1}(u) + \beta\mu_2\pi_{2,2}(u) + a(u)\pi_{0,0}(0), \tag{2}$$

$$-\pi^{(1)}_{1,1}(u) = -(\phi + \beta\nu)\pi_{1,1}(u) + \beta\mu_1\pi_{2,1}(u), \tag{3}$$

$$\begin{aligned} -\pi^{(1)}_{2,1}(u) = &-\Big(\beta(\mu_1 + \nu) + \phi\Big)\pi_{2,1}(u) + \Big(\beta(\mu_1 + \nu) + \alpha\xi_1\Big)\pi_{3,1}(u) \\ &+a(u)\Big(\pi_{1,0}(0) + \pi_{1,1}(0) + \tfrac{2}{N^2}\pi_{2,1}(0)\Big), \end{aligned} \tag{4}$$

$$-\pi_{i,1}^{(1)}(u) = -\Big(\beta(\mu_1+\nu)+\phi+(i-2)\alpha\xi_1\Big)\pi_{i,1}(u)$$
$$+\Big(\beta(\mu_1+\nu)+(i-1)\alpha\xi_1\Big)\pi_{i+1,1}(u) \quad (5)$$
$$+a(u)\Big(\left(1-\tfrac{i-1}{N^2}\right)\pi_{i-1,1}(0)+\tfrac{i}{N^2}\pi_{i,1}(0)\Big),\ 3\le i\le N-1,$$

$$-\pi_{N,1}^{(1)}(u) = -\Big(\beta(\mu_1+\nu)+\phi+(N-2)\alpha\xi_1\Big)\pi_{N,1}(u)$$
$$+a(u)\Big(\left(1-\tfrac{N-1}{N^2}\right)\pi_{N-1,1}(0)+\pi_{N,1}(0)\Big), \quad (6)$$

$$-\pi_{1,2}^{(1)}(u) = -\beta\mu_2\pi_{1,2}(u)+\phi\pi_{1,1}(u)+\beta\mu_1\pi_{2,2}(u), \quad (7)$$

$$-\pi_{i,2}^{(1)}(u) = -\Big(\beta(\mu_1+\mu_2)+(i-2)\alpha\xi_2\Big)\pi_{i,2}(u)+\phi\pi_{i,1}(u)$$
$$+\Big(\beta(\mu_1+\mu_2)+(i-1)\alpha\xi_2\Big)\pi_{i+1,2}(u) \quad (8)$$
$$+a(u)\Big(\left(1-\tfrac{i-1}{N^2}\right)\pi_{i-1,2}(0)+\tfrac{i}{N^2}\pi_{i,2}(0)\Big),\ 2\le i\le N-1,$$

$$-\pi_{N,2}^{(1)}(u) = -\Big(\beta(\mu_1+\mu_2)+(N-2)\alpha\xi_2\Big)\pi_{N,2}(u)+\phi\pi_{N,1}(u)$$
$$+a(u)\Big(\left(1-\tfrac{N-1}{N^2}\right)\pi_{N-1,2}(0)+\pi_{N,2}(0)\Big), \quad (9)$$

Now, define $\zeta_i=\beta(\mu_1+\nu)+\phi+(i-2)\alpha\xi_1$ and $\theta_i=\beta(\mu_1+\mu_2)+(i-2)\alpha\xi_2$, for $2\le i\le N$. Multiplying Equations (1)-(9) by $e^{-\varpi u}$ and integrating over u from 0 to ∞, we get

$$-\varpi\pi_{0,0}^*(\varpi) = -\pi_{0,0}(0)+\beta\mu_1\pi_{1,0}^*(\varpi)+\beta\nu\pi_{1,1}^*(\varpi)+\beta\mu_2\pi_{1,2}^*(\varpi), \quad (10)$$

$$(\beta\mu_1-\varpi)\pi_{1,0}^*(\varpi) = -\pi_{1,0}(0)+\beta\nu\pi_{2,1}^*(\varpi)+\beta\mu_2\pi_{2,2}^*(\varpi)+A^*(\varpi)\pi_{0,0}(0), \quad (11)$$

$$(\phi+\beta\nu-\varpi)\pi_{1,1}^*(\varpi) = -\pi_{1,1}(0)+\beta\mu_1\pi_{2,1}^*(\varpi), \quad (12)$$

$$(\zeta_2-\varpi)\pi_{2,1}^*(\varpi) = -\pi_{2,1}(0)+(\zeta_3-\phi)\pi_{3,1}^*(\varpi)$$
$$+A^*(\varpi)\Big(\pi_{1,0}(0)+\pi_{1,1}(0)+\tfrac{2}{N^2}\pi_{2,1}(0)\Big), \quad (13)$$

$$(\zeta_i-\varpi)\pi^*_{i,1}(\varpi)=-\pi_{i,1}(0)+(\zeta_{i+1}-\phi)\pi^*_{i+1,1}(\varpi)$$
$$+A^*(\varpi)\Big((1-\tfrac{i-1}{N^2})\pi_{i-1,1}(0)+\tfrac{i}{N^2}\pi_{i,1}(0)\Big), \tag{14}$$

$$(\zeta_N-\varpi)\pi^*_{N,1}(\varpi)=-\pi_{N,1}(0)+A^*(\varpi)\Big((1-\frac{N-1}{N^2})\pi_{N-1,1}(0)+\pi_{N,1}(0)\Big), \tag{15}$$

$$(\beta\mu_2-\varpi)\pi^*_{1,2}(\varpi)=-\pi_{1,2}(0)+\phi\pi^*_{1,1}(\varpi)+\beta\mu_1\pi^*_{2,2}(\varpi), \tag{16}$$

$$(\theta_i-\varpi)\pi^*_{i,2}(\varpi)=-\pi_{i,2}(0)+\phi\pi^*_{i,1}(\varpi)+\theta_{i+1}\pi^*_{i+1,2}(\varpi)$$
$$+A^*(\varpi)\Big(\left(1-\tfrac{i-1}{N^2}\right)\pi_{i-1,2}(0)+\tfrac{i}{N^2}\pi_{i,2}(0)\Big), \tag{17}$$

$$(\theta_N-\varpi)\pi^*_{N,2}(\varpi)=-\pi_{N,2}(0)+\phi\pi^*_{N,1}(\varpi)$$
$$+A^*(\varpi)\Big(\left(1-\tfrac{N-1}{N^2}\right)\pi_{N-1,2}(0)+\pi_{N,2}(0)\Big). \tag{18}$$

Next, adding Equations (10)-(18), we get

$$-A^*(\varpi)\Big(\sum_{i=0}^{1}\pi_{i,0}(0)+\sum_{i=1}^{N}(\pi_{i,1}(0)+\pi_{i,2}(0))\Big)=$$
$$\varpi\Big(\sum_{i=0}^{1}\pi^*_{i,0}(\varpi)+\sum_{i=1}^{N}(\pi^*_{i,1}(\varpi)+\pi^*_{i,2}(\varpi))\Big),$$

Then, taking $\varpi\longrightarrow 0$ and using the normalization condition, we obtain

$$\sum_{i=0}^{1}\pi_{i,0}(0)+\sum_{i=1}^{N}(\pi_{i,1}(0)+\pi_{i,2}(0))=\lambda. \tag{19}$$

Next, we have to derive the steady-state probabilities at pre-arrival epoch, to this end we shall establish the relations between system length distributions at arbitrary and pre-arrival epochs. Firstly, we have to connect the pre-arrival epoch probabilities $\pi^-_{i,j}=\lim_{t\to\infty}\mathbb{P}(N_s(t)=i,S(t)=j/I(t)=0)$ ($\pi^-_{i,0},\ i=0,1$ and $\pi^-_{i,j},\ j=1,2;\ 1\le i\le N,$) with the rate probabilities $\pi_{i,0}(0)$ and $\pi_{i,j}(0)$, respectively.

Via Bayes' theorem on conditional probabilities, we obtain

$$\pi^-_{i,j} \quad = \quad \frac{1}{\lambda}\pi_{i,j}(0), j=0,\ \ i=0,1;\ \ j=1,2;\ \ 1\le i\le N. \tag{20}$$

Putting $\varpi=\zeta_N$ in Equation (15), we obtain

$$\pi_{N-1,1}(0)=\psi_{N-1}\pi_{N,1}(0), \tag{21}$$

such that $\psi_{N-1}=\dfrac{(1-A^*(\zeta_N))N^2}{A^*(\zeta_N)(N^2-N+1)}$.
Substituting Equation (21) in Equation (15), we get

$$(\zeta_N-\varpi)\pi^*_{N,1}(\varpi)=\left(A^*(\varpi)\big((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N\big)-\psi_N\right)\pi_{N,1}(0), \tag{22}$$

with $\psi_N=1$.
For $\varpi\neq\zeta_N$, we have

$$\pi^*_{N,1}(\varpi)=\frac{(A^*(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)-\psi_N)}{(\zeta_N-\varpi)}\pi_{N,1}(0). \tag{23}$$

Differentiating Equation (22) with respect to ϖ and taking $\varpi=\zeta_N$, we get

$$\pi^*_{N,1}(\zeta_N)=-A^{*(1)}(\zeta_N)\left(\left(1-\frac{N-1}{N^2}\right)\psi_{N-1}+\psi_N\right)\pi_{N,1}(0). \tag{24}$$

Differentiating (22) with respect to ϖ successively l times, we obtain

$$(\zeta_N-\varpi)\pi^{*(l)}_{N,1}(\varpi)-l\pi^{*(l-1)}_{N,1}(\varpi)=A^{*(l)}(\varpi)\left(\left(1-\frac{N-1}{N^2}\right)\psi_{N-1}+\psi_N\right)\pi_{N,1}(0). \tag{25}$$

From Equations (23)-(25), we get

$$\pi^*_{N,1}(\varpi)=\varsigma_{N,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{N,\varpi}=\begin{cases}\dfrac{A^*(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)-\psi_N}{(\zeta_N-\varpi)}, & \text{if } \varpi\neq\zeta_N;\\ -A^{*(1)}(\varpi)((1-\dfrac{N-1}{N^2})\psi_{N-1}+\psi_N), & \text{if } \varpi=\zeta_N,\end{cases}$$

with

$$\varsigma^{(l)}_{N,\varpi}=\begin{cases}\dfrac{A^{*(l)}(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)+l\varsigma^{(l-1)}_{N,\varpi}}{(\zeta_N-\varpi)}, & \text{if } \varpi\neq\zeta_N;\\ \dfrac{-A^{*(l+1)}(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)}{l+1}, & \text{if } \varpi=\zeta_N,\end{cases}$$

such that $\varsigma^{(l)}_{N,\varpi}$ denotes the l^{th} derivative of $\varsigma_{N,\varpi}$ with respect to ϖ.
For $i=N-1$, taking $\varpi=\zeta_{N-1}$ in Equation (14) and using Equation (21), we obtain

$$\pi_{N-2,1}(0)=\psi_{N-2}\pi_{N,1}(0), \tag{26}$$

with $\psi_{N-2}=\dfrac{(\psi_{N-1}-(\zeta_N-\phi)\varsigma_{N,\zeta_{N-1}}-A^*(\zeta_{N-1})\frac{N-2}{N^2}\psi_{N-1})N^2}{A^*(\zeta_{N-1})(N^2-N+2)}$.
Next, substituting Equation (26) in Equation (14), for $i=N-1$, we obtain

$$\pi^*_{N-1,1}(\varpi)=\varsigma_{N-1,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{N-1,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2} + \frac{N-1}{N^2}\psi_{N-1}) + (\zeta_N - \phi)\varsigma_{N,\varpi} - \psi_{N-1}}{(\zeta_{N-1}-\varpi)}, & \text{if } \varpi \neq \zeta_{N-1}; \\ -(A^{*(1)}(\varpi)((1-\dfrac{N-2}{N^2})\psi_{N-2} + \dfrac{N-1}{N^2}\psi_{N-1}) + (\zeta_N - \phi)\varsigma^{(1)}_{N,\varpi}), & \text{if } \varpi = \zeta_{N-1}, \end{cases}$$

with

$$\varsigma^{(l)}_{N-1,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2} + \frac{N-1}{N^2}\psi_{N-1}) + (\zeta_N - \phi)\varsigma^{(l)}_{N,\varpi} + l\varsigma^{(l-1)}_{N-1,\varpi}}{(\zeta_{N-1}-\varpi)}, & \text{if } \varpi \neq \zeta_{N-1}; \\ -\dfrac{A^{*(l+1)}(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2} + \frac{N-1}{N^2}\psi_{N-1}) + (\zeta_N - \phi)\varsigma^{(l+1)}_{N,\varpi}}{l+1}, & \text{if } \varpi = \zeta_{N-1}. \end{cases}$$

In the same way, for $i = N-2, N-3, ..., 3$ in Equation (14), it yields

$$\pi_{i-1,1}(0) = \psi_{i-1}\pi_{N,1}(0),\ i = N-2, N-3, ..., 3. \tag{27}$$

where

$$\psi_{i-1} = \frac{(\psi_i - (\zeta_{i+1} - \phi)\varsigma_{i+1,\zeta_i} - A^*(\zeta_i)\frac{i}{N^2}\psi_i)N^2}{A^*(\zeta_i)(N^2 - i - 1)}, \qquad i = N-2, N-3, ..., 3,$$

and

$$\pi^*_{i,1}(\varpi) = \varsigma_{i,\varpi}\pi_{N,1}(0),\ i = N-2, N-3, ..., 3,$$

where

$$\varsigma_{i,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma_{i+1,\varpi} - \psi_i}{(\zeta_i - \varpi)}, & \text{if } \varpi \neq \zeta_i; \\ -(A^{*(1)}(\varpi)((1-\dfrac{i-1}{N^2})\psi_{i-1} + \dfrac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma^{(1)}_{i+1,\varpi}), & \text{if } \varpi = \zeta_i, \end{cases}$$

with

$$\varsigma^{(l)}_{i,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma^{(l)}_{i+1,\varpi} - l\varsigma^{(l-1)}_{i,\varpi}}{(\zeta_i - \varpi)}, & \text{if } \varpi \neq \zeta_i; \\ -\dfrac{(\zeta_{i+1} - \phi)\varsigma^{(l+1)}_{i+1,\zeta_i} + A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i)}{l+1}, & \text{if } \varpi = \zeta_i. \end{cases}$$

Taking $\varpi = \zeta_2$ in Equation (13), we find

$$\pi_{1,1}(0) = \psi_1\pi_{N,1}(0) + \omega\pi_{1,0}(0), \tag{28}$$

where
$\psi_1 = \dfrac{\psi_2 - (\zeta_3 - \phi)\varsigma_{3,\zeta_2} - A^*(\zeta_2)\frac{2}{N^2}\psi_2}{A^*(\zeta_2)}$ and $\omega = -\dfrac{A^*(\zeta_2)}{A^*(\zeta_2)} = -1$.
Now, substituting Equation (28) in Equation (13), we obtain

$$\pi^*_{2,1}(\varpi) = \varsigma_{2,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{2,\varpi} = \begin{cases} \dfrac{-\psi_2 + (\zeta_3 - \phi)\varsigma_{3,\varpi} + A^*(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2)}{(\zeta_2 - \varpi)}, & \text{if } \varpi \neq \zeta_2; \\ -((\zeta_3 - \phi)\varsigma_{3,\varpi}^{(1)} + A^{*(1)}(\varpi)(\psi_1 + \dfrac{2}{N^2}\psi_2), & \text{if } \varpi = \zeta_2, \end{cases}$$

with

$$\varsigma_{2,\varpi}^{(l)} = \begin{cases} \dfrac{(\zeta_3 - \phi)\varsigma_{3,\varpi}^{(l)} + A^{*(l)}(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2) - l\varsigma_{2,\varpi}^{(l-1)}}{(\zeta_2 - \varpi)}, & \text{if } \varpi \neq \zeta_2; \\ -\dfrac{(\zeta_3 - \phi)\varsigma_{3,\varpi}^{(l+1)} + A^{*(l+1)}(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2)}{l+1}, & \text{if } \varpi = \zeta_2. \end{cases}$$

From Equation (12), we have

$$\pi_{1,1}^*(\varpi) = \varsigma_{1,\varpi}\pi_{N,1}(0) + \tau_{1,\varpi}\pi_{1,0}(0)),$$

where

$$\varsigma_{1,\varpi} = \begin{cases} \dfrac{\beta\mu_1\varsigma_{2,\varpi} - \psi_1}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ -\beta\mu_1\varsigma_{2,\varpi}^{(1)}, & \text{if } \varpi = \phi + \beta\nu, \end{cases} ; \varsigma_{1,\varpi}^{(l)} = \begin{cases} \dfrac{\beta\mu_1\varsigma_{2,\varpi}^{(l)} - l\varsigma_{1,\varpi}^{(l-1)}}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ -\dfrac{\beta\mu_1\varsigma_{2,\varpi}^{(l+1)}}{l+1}, & \text{if } \varpi = \phi + \beta\nu, \end{cases}$$

$$\tau_{1,\varpi} = \begin{cases} -\dfrac{\omega}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ 0, & \text{if } \varpi = \phi + \beta\nu, \end{cases} ; \tau_{1,\varpi}^{(l)} = \begin{cases} \dfrac{l\tau_{1,\varpi}^{(l-1)}}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ 0, & \text{if } \varpi = \phi + \beta\nu. \end{cases}$$

Putting $\theta_N = \varpi$ in Equation (18) and using $\pi_{N,1}^*(\varpi)$, we obtain

$$\pi_{N-1,2}(0) = \eta_{N-1}\pi_{N,2}(0) + \gamma_{N-1}\pi_{N,1}(0), \tag{29}$$

where
$\eta_{N-1} = \dfrac{1 - A^*(\theta_N)N^2}{A^*(\theta_N)(N^2 - N - 1)}$ and $\gamma_{N-1} = -\dfrac{\phi\varsigma_{N,\theta_N}N^2}{A^*(\theta_N)(N^2 - N - 1)}$.
Substituting Equation (29) in Equation (18), we get

$$\pi_{N,2}^*(\varpi) = \rho_{N,\varpi}\pi_{N,2}(0) + \chi_{N,\varpi}\pi_{N,1}(0),$$

where

$$\rho_{N,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1 - \frac{N-1}{N^2})\eta_{N-1} + \eta_N) - \eta_N}{\theta_N - \varpi}, & \text{if } \theta_N \neq \varpi; \\ -A^{*(1)}(\varpi)((1 - \dfrac{N-1}{N^2})\eta_{N-1} + \eta_N), & \text{if } \theta_N = \varpi, \end{cases}$$

$$\chi_{N,\varpi} = \begin{cases} \dfrac{\phi\varsigma_{N,\varpi} + A^*(\varpi)(1 - \frac{N-1}{N^2})\gamma_{N-1}}{\theta_N - \varpi}, & \text{if } \theta_N \neq \varpi; \\ -(\phi\varsigma_{N,\varpi}^{(1)} + A^{*(1)}(\varpi)(1 - \dfrac{N-1}{N^2})\gamma_{N-1}), & \text{if } \theta_N = \varpi, \end{cases}$$

with

$$\rho_{N,\varpi}^{(l)} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{N-1}{N^2})\eta_{N-1}+\eta_N)+l\rho_{N,\varpi}^{(l-1)}}{(\theta_N-\varpi)}, & \text{if } \theta_N \neq \varpi; \\ -\dfrac{A^{*(l+1)}(\theta_N)((1-\frac{N-1}{N^2})\eta_{N-1}+\eta_N)}{l+1}, & \text{if } \theta_N = \varpi, \end{cases}$$

$$\chi_{N,\varpi}^{(l)} = \begin{cases} \dfrac{\phi\varsigma_{N,\varpi}^{(l)}+A^{*(l)}(\varpi)(1-\frac{N-1}{N^2})\gamma_{N-1}+l\chi_{N,\varpi}^{(l-1)}}{\theta_N-\varpi}, & \text{if } \theta_N \neq \varpi; \\ -\dfrac{\phi\varsigma_{N,\theta_N}^{(l+1)}+A^{*(l+1)}(\theta_N)(1-\frac{N-1}{N^2})\gamma_{N-1}}{l+1}, & \text{if } \theta_N = \varpi, \end{cases}$$

$\eta_N = 1$ and $\gamma_N = 0$.
In the same manner, we obtain $\pi_{i,2}(0)$ and $\pi_{i,2}^*(\varpi)$ using Equation (17). Thus

$$\pi_{i-1,2}(0) = \eta_{i-1}\pi_{N,2}(0)+\gamma_{i-1}\pi_{N,1}(0), \;\; 2 \leq i \leq N-1, \tag{30}$$

with

$$\eta_{i-1} = N^2\frac{\eta_i-\theta_{i+1}\rho_{i+1,\theta_i}-A^*(\theta_i)\frac{i}{N^2}\eta_i}{A^*(\theta_i)(N^2-i+1)},$$

$$\gamma_{i-1} = N^2\frac{\gamma_i-\theta_{i+1}\chi_{i+1,\theta_i}-A^*(\theta_i)\frac{i}{N^2}\gamma_i-\phi\varsigma_{i,\theta_i}}{A^*(\theta_i)(N^2-i+1)}.$$

Substituting Equation (30) in Equation(17)

$$\pi_{i,2}^*(\varpi) = \rho_{i,\varpi}\pi_{N,2}(0)+\chi_{i,\varpi}\pi_{N,1}(0),$$

with

$$\rho_{i,\varpi} = \begin{cases} \dfrac{-\eta_i+\theta_{i+1}\rho_{i+1,\varpi}+A^*(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1}+\frac{i}{N^2}\eta_i)}{(\theta_i-\varpi)}, & \text{if } \theta_i \neq \varpi; \\ -(\theta_{i+1}\rho_{i+1,\varpi}^{(1)}+A^{*(1)}(\varpi)((1-\dfrac{i-1}{N^2})\eta_{i-1}+\dfrac{i}{N^2}\eta_i)), & \text{if } \theta_i = \varpi, \end{cases}$$

$$\chi_{i,\varpi} = \begin{cases} \dfrac{-\gamma_i+\phi\varsigma_{i,\varpi}+\theta_{i+1}\chi_{i+1,\varpi}+A^*(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1}+\frac{i}{N^2}\gamma_i)}{\theta_i-\varpi}, & \text{if } \theta_i \neq \varpi; \\ -(\phi\varsigma_{i,\varpi}^{(1)}+\theta_{i+1}\chi_{i+1,\varpi}^{(1)}+A^{*(1)}(\varpi)((1-\dfrac{i-1}{N^2})\gamma_{i-1}+\dfrac{i}{N^2}\gamma_i)), & \text{if } \theta_i = \varpi, \end{cases}$$

where

$$\rho_{i,\varpi}^{(l)} = \begin{cases} \dfrac{\theta_{i+1}\rho_{i+1,\varpi}^{(l)}+A^{*(l)}(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1}+\frac{i}{N^2}\eta_i)+l\rho_{i,\varpi}^{(l-1)}}{(\theta_i-\varpi)}, & \text{if } \theta_i \neq \varpi; \\ -\dfrac{\theta_{i+1}\rho_{i+1,\varpi}^{(l+1)}+A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1}+\frac{i}{N^2}\eta_i)}{l+1}, & \text{if } \theta_i = \varpi, \end{cases}$$

$$\chi_{i,\varpi}^{(l)} = \begin{cases} \dfrac{\phi\varsigma_{i,\varpi}^{(l)}+\theta_{i+1}\chi_{i+1,\varpi}^{(l)}+A^{*(1)}(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1}+\frac{i}{N^2}\gamma_i)+l\chi_{i,\varpi}^{(l-1)}}{\theta_i-\varpi}, & \text{if } \theta_i \neq \varpi; \\ -\dfrac{\phi\varsigma_{i,\varpi}^{(l+1)}+\theta_{i+1}\chi_{i+1,\varpi}^{(l+1)}+A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1}+\frac{i}{N^2}\gamma_i)}{l+1}, & \text{if } \theta_i = \varpi. \end{cases}$$

Putting $\varpi = \beta\mu_1$ in Equation(11), we get

$$\pi_{0,0}(0) = \varepsilon_0 \pi_{1,0}(0) + \sigma_0 \pi_{N,1}(0) + \triangle_0 \pi_{N,2}(0), \tag{31}$$

where $\varepsilon_0 = \dfrac{1}{A^*(\beta\mu_1)}$, $\sigma_0 = \dfrac{-\beta\nu\varsigma_{2,\beta\mu_1} - \beta\mu_2\chi_{2,\beta\mu_1}}{A^*(\beta\mu_1)}$, and $\triangle_0 = \dfrac{-\beta\mu_2\rho_{2,\beta\mu_1}}{A^*(\beta\mu_1)}$.
Now, let $\varpi = \phi + \beta\nu$, using (30), we get

$$\pi_{1,0}(0) = \kappa_1 \pi_{N,1}(0), \tag{32}$$

where $\kappa_1 = \psi_1 - \beta\mu_1\varsigma_{2,\phi+\beta\nu}$.
Putting $\beta\mu_2 = \varpi$ in Equation (16)

$$\pi_{N,2}(0) = \kappa_2 \pi_{N,1}(0), \tag{33}$$

where $\kappa_2 = \dfrac{\phi\varsigma_{1,\beta\mu_2} + \beta\mu_1\chi_{2,\beta\mu_2} + \phi\kappa_1\tau_{1,\beta\mu_2} - \gamma_1}{\eta_1 - \beta\mu_1\rho_{2,\beta\mu_2}}$.
From Equations (19),(21), and (26)-(33), it yields

$$\pi_{N,1}(0) = \lambda\left(\kappa_1\varepsilon_0 + \sigma_0 + \kappa_2\triangle_0 + \psi_1 + \sum_{i=2}^{N}\psi_i + \sum_{i=2}^{N}(\gamma_i + \kappa_2\eta_i)\right)^{-1}.$$

Now, from the rate probabilities ($\pi_{i,j}(0)$) using Equation (20), the pre-arrival epoch probabilities ($\pi^-_{i,j}$) can be derived easily.

Next, setting $\varpi = 0$ in the Equations (11)-(18) and using (20). We obtain after slight simplification.

$$\begin{aligned}
\pi_{N,1} &= \frac{\lambda}{\zeta_N}\left(1 - \frac{N-1}{N^2}\right)\pi^-_{N-1,1},\\
\pi_{i,1} &= \left(\frac{\zeta_{i+1} - \phi}{\zeta_i}\right)\pi_{i+1,1} + \frac{\lambda}{\zeta_i}\left(\left(1 - \frac{i-1}{N^2}\right)\pi^-_{i-1,1} - \left(1 - \frac{i}{N^2}\right)\pi^-_{i,1}\right), i = N-1, ..., 3,\\
\pi_{2,1} &= \left(\frac{\zeta_3 - \phi}{\zeta_2}\right)\pi_{3,1} + \frac{\lambda}{\zeta_2}\left(\pi^-_{1,0} + \pi^-_{1,1} - \left(1 - \frac{2}{N^2}\right)\pi^-_{2,1}\right),\\
\pi_{1,1} &= \left(\frac{\beta\mu_1}{\phi + \beta\nu}\right)\pi_{2,1} - \left(\frac{\lambda}{\phi + \beta\nu}\right)\pi^-_{1,1},\\
\pi_{N,2} &= \frac{\phi}{\theta_N}\pi_{N,1} + \frac{\lambda}{\theta_N}\left(1 - \frac{N-1}{N^2}\right)\pi^-_{N-1,2},\\
\pi_{i,2} &= \left(\frac{\theta_{i+1}}{\theta_i}\right)\pi_{i+1,2} + \frac{\phi}{\theta_i}\pi_{i,1} + \frac{\lambda}{\theta_i}\left(\left(1 - \frac{i-1}{N^2}\right)\pi^-_{i-1,2} - \left(1 - \frac{i}{N^2}\right)\pi^-_{i,2}\right), i = N-1, ..., 2,\\
\pi_{1,2} &= \frac{\mu_1}{\mu_2}\pi_{2,2} + \frac{\phi}{\beta\mu_2}\pi_{1,1} - \frac{\lambda}{\beta\mu_2}\pi^-_{1,2},\\
\pi_{1,0} &= \frac{\nu}{\mu_1}\pi_{2,1} + \frac{\mu_2}{\mu_1}\pi_{2,2} + \frac{\lambda}{\beta\mu_1}\left(\pi^-_{0,0} - \pi^-_{1,0}\right).
\end{aligned}$$

Finally, the explicit expressions of $\pi_{0,0}$ can be computed by using the normalization condition, that is,

$$\pi_{0,0} = 1 - \pi_{1,0} - \sum_{i=1}^{N}(\pi_{i,1} + \pi_{i,2}).$$

4. Measures of Performance

– The mean number of customers in the system.

$$L_s = \pi_{1,0} + \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,2}).$$

– The mean number of customers waiting for service.

$$L_q = \sum_{i=2}^{N} (i-2)(\pi_{1,1} + \pi_{i,2}).$$

– The mean waiting time of customers in the system.

$$W_s = \frac{L_s}{\lambda'}, \text{ where } \quad \lambda' = \lambda(1 - (\pi_{N,1} + \pi_{N,2})) \text{ is the effective arrival rate.}$$

– The mean rate of joining the system.

$$J_s = \lambda(\pi_{0,0} + \pi_{1,0} + \pi_{1,1} + \pi_{1,2}) + \sum_{i=2}^{N} \lambda \left(1 - \frac{i}{N^2}\right)(\pi_{i,1} + \pi_{i,2}).$$

– The probability that server 2 is idle, in working vacation period and in normal busy period, respectively.

$$P_{idle} = \sum_{i=0}^{1} \pi_{i,0}; \;\; P_w = \sum_{i=1}^{N} \pi_{i,1}; \;\; P_b = \sum_{i=1}^{N} \pi_{i,2}.$$

– The average balking rate.

$$B_r = \frac{\lambda}{N^2} \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,2})$$

– The average reneging rates during busy period and working vacation period, respectively.

$$R_{ren1} = \alpha\xi_1 \sum_{i=2}^{N} (i-2)\pi_{i,1}, \;\; R_{ren2} = \alpha\xi_2 \sum_{i=2}^{N} (i-2)\pi_{i,2}.$$

– The average retention rates during busy period and working vacation period, respectively.

$$R_{ret1} = \alpha'\xi_1 \sum_{i=2}^{N} (i-2)\pi_{i,1}, \;\; R_{ret2} = \alpha'\xi_2 \sum_{i=2}^{N} (i-2)\pi_{i,2}.$$

5. Economic analysis

In this section, we develop a model for the costs incurred in the queueing system under consideration using the following symbols:

- C_1 : Cost per unit time when server 2 is on normal busy period.
- C_2 : Cost per unit time when server 2 is on working vacation period.
- C_3 : Cost per unit time when server 2 is idle during working vacation.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.

- C_5 : Cost per unit time when a customer balks.
- C_6 : Cost per service per unit time during busy period.
- C_7 : Cost per service per unit time during working vacation period.
- C_8 : Cost per unit time when a customer reneges during the working vacation period of server 2.
- C_9 : Cost per unit time when a customer reneges during normal busy period of server 2.
- C_{10} : Cost per unit time when a customer is retained during the working vacation period of server 2.
- C_{11} : Cost per unit time when a customer is retained during normal busy period of server 2.
- C_{12} : Cost per unit time when a customer returns to the system as a feedback customer.
- C_{13} : Fixed server purchase cost per unit.

Let
R be the revenue earned by providing service to a customer.
Γ be the total expected cost per unit time of the system.
Δ be the total expected revenue per unit time of the system.
Θ be the total expected profit per unit time of the system.
Thus

$$\Gamma = C_1P_b + C_2P_w + C_3P_{idle} + C_4L_q + C_5B_r + C_8R_{ren1} + C_9R_{ren2}$$

$$+C_{10}R_{ret1} + C_{11}R_{ret2} + (\mu_1 + \mu_2)C_6 + \nu C_7 + \beta'(\mu_1 + \mu_2 + \nu)C_{12} + 2C_{13}.$$

The total expected revenue per unit time of the system is given by:

$$\Delta = R\big(\mu_1\pi_{1,0} + (\mu_1 + \nu)P_w + (\mu_1 + \mu_2)P_b\big)$$

Now, the total expected profit is presented as

$$\Theta = \Delta - \Gamma.$$

6. Numerical analysis

6.1. Effect of different parameters on the performance measures of the system

Case 1: Effect of arrival rate (λ).

We check the behavior of the system characteristics for various values of (λ) by keeping all other variables fixed. Put $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\alpha = 0.4$, $\beta = 0.6$, and $N = 5$.

Table 1: Variation in system performance measures vs. λ

λ	1,4	2,2	3	3,8	4,2	4,8
L_s	1.14991	1.91543	2.59842	3.12838	3.33741	3.59305
J_s	1.34147	1.95358	2.38493	2.66080	2.75654	2.86360
B_r	0.05852	0.24641	0.61506	1.13919	1.44346	1.93639
R_{ren1}	0.00231	0.01641	0.02729	0.03161	0.03187	0.03077
R_{ren2}	0.01089	0.05901	0.12314	0.18468	0.21173	0.24702
R_{ret1}	0.00347	0.02462	0.04094	0.04742	0.04780	0.04616
R_{ret2}	0.01634	0.08852	0.18472	0.27702	0.31759	0.37054
W_s	0.07773	0.43723	0.88339	1.28599	1.45611	1.67216
P_{idle}	0.58306	0.34822	0.19872	0.11374	0.08692	0.05907
P_w	0.16708	0.20504	0.19084	0.15799	0.14092	0.11745
P_b	0.24986	0.44674	0.61044	0.72827	0.77216	0.82348

According to Table 1, we observe that along the increasing of the arrival rate λ, the characteristics B_r, L_s, J_s, P_b, R_{ren1}, R_{ren2}, R_{ret1}, R_{ret2}, P_w all increase. While P_{idle} decreases monotonically. This is due to the fact that along the increases of the arrival rate, the queue of the system becomes large. Thus, the normal busy period becomes significant, while the probability that the server 2 becomes idle P_{idle} decreases. Furthermore, the average balking rate increases with λ because of the size of the system.

Case 2: Effect of service rates $(\mu_1), (\mu_2)$ *and* (ν).

We examine the behavior of the characteristics of the system for various values of $(\mu_1), (\mu_2)$ and (ν), respectively by keeping all other variables fixed. To this end, we consider the following cases

- $\lambda = 2.5$, $\mu_2 = 1.9$, $\nu = 1.4$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 1.2$, and $N = 5$.
- $\lambda = 2.5$, $\mu_1 = 3$, $\nu = 1.4$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 1.2$, and $N = 5$.
- $\lambda = 2.5$, $\mu_1 = 3$, $\mu_2 = 2.5$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 0.5$, and $N = 5$.

Table 2: Variation in system performance measures vs. μ_1

μ_1	2.1	2.5	2.9	3.3	3.5	3.7
L_s	2.59636	2.37738	2.18182	2.00876	1.93012	1.85638
J_s	1.97894	2.06061	2.12774	2.18275	2.20640	2.22782
B_r	0.52105	0.43938	0.37225	0.31724	0.29359	0.27217
R_{ren1}	0.00409	0.00379	0.00339	0.00293	0.00269	0.00245
R_{ren2}	0.06321	0.05109	0.04092	0.03246	0.02879	0.02546
R_{ret1}	0.00613	0.00568	0.00509	0.00440	0.00404	0.00367
R_{ret2}	0.09482	0.07664	0.06139	0.04869	0.04319	0.03819
W_s	0.89249	0.73355	0.59642	0.47920	0.42735	0.37959
P_{idle}	0.20634	0.24159	0.27501	0.30622	0.32095	0.33510
P_w	0.17391	0.18698	0.19688	0.20405	0.20676	0.20896
P_b	0.61974	0.57143	0.52811	0.48972	0.47228	0.45594

Table 3: Variation in system performance measures vs. μ_2

μ_2	1.7	1.9	2.1	2.3	2.5	2.7
L_s	2.20418	2.13650	2.07650	2.02313	1.97550	1.93286
J_s	2.11974	2.14254	2.16232	2.17956	2.19464	2.20789
B_r	0.38025	0.35745	0.33767	0.32043	0.30535	0.29210
R_{ren1}	0.00306	0.00328	0.00348	0.00367	0.00384	0.00399
R_{ren2}	0.04224	0.03865	0.03550	0.03273	0.03027	0.02809
R_{ret1}	0.00459	0.00492	0.00523	0.00550	0.00576	0.00599
R_{ret2}	0.06336	0.05798	0.05326	0.04909	0.04541	0.04214
W_s	0.60457	0.56533	0.53104	0.50096	0.47448	0.45108
P_{idle}	0.26413	0.28303	0.30013	0.31564	0.32975	0.34260
P_w	0.18537	0.19891	0.21123	0.22244	0.23268	0.24204
P_b	0.55050	0.51806	0.48864	0.46192	0.43757	0.41536

Table 4: Variation in system performance measures vs. ν

ν	1.3	1.5	1.7	1.9	2.1	2.3
L_s	2.10375	2.05437	2.00692	1.96140	1.91775	1.87594
J_s	2.15324	2.16818	2.18221	2.19538	2.20775	2.21936
B_r	0.34675	0.33181	0.31778	0.30461	0.29224	0.28063
R_{ren1}	0.00968	0.00911	0.00857	0.00807	0.00760	0.00716
R_{ren2}	0.02424	0.02337	0.02255	0.02176	0.02101	0.02029
R_{ret1}	0.01453	0.01367	0.01286	0.01211	0.01141	0.01075
R_{ret2}	0.03636	0.03506	0.03382	0.03264	0.03151	0.03044
W_s	0.54523	0.52009	0.49635	0.47394	0.45281	0.43290
P_{idle}	0.29148	0.30771	0.32362	0.33918	0.35437	0.36917
P_w	0.38610	0.37849	0.37095	0.36351	0.35618	0.34897
P_b	0.32242	0.31380	0.30543	0.29731	0.28945	0.28186

From Tables 2–3–4, we observe that

– with the increases of μ_1, μ_2 and ν, B_r decreases, while J_s increases, as it should be. Therefore, customers are served faster with μ_1, μ_2 and ν. This implies a decrease in the mean number of customers in the system L_s, in the probability that the server 2 is on normal busy period P_b and in the mean waiting time W_s. Consequently, the probability that the server 2 becomes idle P_{idle} increases with the service rates.

– the probability of working vacation of server 2, P_w increases with both μ_1 and μ_2 because customers are served faster. Then, the mean system size decreases. Hence, the server 2 switches to vacation period. On the other hand, P_w decreases with ν, as intuitively expected.

– when μ_1 and ν increase, the average reneging rates during working vacation and during normal busy period R_{ren1} and R_{ren2}, average retention rates in working vacation and in normal busy period R_{ret1} and R_{ret1} decrease. This agree absolutely with our intuition. While when μ_2 increases, R_{ren2} and R_{ret2} decrease because customers are served faster. Thus, the size of the system is reduced, hence, server 2 goes on vacation. Consequently, the probability of working vacation increases which leads to an increase in the average reneging and retention rates R_{ren1} and R_{ret1}, respectively.

Case 3: Effect of reneging rates (ξ_1) and (ξ_2).

We check the behavior of the performance measures of the system for various values of (ξ_1) and (ξ_2), respectively by keeping all other variables fixed. Let

- $\lambda = 3.5$, $\mu_1 = 2.1$, $\mu_2 = 1.7$, $\nu = 1.3$, $\beta = 0.6$, $\xi_2 = 2$, $\alpha = 0.6$, $\phi = 0.1$, and $N = 5$.
- $\lambda = 3.5$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$, $\xi_1 = 1$, $\alpha = 0.6$, $\phi = 1.2$, and $N = 5$.

Table 5: Variation in system performance measures vs. ξ_1

ξ_1	3.5	3.7	3.9	4.1	4.3	4.5
L_s	2.38620	2.36189	2.33912	2.31773	2.29763	2.27868
J_s	3.04204	3.05417	3.06521	3.07529	3.08453	3.09301
B_r	0.45795	0.44582	0.43478	0.42470	0.41546	0.40698
R_{ren1}	0.86827	0.87516	0.88040	0.88415	0.88655	0.88775
R_{ren2}	0.21529	0.21313	0.21109	0.20917	0.20736	0.20564
R_{ret1}	0.57884	0.58344	0.58693	0.58943	0.59103	0.59183
R_{ret2}	0.14353	0.14208	0.14072	0.13944	0.13824	0.13709
W_s	0.59288	0.57183	0.55215	0.53372	0.51642	0.50016
P_{idle}	0.15090	0.15327	0.15554	0.15769	0.15974	0.16170
P_w	0.66838	0.66710	0.66586	0.66468	0.66355	0.66247
P_b	0.18072	0.17963	0.17860	0.17762	0.17671	0.17583

Table 6: Variation in system performance measures vs. ξ_2

ξ_2	3.5	3.7	3.9	4.1	4.3	4.5
L_s	2.19085	2.17338	2.15715	2.14203	2.12791	2.11470
J_s	3.09108	3.09919	3.10654	3.11320	3.11926	3.12479
B_r	0.40891	0.40080	0.39345	0.38679	0.38073	0.37520
R_{ren1}	0.10338	0.10446	0.10547	0.10642	0.10733	0.10818
R_{ren2}	0.69007	0.69389	0.69657	0.69822	0.69896	0.69889
R_{ret1}	0.06892	0.06964	0.07031	0.07095	0.07155	0.07212
R_{ret2}	0.46004	0.46259	0.46438	0.46548	0.46597	0.46592
W_s	0.50091	0.48666	0.47347	0.46121	0.44980	0.43915
P_{idle}	0.22108	0.22338	0.22554	0.22758	0.22951	0.23134
P_w	0.26548	0.26824	0.27084	0.27330	0.27561	0.27780
P_b	0.51344	0.50838	0.50362	0.49912	0.49488	0.49086

According to Tables 5–6, we observe that

– with the increases of reneging rates ξ_1 and ξ_2, the characteristics L_s, W_s and B_r decrease, while J_s increases, as intuitively expected.

– along the increasing of ξ_1, the average reneging rate during working vacation R_{ren1} increases, while the average rate of reneging in the normal busy period of server 2, R_{ren2} decreases.

– along the increasing of ξ_1, the probability of working vacation P_w, the probability of normal busy period P_b decrease because of the size of the system which becomes small due to reneging. Consequently, the probability that the server 2 becomes idle P_{idle} increases with ξ_1.

– when the reneging rate ξ_2 increases, the average reneging rate during normal busy period R_{ren2} and the average rate of reneging in the busy period of server 2 during his vacation R_{ren1} increase.

– the increases of ξ_2 implies a decreasing of P_b and an increasing of P_w, which can be explained by the fact that when reneging rate increases in the normal busy period of server 2, more customers are lost. Thus, server 2 goes on vacation, consequently P_w and P_{idle} increase.

Case 4: Effect of vacation rate (ϕ).

We study the behavior of the performance measures of the system for various values of (ϕ) by keeping all other variables fixed. Put $\lambda = 3$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.5$, $\alpha = 0.6$, and $N = 5$.

Table 7: Variation in system performance measures vs. ϕ

ϕ	0.5	0.7	0.9	1.1	1.3	1.5
L_s	2.60098	2.56891	2.54766	2.53259	2.52136	2.51270
J_s	2.38962	2.40830	2.42040	2.42879	2.43490	2.43952
B_r	0.61037	0.59169	0.57959	0.57120	0.56509	0.56047
R_{ren1}	0.01925	0.01413	0.01087	0.00864	0.00705	0.00586
R_{ren2}	0.16768	0.18586	0.19731	0.20502	0.21048	0.21448
R_{ret1}	0.01283	0.00942	0.00724	0.00576	0.00470	0.00391
R_{ret2}	0.11179	0.12391	0.13154	0.13668	0.14032	0.14299
W_s	0.87984	0.85509	0.83889	0.82753	0.81917	0.81278
P_{idle}	0.19442	0.19954	0.20307	0.20567	0.20767	0.20927
P_w	0.34910	0.28653	0.24394	0.21293	0.18926	0.17055
P_b	0.45647	0.51392	0.55298	0.58139	0.60305	0.62017

From Table 7, we remark that along the increasing of the vacation rate ϕ, L_s and W_s decrease. Therefore, the average balking rate B_r decreases, while the average rate of joining the system J_s increases with ϕ. Further, the increase in vacation rate implies that P_b increases, while, the probability that the system goes on working vacation P_w decreases. This implies an increase in the mean number of customers served. Therefore, the probability that the server 2 becomes idle P_{idle} increases. Further, with the increases of ϕ, R_{ren1} and R_{ret1} (resp. R_{ren2} and R_{ret2}) decreases (resp. increase), as intuitively expected.

Case 5: Effect of non-feedback probability (β).

We examine the behavior of the performance measures of the system for various values of (β) by keeping all other variables fixed. Put $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\alpha = 0.4$, $\lambda = 3$, and $N = 5$.

Table 8: Variation in system performance measures vs. β

β	0.1	0.3	0.5	0.7	0.9	1
L_s	4.54732	3.81878	2.97531	2.27287	1.77073	1.58122
J_s	0.86754	1.63389	2.19464	2.52636	2.70701	2.76395
B_r	2.13245	1.36610	0.80535	0.47363	0.29298	0.23604
R_{ren1}	0.00085	0.01173	0.02482	0.02695	0.02132	0.01758
R_{ren2}	0.40725	0.29047	0.16948	0.08786	0.04352	0.03040
R_{ret1}	0.00128	0.01760	0.03724	0.04042	0.03198	0.02637
R_{ret2}	0.61088	0.43570	0.25423	0.13179	0.06529	0.04560
W_s	2.54892	1.86435	1.16273	0.66146	0.36090	0.26329
P_{idle}	0.00141	0.03512	0.13419	0.26505	0.38866	0.44283
P_w	0.00291	0.05196	0.14789	0.22317	0.25612	0.26039
P_b	0.99568	0.91292	0.71792	0.51177	0.35522	0.29678

Thought Table 8, we see that when the non-feedback probability β increases, L_s and W_s decrease, this results in the decreasing of the average balking rate B_r and in the increasing of the average rate of joining the system J_s. Moreover, along the increases of the non-feedback probability, R_{ren1} and R_{ret1} increase, while R_{ren2} and R_{ret2} decreases. Further, obviously, the probability of normal busy period P_b decreases, the probability that the system is on working vacation P_w and the probability that the server is idle P_{idle} increase, as it should be.

Case 6: Effect of non-retention probability (α).

We examine the behavior of the characteristics of the system for various values of (α) by keeping all other variables fixed. We take $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\lambda = 3$, $\beta = 0.6$, and $N = 5$.

Table 9: Variation in system performance measures vs. α

α	0,1	0.3	0.5	0.7	0.9	1.0
L_s	2.72298	2.63780	2.56099	2.49153	2.42851	2.39916
J_s	2.30217	2.35932	2.40879	2.45182	2.48939	2.50639
B_r	0.69782	0.64067	0.59120	0.54817	0.51060	0.49360
R_{ren1}	0.00697	0.02064	0.03379	0.04632	0.05815	0.06379
R_{ren2}	0.03488	0.09622	0.14786	0.19138	0.22809	0.24423
R_{ret1}	0.06277	0.04816	0.03379	0.01985	0.00646	0.00000
R_{ret2}	0.31400	0.22452	0.14786	0.08202	0.02534	0.00000
W_s	0.98850	0.91654	0.85196	0.79381	0.74127	0.71689
P_{idle}	0.18516	0.19437	0.20289	0.21078	0.21809	0.22155
P_w	0.17945	0.18723	0.19428	0.20069	0.20653	0.20924
P_b	0.63539	0.61840	0.60283	0.58853	0.57538	0.56921

Through Table 9, we remark that when the non-retention probability α increases, the size of the system L_s, the mean waiting time W_s and the average balking rate B_r decrease, while the probability that customers join the system increases. Moreover, the average reneging rates R_{ren1} and R_{ren2} increase with α, while average retention rates R_{ret1} and R_{ret2} decrease with the increasing of α, which absolutely agree with our intuition. This implies that the probability of normal busy period P_b decreases. Consequently, P_w and P_{idle} increase with α, as it should be.

Case 7: Effect of system capacity (N).

We analyze the behavior of the performance measures of the system for various values of (N) by keeping all other variables fixed. Let $\lambda = 3$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, and $\phi = 1.1$.

Table 10: Variation in system performance measures vs. N

N	3	4	5	6	7	8
L_s	1.73287	2.21507	2.69421	3.16073	3.60819	4.03161
J_s	1.97401	2.17590	2.32287	2.43554	2.52532	2.59887
B_r	1.02598	0.82409	0.67712	0.56445	0.47467	0.40112
R_{ren1}	0.00201	0.00446	0.00532	0.00543	0.00524	0.00496
R_{ren2}	0.00850	0.03470	0.06644	0.10020	0.13399	0.16668
R_{ret1}	0.00301	0.00669	0.00798	0.00814	0.00786	0.00744
R_{ret2}	0.01275	0.05205	0.09966	0.15030	0.20098	0.25002
W_s	0.15645	0.54544	0.96360	1.38828	1.80595	2.20759
P_{idle}	0.29531	0.23036	0.18785	0.15868	0.13791	0.12272
P_w	0.28102	0.23321	0.19472	0.16616	0.14516	0.12955
P_b	0.42367	0.53643	0.61743	0.67516	0.71693	0.74773

From Tables 10, we remark that along the increasing of N, the average balking rate B_r decreases due to the large capacity of the system. Then, the means system size L_s, and the mean waiting time W_s increase. Consequently, P_b increases, while, P_w and P_{idle} decrease, this implies an increase in the mean number of customers served with N. Moreover, the average reneging and retention rates R_{ren2} and R_{ret2} increase due to the significant number of customers in the system. While the behaviour of R_{ren1} and R_{ret1} is not monotonic, it increases, then decreases when N is above a certain threshold.

6.2. Economic analysis

In this part we present the variation in total expected cost, total expected revenue and total expected profit with the change in diverse parameters of the system. For the whole numerical study we fix the costs at $C_1 = 4$, $C_2 = 2$, $C_3 = 2$, $C_4 = 3$, $C_5 = 3$, $C_6 = 4$, $C_7 = 4$, $C_8 = 2$, $C_9 = 2$, $C_{10} = 3$, $C_{11} = 3$, $C_{12} = 2$, $C_{13} = 5$, $R = 25$. And consider the following Tables

- Table 11: $\lambda = 1.4 : 0.8 : 4.8$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 12: $\lambda = 2.5$, $\mu_1 = 2.1 : 0.4 : 3.7$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 13: $\lambda = 2.5$, $\mu_1 = 3.0$, $\mu_2 = 1.7 : 0.2 : 2.7$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 14: $\lambda = 2.5$, $\mu_1 = 3.0$, $\mu_2 = 2.5$, $\nu = 1.3 : 0.2 : 2.3$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 15: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.3 : 0.2 : 1.3$, $\xi_2 = 0.1$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 16: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.3 : 0.2 : 1.3$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 17: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 0.3 : 0.2 : 1.3$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 18: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.1 : 0.2 : 1$, $\alpha = 0.6$, $N = 10$,

- Table 19: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.6$, $\alpha = 0.1 : 0.2 : 1$, $N = 10$,
- Table 20: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.1$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 3 : 2 : 11$.

The numerical results are presented in following Tables and Graphes.

Table 11: Γ, Δ and Θ vs. λ.

λ	1.4	2.2	3	3.8	4.2	4.8
Γ	43.41467	47.16563	53.81363	61.46361	64.93732	69.43411
Δ	62.44631	87.83599	103.58542	110.98226	112.68849	113.98970
Θ	19.03164	40.67037	49.77179	49.51866	47.75117	44.55559

Table 12: Γ, Δ and Θ vs. μ_1.

μ_1	2.1	2.5	2.9	3.3	3.5	3.7
Γ	49.04408	48.72019	48.82543	49.31085	49.67482	50.10801
Δ	89.84638	95.38482	100.32531	104.82918	106.95748	109.01970
Θ	40.80231	46.66463	51.49987	55.51833	57.28266	58.91169

Table 13: Γ, Δ and Θ vs. μ_2.

μ_2	1.7	1.9	2.1	2.3	2.5	2.7
Γ	48.64273	48.91370	49.2815	49.73212	50.25305	50.83345
Δ	100.27834	101.48669	102.4811	103.30047	103.97722	104.53800
Θ	51.63561	52.57298	53.1996	53.56835	53.72417	53.70455

Table 14: Γ, Δ and Θ vs. ν.

ν	1.3	1.5	1.7	1.9	2.1	2.3
Γ	60.006248	60.42808	60.83935	61.24349	61.64399	62.04501
Δ	112.06111	113.81396	115.51876	117.17090	118.76361	120.29314
Θ	52.05486	53.38588	54.67941	55.92741	57.11961	58.24813

Table 15: Γ, Δ and Θ vs. ξ_1.

ξ_1	0.3	0.5	0.7	0.9	1.1	1.3
Γ	55.60031	55.55241	55.50744	55.46520	55.42550	55.38809
Δ	106.73059	106.62743	106.52766	106.43137	106.33855	106.24913
Θ	51.13028	51.07502	51.02023	50.96617	50.91305	50.86104

Table 16: Γ, Δ and Θ vs. ξ_2.

ξ_2	0.3	0.5	0.7	0.9	1.1	1.3
Γ	54.42735	54.18721	53.50838	52.98109	52.05005	52.20927
Δ	104.77963	104.26948	102.84655	101.58610	99.60370	99.46025
Θ	50.35228	50.08227	49.33816	48.60501	47.55365	47.25098

Table 17: Γ, Δ and Θ vs. ϕ.

ϕ	0.1	0.5	0.9	1.3	1.7	2.1
Γ	55.52628	55.08756	55.01420	54.98712	54.97424	54.96726
Δ	103.76382	105.37013	105.66036	105.78134	105.84754	105.88922
Θ	48.23755	50.28258	50.64616	50.79423	50.87330	50.92196

Table 18: Γ, Δ and Θ vs. β.

β	0.1	0.3	0.5	0.7	0.9	1
Γ	82.79219	72.35272	59.53657	49.17583	42.63061	40.23601
Δ	114.99820	114.53007	109.25291	96.80668	83.22630	77.19281
Θ	32.20601	42.17735	49.71635	47.63085	40.59570	36.95680

Table 19: Γ, Δ and Θ vs. α

α	0.1	0.3	0.5	0.7	0.9	1
Γ	58.62700	55.12727	52.71747	51.01291	49.76362	49.25887
Δ	106.81003	104.56888	102.68686	101.11251	99.78247	99.19185
Θ	48.18303	49.44161	49.96940	50.09960	50.01886	49.93298

Table 20: Γ, Δ and Θ vs. N.

N	3	5	6	7	9	11
Γ	46.70298	48.86349	50.13514	51.42076	53.87616	56.02955
Δ	88.66901	98.09453	100.70350	102.56714	104.95503	106.32773
Θ	41.96604	49.23104	50.56836	51.14637	51.07887	50.29818

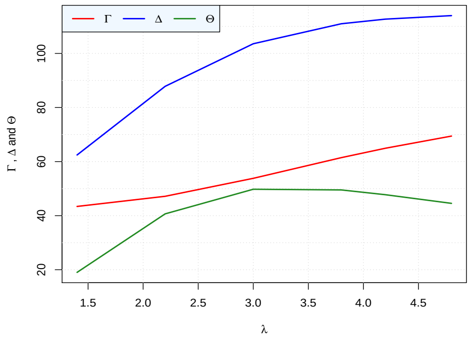

Figure 1: Γ, Δ and Θ vs. λ.

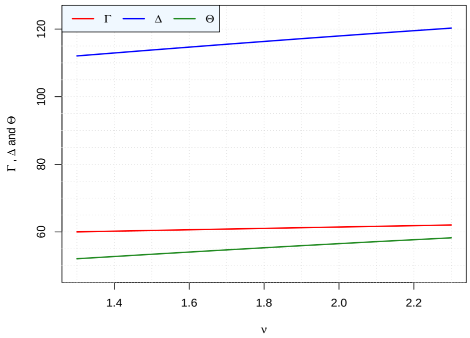

Figure 2: Γ, Δ and Θ vs. ν

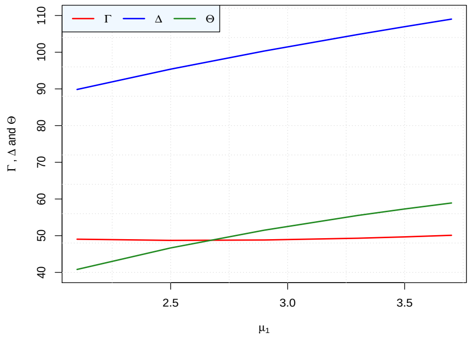

Figure 3: Γ, Δ and Θ vs. μ_1.

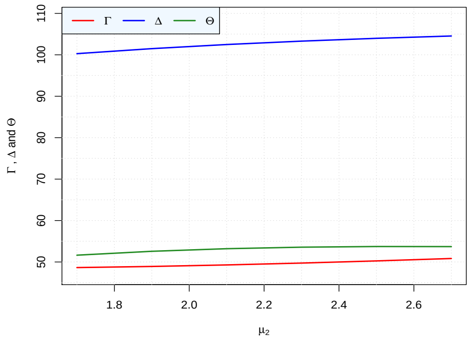

Figure 4: Γ, Δ and Θ vs. μ_2.

General comments

– According to Table 11 and Figure 1, we remark that the increases of λ generates an increase in Γ and Δ, this is quite obvious. While the behavior of Θ is not monotonic, it increases, then decreases when λ is above a certain threshold, this can be explicable by the fact that a large number of incoming customers engenders a large number of customers served, and consequently the total expected profit increases, but when λ is large enough, the customers in the system may renege due to the long queue length, this implies a decreases in Θ. Furthermore, the non-monotonicity of the total expected profit can be due to the choice of the system parameters.

– From Tables 12-14 and Figures 2-4, we remark that Γ, Δ and Θ all increase with the increasing of μ_1, μ_2, and ν. We can explain this by the fact that with the increasing of the service rates, the average balking rate B_r decreases. The customers are served faster, this leads to a decrease in the mean number of

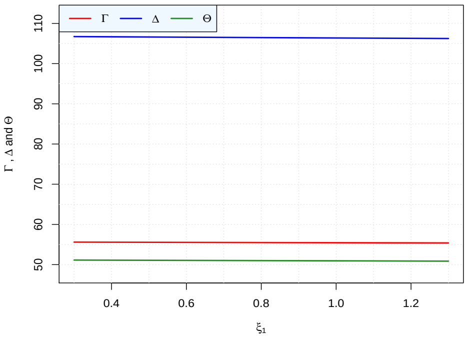

Figure 5: Γ, Δ and Θ vs. ξ_1.

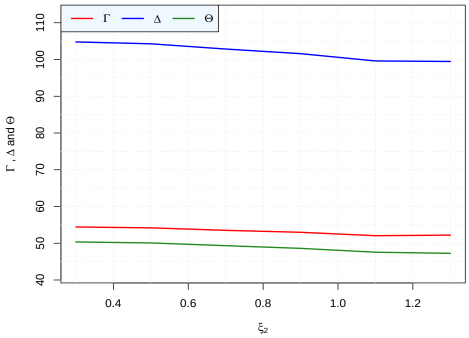

Figure 6: Γ, Δ and Θ vs. ξ_2.

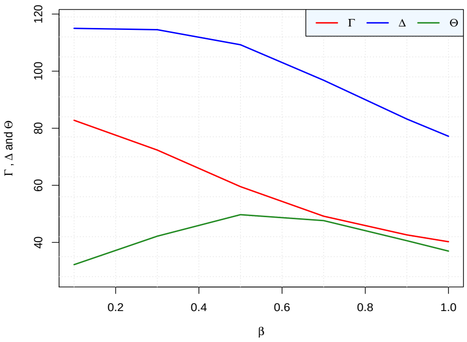

Figure 7: Γ, Δ and Θ vs. β.

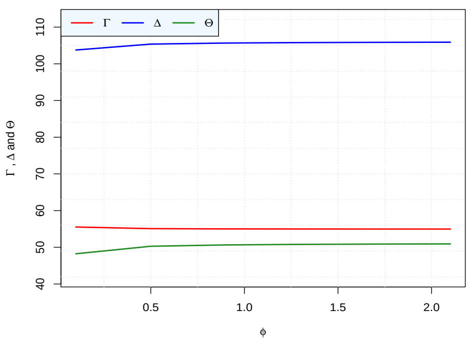

Figure 8: Γ, Δ and Θ vs. ϕ.

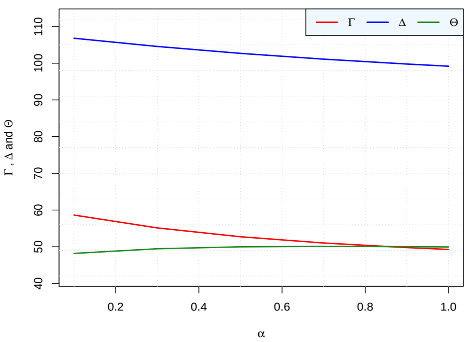

Figure 9: Γ, Δ and Θ vs. α.

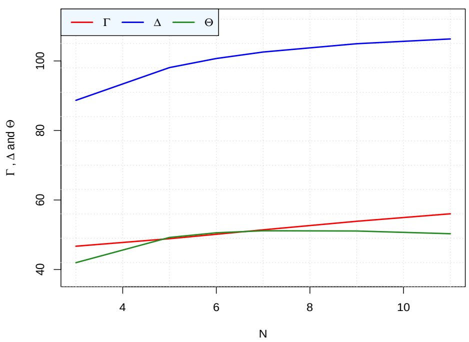

Figure 10: Γ, Δ and Θ vs. N.

customers in the system L_s, in the probability that the system is idle P_{idle}, in the mean waiting time W_s, in average reneging rates R_{ren1} and R_{ren1}. Therefore, the expected total profit increases.

– From Tables 15-16 and Figures 5-6, we remark that Δ, and Θ decrease along the increasing of impatience rates ξ_1 and ξ_2. This is due to the fact that the mean waiting time of impatient customers decreases with the increasing of ξ_1 and ξ_2. Therefore, the average rate of loss customers increases, while the mean number of customers waiting for service and the busy period probability decrease which results in the decreasing of the total expect cost Γ. Consequently, this later generates a decrease in the total expected profit Θ. Thus, it is quite clear that impatient phenomenon has a negative impact in the economy.

– From Table 17 and Figure 8, we see that Γ decreases with ϕ, while Δ and Θ increase along the increasing of the vacation rate ϕ. Obviously, the decrease in the mean vacation time implies a diminution in probability of loss customers, this leads to a high rate of customers served. Therefore, the total expected profit becomes significant.

– From Table 18 and Figure 7, we remark that along the increasing of non-feedback probability β, total expected cost Γ and total expected revenue Δ decrease. While, the total expected profit Θ is not monotonic with β, it first increases, then, decreases significantly. The non-monotonicity can be due to the choice of the system parameters. Therefore, one can deduce easily the negative impact of this probability on different costs of the system.

– Through Table 19 and Figure 9, we observe that the increasing of non-retention probability α generates a decrease in Γ and Δ. While, the behavior of the total expected profit Θ is not monotone with α, it increases, then, it decreases, when α is above a certain threshold. This can be explained by the fact that when the non-retention probability α increases, the size of the system and the mean waiting time decrease, while the average reneging rate increases. This implies also that the probability of normal busy period P_b decreases. Therefore, the mean number of customers served is reduced. Moreover, the increase of Θ can be due to the choice of $\xi_1 = 0.6$ and $\xi_2 = 0.4$ So, it is quite evident that retention probability has a positive effect on the revenue generation and on the total expected profit of the system.

– From Table 20 and Figure 10, we remark that along the increasing N, total expected cost Γ, total expected revenue Δ increase. While, total expected profit Θ is not monotonic, it increases, then decreases when N is above a certain threshold. Obviously, the larger the size of the system, the smaller the average rate of balking, this generates a large number of customers served which engenders a positive impact on the costs of the system and consequently on the economy of any firm. Note that the non-monotonicity of Θ can be due to the choice of the impatience rates ξ_1 and ξ_2.

7. Conclusion

In this paper, we present a study of heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers. It is supposed that impatience timers of customers in the system depend on the state of the server. The equations of the steady state probabilities are developed. The most important performance measures of the system are given. Then, based on the performance analysis, we formulate a cost model to determine the effect of different system parameters on the different characteristics as well as on total expected cost, total expected revenue, and total expected profit of the system.

In this study, the positive impact of retention probability on both characteristics and costs of the system under consideration has been shown. The present analysis has a large application in many real world systems as telecommunication networks, call centers and production-inventory systems. For further work, it will be interesting to consider a multiserver queueing system with heterogeneous service times, multiple working vacations, and impatient customers depending on the state of the servers. Moreover, one can develop a similar model wherein the servers are subject to sudden halt.

Acknowledgment

Authors thank the referee for the suggestions for a better presentation.

References

[1] Abidini, M. A., Boxma, O. and Resing, J. (2016). Analysis and optimization of vacation and polling models with retrials, *Performance Evaluation*, **98**, 52–69.

[2] Altman, E. and Yechiali, U. (2008). Infinite-server queues with system's additional tasks and impatient customers, *Probab. Eng. Inform. Sci.*, **22**, 477–493.

[3] Ammar, S. I. (2014). Transient analysis of a two-heterogeneous servers queue with impatient behavior, *Journal of the Egyptian Mathematical Society*, **22**(1), 90–95.

[4] Bae, J. and Kim, S. (2010). The stationary workload of the $G/M/1$ queue with impatient customers, *Queueing Syst.*, **64**, 253–265.

[5] Bouchentouf, A. A. and Guendouzi, A. (2018). Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers, *SeMA Journal*, https://doi.org/10.1007/s40324-018-0180-2

[6] Bouchentouf, A. A., Kadi, M. and Rabhi, A. (2014). Analysis of two heterogeneous server queueing model with balking, reneging and feedback, *Mathematical Sciences And Applications E-Notes*, **2**(2), 10–21.

[7] Bouchentouf, A. A. and Yahiaoui, L. (2017). On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, *Arab. J. Math.*, **6**(1), 1–11.

[8] Boxma, O., Perry, D., Stadje W. and Zacks, S. (2010). The busy period of an $M/G/1$ queue with customer impatience, *J. Appl. Probab.*, **47**, 130–145.

[9] Choudhury, A. and Medhi, P. (2011). Balking and reneging in multiserver Markovian queuing system, *Int. J. Math. Oper. Res.*, **3**, 377–394.

[10] Goswami, V. (2014). Analysis of Impatient Customers in Queues with Bernoulli Schedule Working Vacations and Vacation Interruption, *Journal of Stochastics*, Volume 2014, Article ID 207285, 10 pages.

[11] Gupta, N., Mishra, G. D. and Choubey, A. (2008). Performance analysis of an queueing model $M/M/c/N$ with balking and reneging, *International Journal of Computer Mathematical Sciences and Applications*, **2**(4), 335–339.

[12] Gupta, N., Mishra, G. D. and Choubey, A. (2009). Performance analysis of queueing model $M/M/1/N$ with balking and reneging, *International Journal of Pure Applied Mathematical Sciences*, **LXX(1-2)**, 59–65.

[13] Jose, Joby K., Manoharan, M. (2011). Markovian queueing system with random balking, *OPSEARCH*, **48**(3), 236-246.

[14] Jose, Joby K., Manoharan, M. (2014) Optimum system capacity for Markovian queuing system with an adaptive balking, *Journal of Statistics & Management Systems,* **17**(2), 155-164.

[15] Krishnamoorthy, A. and Sreenivasan, C. (2012). An M/M/2 Queueing System with Heterogeneous Servers Including One with Working Vacation, *International Journal of Stochastic Analysis*, Volume 2012, Article ID 145867, 16 pages.

[16] Kumar, R. and Sharma, S. K. (2014). Optimization of an $M/M/1/N$ feedback queue with retention of reneged customers, *Operations research and decisions*, **24**(3), 45–58.

[17] Kumar, R. and Sharma, S. K. (2014). A Multi-Server Markovian Feedback Queue with Balking Reneging and Retention of Reneged Customers, *AMO-Advanced Modeling and Optimization*, **16**(2), 395–406.

[18] Laxmi, P. V. and Jyothsna, K. (2015). Balking and reneging multiple working vacations queue with heterogeneous servers, *J Math Model Algor*, **14**, 267–285.

[19] Laxmi, P. V., Goswami, V. and Jyothsna, K. (2013). Analysis of finite buffer Markovian queue with balking, reneging and working vacations, *International Journal of Strategic Decision Sciences*, **4**(1), 1–24.

[20] Madhu, J. and Anamika, J. (2010). Working vacations queueing model with multiple types of server breakdowns, *Applied Mathematical Modelling*, **34**(1), 1–13.

[21] Panda, G. and Goswami, V. (2016). Equilibrium balking strategies in renewal input queue with bernoulli-schedule controlled vacation and vacation interruption, *Journal of industrial and management optimization*, **12**(3), 851–878.

[22] Yue, D. and Yue, W. (2010). A heterogeneous two-server network system with balking and a Bernoulli vacation schedule, *J. Ind. Manag. Optim.*, **6**, 501–516.