ProbStat Forum, Volume 12, January 2019, Pages 36–46
ISSN 0974-3235

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For details please visit www.probstat.org.in

Moments of progressively type-II right censored order statistics from additive Weibull distribution

B. Singh[a], **R.U. Khan**[a]

[a] *Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202002, India.*

Abstract. Some new recurrence relations for the single and product moments of progressively Type-II right censored order statistics from additive Weibull distribution have been established. These relations generalize the results given by Aggarwala and Balakrishnan (1996) and Mahmoud *et al.* (2017) for the progressively Type-II right censored order statistics for exponential and modified-Weibull distributions. Further, various deductions and related results are discussed and identified some of these recurrence relations for single moments of Progressively Type-II right censored order statistics which characterize this distribution. Finally this distribution is characterized via truncated moment.

1. Introduction

There are many scenarios in life-testing and reliability experiments in which units are lost or eliminated from the experimentation before failure. For this reason, several censoring scheme was introduced. A more general censoring scheme called progressive Type-II right censoring. This progressive censoring scheme can be described as follows:

Consider an experiment in which n independent items are placed on a life-test with continuous, identically distributed failure times $X_1, X_2, ..., X_n$. Suppose further that a censoring scheme $(R_1, R_2, ..., R_m)$ is prefixed such that immediately following the first failure X_1, R_1 of $n-1$ surviving units are removed from the test at random, then immediately following the second failure X_2, R_2 of $n-R_1-2$ surviving units are removed from the test at random. This process continues until, at the time of m-th observed failure X_m, the remaining $R_m = n - R_1 - ... - R_{m-1} - m$ units are removed from the experiment. The m ordered observed failure times denoted by $X_{1:m:n}^{(R_1,...,R_m)}$, $X_{2:m:n}^{(R_1,...,R_m)}$,..., $X_{m:m:n}^{(R_1,...,R_m)}$ are called progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$. If the failure times of the n items are based on continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$, then the joint pdf of $X_{1:m:n}^{(R_1,...,R_m)}$, $X_{2:m:n}^{(R_1,...,R_m)}$,..., $X_{m:m:n}^{(R_1,...,R_m)}$ is given by (Balakrishnan and Sandhu (1995))

$$f_{X_{1:m:n},...,X_{m:m:n}}(x_1, x_2, ..., x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$
$$-\infty < x_1 < x_2 < ... < x_m < \infty, \tag{1}$$

2010 *Mathematics Subject Classification.* MSC 2010: 62G30, 62E10, 60E05.
Keywords. Order statistics, Progressively Type-II right censored order statistics, additive Weibull distribution, Recurrence relations, Truncated moment, characterization.
Received: 21 February 2018; Revised: 18 January 2019, Accepted: 30 January 2019.

Email addresses: bavita43@gmail.com (B. Singh), aruke@rediffmail.com (R.U. Khan)

where $A(n,m-1)=n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{m-1}-m+1)$, with $A(n,0)=n$. Note that all the factors in $A(n,m-1)$ are positive integers. When $R_1=R_2=...=R_m=0$, so that $m=n$, this censoring scheme reduces to the case of no censoring (ordinary order statistics). For more details see Balakrishnan and Aggarwala(2000).

The $k-$th single moments of the $i-$th progressively Type-II right censored order statistic can be expressed from (1), as given by (Aggarwala and Balakrishnan (1996))

$$\begin{aligned}\mu_{i:m:n}^{(R_1,...,R_m)^{(k)}} &= E[X_{i:m:n}^{(R_1,...,R_m)}]^k \\ &= A(n,m-1)\int\int...\int_{0<x_1<...<x_m<\infty} x_i^k f(x_1)[1-F(x_1)]^{R_1} \\ &\times f(x_2)[1-F(x_2)]^{R_2}...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m. \qquad (2)\end{aligned}$$

Several authors obtained the recurrence relations under progressively Type-II right censored order statistics with special reference to those arising from exponential, Pareto, power function, Burr, logistic, half-logistic, log-logistic, generalized half-logistic and modified Weibull distributions, see Aggarwala and Balakrishnan (1996), Balakrishnan and Aggarwala (2000), Saran and Pushkarna (2001, 2014), Balakrishnan *et al.* (2001, 2011), Balakrishnan and Saleh (2011, 2012, 2013, 2017) and Mahmoud *et al.* (2014, 2017). In this work we mainly focus on the study of progressively type-II right censored order statistics arising from the additive Weibull distribution.

A random variable X is said to have additive Weibull distribution (Lemonte *et al.* (2014)) if its *pdf* is of the form

$$f(x)=(\alpha\beta x^{\beta-1}+\theta\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\theta x^{\delta})}, \quad x>0,\ \alpha,\beta,\theta,\delta>0. \qquad (3)$$

with the corresponding *df*

$$F(x)=1-e^{-(\alpha x^{\beta}+\theta x^{\delta})}. \qquad (4)$$

Also, the characterizing differential equation given by

$$f(x)=(\alpha\beta x^{\beta-1}+\theta\delta x^{\delta-1})[1-F(x)]. \qquad (5)$$

Additive Weibull model is the combination of two Weibull models in which one has increasing failure rate while other has decreasing failure rate. The exponential-Weibull and Weibull distributions are the special cases for $\delta=1$ or $\beta=1$ and $\theta=0$ or $\alpha=0$, respectively. The exponential distribution arises when $\beta=1$, $\theta=0$ or $\alpha=0$, $\delta=1$. The Rayleigh and two-parameter linear failure rate distributions are obtained when $\alpha=0,\ \delta=2$ or $\theta=0,\ \beta=2$ and $\beta=2,\delta=1$ or $\beta=1,\ \delta=2$, respectively.

2. Single moments of progressively Type-II censored order statistics

In this section, we establish several new recurrence relation for the single moments of progressively Type-II right censored order statistics from additive Weibull distribution by using characterizing differential equation given in (5).

Theorem 2.1 For $2\le m\le n$ and $k\ge 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,...,R_m)^{(k)}} &= \left(\frac{\alpha\beta}{k+\beta}\right)\Big[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,...,R_m)^{(k+\beta)}}+(1+R_1)\mu_{1:m:n}^{(R_1,...,R_m)^{(k+\beta)}}\Big] \\ &+\left(\frac{\theta\delta}{k+\delta}\right)\times\Big[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,...,R_m)^{(k+\delta)}}+(1+R_1)\mu_{1:m:n}^{(R_1,...,R_m)^{(k+\delta)}}\Big], \qquad (6)\end{aligned}$$

and for $m = 1,\ n = 1, 2, ...$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)^{(k)}} = \left(\frac{n\alpha\beta}{k+\beta}\right)\mu_{1:1:n}^{(n-1)^{(k+\beta)}} + \left(\frac{n\theta\delta}{k+\delta}\right)\mu_{1:1:n}^{(n-1)^{(k+\delta)}}. \tag{7}$$

Proof. From (2), we have

$$\mu_{1:m:n}^{(R_1,...,R_m)^{(k)}} = A(n, m-1)\int\int ... \int_{0<x_2<...<x_m<\infty} L(x_2)f(x_2)[1-F(x_2)]^{R_2}...f(x_m)[1-F(x_m)]^{R_m}dx_2...dx_m, \tag{8}$$

where

$$L(x_2) = \int_0^{x_2} x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1. \tag{9}$$

On using (5), (9) can be rewrite as

$$L(x_2) = \alpha\beta\int_0^{x_2} x_1^{k+\beta-1}[1-F(x_1)]^{R_1+1}dx_1 + \theta\delta\int_0^{x_2} x_1^{k+\delta-1}[1-F(x_1)]^{R_1+1}dx_1. \tag{10}$$

Integrating (10) by parts, now yields

$$L(x_2) = \left(\frac{\alpha\beta}{k+\beta}\right)\Bigg[x_2^{k+\beta}[1-F(x_2)]^{R_1+1} + (1+R_1)\int_0^{x_2} x_1^{k+\beta}f(x_1) \times[1-F(x_1)]^{R_1}dx_1\Bigg] + \left(\frac{\theta\delta}{k+\delta}\right)\Bigg[x_2^{k+\delta}[1-F(x_2)]^{R_1+1} +(1+R_1)\int_0^{x_2} x_1^{k+\delta}f(x_1)[1-F(x_1)]^{R_1}dx_1\Bigg].$$

Now substituting for $L(x_2)$ in (8) and simplifying the resulting expression, we derive the relation in (6).

Remark 2.1 When $R_1 = R_2 = ... = R_m = 0$ in (6), we get the recurrence relation for single moments of order statistics from additive Weibull distribution

$$\mu_{1:n}^{(k)} = \left(\frac{\alpha\beta}{k+\beta}\right)\left[(n-1)\mu_{1:n-1:n}^{(1,0,...,0)^{(k+\beta)}} + \mu_{1:n}^{(k+\beta)}\right] +\left(\frac{\theta\delta}{k+\delta}\right)\left[(n-1)\mu_{1:n-1:n}^{(1,0,...,0)^{(k+\delta)}} + \mu_{1:n}^{(k+\delta)}\right].$$

Theorem 2.2 For $2 \leq i \leq m-1$, $m \leq n$ and $k \geq 0$,

$$\mu_{i:m:n}^{(R_1,...,R_m)^{(k)}} = \left(\frac{\alpha\beta}{k+\beta}\right)\Bigg[(n-R_1-R_2-...-R_i-i)\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(k+\beta)}} -(n-R_1-R_2-...-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(k+\beta)}} + (1+R_i)\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\beta)}}\Bigg] +\left(\frac{\theta\delta}{k+\delta}\right)\Bigg[(n-R_1-R_2-...-R_i-i)\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(k+\delta)}} -(n-R_1-R_2-...-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(k+\delta)}} + (1+R_i)\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\delta)}}\Bigg]. \tag{11}$$

Proof. From (2), we have

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = A(n,m-1)\int\int\ldots\int_{0<x_1<\ldots<x_{i-1}<x_{i+1}<\ldots<x_m<\infty} I(x_{i-1},x_{i+1})$$

$$\times f(x_1)[1-F(x_1)]^{R_1}\ldots f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} f(x_{i+1})$$

$$\times[1-F(x_{i+1})]^{R_{i+1}}\ldots f(x_m)[1-F(x_m)]^{R_m} dx_1\ldots dx_m, \tag{12}$$

where

$$I(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i} dx_i. \tag{13}$$

Making use of relation in (5) and splitting the integral according with form, we have

$$I(x_{i-1},x_{i+1}) = \alpha\beta\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta-1}[1-F(x_i)]^{R_i+1}dx_i + \theta\delta\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\delta-1}$$

$$\times[1-F(x_i)]^{R_i+1}dx_i. \tag{14}$$

Integrating (14) by parts, after simplification, we find that

$$I(x_{i-1},x_{i+1}) = \left(\frac{\alpha\beta}{k+\beta}\right)\Big[x_{i+1}^{k+\beta}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\beta}[1-F(x_{i-1})]^{R_i+1}$$

$$+(1+R_i)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta} f(x_i)[1-F(x_i)]^{R_i}dx_i\Big]$$

$$+\left(\frac{\theta\delta}{k+\delta}\right)\Big[x_{i+1}^{k+\delta}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\delta}[1-F(x_{i-1})]^{R_i+1}$$

$$+(1+R_i)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\delta} f(x_i)[1-F(x_i)]^{R_i}dx_i\Big].$$

Now, substituting the above resulting expression of $I(x_{i-1},x_{i+1})$ in (12), and simplifying, it leads to (11).

Corollary 2.1 For $i=m$, $n=1,2,\ldots$ and $k\geq 0$,

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} = \left(\frac{\alpha\beta}{k+\beta}\right)\Big[-(n-R_1-R_2-\ldots-R_{m-1}-m+1)$$

$$\times\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)^{(k+\beta)}} + (1+R_m)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+\beta)}}\Big] + \left(\frac{\theta\delta}{k+\delta}\right)$$

$$\times\Big[-(n-R_1-R_2-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)^{(k+\delta)}}$$

$$+(1+R_m)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+\delta)}}\Big].$$

Remark 2.2
i) Setting $\alpha=1,\beta=1,\ \theta=0$ or $\alpha=0,\delta=1,\theta=1$ in (11), we get the recurrence relation for single moments of progressively Type-II right censored order statistics from the standard exponential distribution as obtained by Aggarwala and Balakrishnan (1996).
ii) Putting $\beta=1$ or $\delta=1$ in (11), we deduce the recurrence relation for single moments of progressively Type-II right censored order statistics from the modified Weibull distribution, as established by Mahmoud *et al.* (2017).

iii) For $\alpha = 0, \delta = 2$ or $\theta = 0$, $\beta = 2$ in (11), the result for single moments of progressively Type-II right censored order statistics is deduced for the Rayleigh distribution as obtained by Mahmoud *et al.* (2017).
iv) At $\theta = 0$ or $\alpha = 0$ in (11), we deduce the recurrence relation for single moments of progressively Type-II right censored order statistics from the Weibull distribution as established by Athar and Akhter (2016).
v) Putting $\beta = 2$, $\delta = 1$ or $\beta = 1$, $\delta = 2$ in (11), we deduce the recurrence relation for single moments of progressively Type-II right censored order statistics from the linear failure rate distribution as obtained by Mahmoud *et al.* (2017).

Remark 2.3 When $R_1 = R_2 = ... = R_m = 0$ in (11), the recurrence relation for single moments of order statistics from additive Weibull distribution is established in the form

$$\mu_{i:n}^{(k)} = \left(\frac{\alpha\beta}{k+\beta}\right)\left[(n-i)\mu_{i:n-1:n}^{(0,0,...,1,...,0)^{(k+\beta)}} - (n-i+1)\mu_{i-1:n-1:n}^{(0,0,...,1,...,0)^{(k+\beta)}} + \mu_{i:n}^{(k+\beta)}\right]$$
$$+\left(\frac{\theta\delta}{k+\delta}\right)\left[(n-i)\mu_{i:n-1:n}^{(0,0,...,1,..0)^{(k+\delta)}} - (n-i+1)\mu_{i-1:n-1:n}^{(0,0,...,1,...,0)^{(k+\delta)}} + \mu_{i:n}^{(k+\delta)}\right].$$

3. Product moments of progressively Type-II censored order statistics

In this section, we establish some recurrence relations for product moments of progressive Type-II right censored order statistics from additive Weibull distribution using the characterizing differential equation. From the joint *pdf* given in (1), we can express the product moments of the i-th and $j-$th progressively Type-II right censored order statistics as

$$\mu_{i:j:m:n}^{(R_1,...,R_m)^{(r,s)}} = E[X_{i:m:n}^{(R_1,...,R_m)^{(r)}} X_{j:m:n}^{(R_1,...,R_m)^{(s)}}]$$
$$= A(n,m-1)\int\int...\int_{0<x_1<...<x_m<\infty} x_i^r x_j^s f(x_1)[1-F(x_1)]^{R_1}$$
$$\times f(x_2)[1-F(x_2)]^{R_2}...f(x_m)[1-F(x_m)]^{R_m} dx_1...dx_m. \tag{15}$$

Theorem 3.1 For $1 \le i < j \le m-1$, $m \le n$,

$$\mu_{i,j:m:n}^{(R_1,...,R_m)^{(r,s)}} = \left(\frac{\alpha\beta}{s+\beta}\right)\Bigg[(n-R_1-R_2-...-R_j-j)$$
$$\times\mu_{i,j:m-1:n}^{(R_1,...,R_{j-1},R_j+R_{j+1}+1,...,R_m)^{(r,s+\beta)}} - (n-R_1-R_2-...-R_{j-1}-j+1)$$
$$\times\mu_{i,j-1:m-1:n}^{(R_1,...,R_{j-1}+R_j+1,R_{j+1},...,R_m)^{(r,s+\beta)}} + (1+R_j)\mu_{i,j:m:n}^{(R_1,...,R_m)^{(r,s+\beta)}}\Bigg] + \left(\frac{\theta\delta}{s+\delta}\right)$$
$$\times\Bigg[(n-R_1-R_2-...-R_j-j)\mu_{i,j:m-1:n}^{(R_1,...,R_{j-1},R_j+R_{j+1}+1,...,R_m)^{(r,s+\delta)}}$$
$$-(n-R_1-R_2-...-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,...,R_{j-1}+R_j+1,...,R_m)^{(r,s+\delta)}}$$
$$+(1+R_j)\mu_{i,j:m:n}^{(R_1,...,R_m)^{(r,s+\delta)}}\Bigg]. \tag{16}$$

Proof. From (15), we have

$$\mu_{i,j:m:n}^{(R_1,...,R_m)^{(r,s)}} = A(n,m-1)\int\int...\int_{0<x_1<...<x_m<\infty} x_i^r I(x_{j-1},x_{j+1})$$
$$\times f(x_1)[1-F(x_1)]^{R_1}...f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})$$

$$\times[1-F(x_{j+1})]^{R_{j+1}}...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m, \tag{17}$$

$$\text{where} \quad I(x_{j-1},x_{j+1})=\int_{x_{j-1}}^{x_{j+1}} x_j^s f(x_j)[1-F(x_j)]^{R_j}dx_j. \tag{18}$$

On using (5) in (18), we get

$$I(x_{j-1},x_{j+1})=\alpha\beta\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta-1}[1-F(x_j)]^{R_j+1}dx_j+\theta\delta\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\delta-1}$$
$$\times[1-F(x_j)]^{R_j+1}dx_j. \tag{19}$$

Integrating (19) by parts, we have

$$I(x_{j-1},x_{j+1})=\left(\frac{\alpha\beta}{s+\beta}\right)\Bigg[x_{j+1}^{s+\beta}[1-F(x_{j+1})]^{R_j+1}-x_{j-1}^{s+\beta}[1-F(x_{j-1})]^{R_j+1}$$
$$+(1+R_j)\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta}f(x_j)[1-F(x_j)]^{R_j}dx_j\Bigg]+\left(\frac{\theta\delta}{s+\delta}\right)$$
$$\times\Bigg[x_{j+1}^{s+\delta}[1-F(x_{j+1})]^{R_j+1}-x_{j-1}^{s+\delta}[1-F(x_{j-1})]^{R_j+1}$$
$$+(1+R_j)\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\delta}f(x_j)[1-F(x_j)]^{R_j}dx_j\Bigg]. \tag{20}$$

Putting the resulting expression of $I(x_{j-1},x_{j+1})$ given in (20) in (17), we get

$$\mu_{i,j:m:n}^{(R_1,\ldots,R_m)^{(r,s)}}=\left(\frac{\alpha\beta}{s+\beta}\right)\Bigg[A(n,m-1)\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_{j+1}^{s+\beta}f(x_1)$$
$$\times[1-F(x_1)]^{R_1}...f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}}f(x_{j+1})[1-F(x_{j+1})]^{R_j+R_{j+1}+1}...f(x_m)$$
$$\times[1-F(x_m)]^{R_m}dx_1...dx_m-A(n,m-1)\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_{j-1}^{s+\beta}$$
$$\times f(x_1)[1-F(x_1)]^{R_1}...f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}+R_j+1}f(x_{j+1})$$
$$\times[1-F(x_{j+1})]^{R_{j+1}}...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m$$
$$+(1+R_j)A(n,m-1)\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_j^{s+\beta}f(x_1)[1-F(x_1)]^{R_1}$$
$$...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m\Bigg]$$
$$+\left(\frac{\theta\delta}{s+\delta}\right)\Bigg[A(n,m-1)\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_{j+1}^{s+\delta}f(x_1)[1-F(x_1)]^{R_1}$$
$$...f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}}f(x_{j+1})[1-F(x_{j+1})]^{R_j+R_{j+1}+1}...f(x_m)$$
$$\times[1-F(x_m)]^{R_m}dx_1...dx_m-A(n,m-1)\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_{j-1}^{s+\delta}$$
$$\times f(x_1)[1-F(x_1)]^{R_1}...f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}+R_j+1}f(x_{j+1})$$
$$\times[1-F(x_{j+1})]^{R_{j+1}}...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m+(1+R_j)A(n,m-1)$$
$$\times\int\int...\int_{0<x_1<\ldots<x_m<\infty} x_i^r x_j^{s+\delta}f(x_1)[1-F(x_1)]^{R_1}...f(x_m)\times[1-F(x_m)]^{R_m}dx_1...dx_m\Bigg]. \tag{21}$$

Simplifying the resulting expression in (21) yields (16).
One can also note that Theorem 2.2 can be deduced from Theorem 3.1 by putting $r = 0$.

Remark 3.1
i) Setting $\alpha = 1, \beta = 1,\ \theta = 0$ or $\alpha = 0, \delta = 1, \theta = 1$ in (16), we get recurrence relation for product moments of progressively Type-II right censored order statistics from the standard exponential distribution as obtained by Aggarwala and Balakrishnan (1996).
ii) Putting $\beta = 1$ or $\delta = 1$ in (16), we deduce the recurrence relation for product moments of progressively Type-II right censored order statistics from the modified Weibull distribution, as established by Mahmoud *et al.* (2017).
iii) For $\alpha = 0, \delta = 2$ or $\theta = 0,\ \beta = 2$ in (16), the result for product moments of progressively Type-II right censored order statistics is deduced for the Rayleigh distribution as obtained by Mahmoud *et al.* (2017).
iv) At $\theta = 0$ or $\alpha = 0$ in (16), we deduce the recurrence relation for product moments of progressively Type-II right censored order statistics from the Weibull distribution as established by Athar and Akhter (2016).
v) Putting $\beta = 2, \delta = 1$ or $\beta = 1, \delta = 2$ in (16), we deduce the recurrence relation for product moments of progressively Type-II right censored order statistics from the linear failure rate distribution as obtained by Mahmoud *et al.* (2017).

Remark 3.2 When $R_1 = R_2 = ... = R_m = 0$ in (16), the recurrence relation for product moments of order statistics from additive Weibull distribution is established as

$$\mu_{i,j:n}^{(r,s)} = \left(\frac{\alpha\beta}{s+\beta}\right)\Bigg[(n-j)\mu_{i,j:n-1:n}^{(0,0,...,1,...,0)^{(r,s+\beta)}} - (n-j+1)\mu_{i,j-1:n-1:n}^{(0,0,...,1,...,0)^{(r,s+\beta)}}$$
$$+\mu_{i,j:n}^{(r,s+\beta)}\Bigg] + \left(\frac{\theta\delta}{s+\delta}\right)\Bigg[(n-j)\mu_{i,j:n-1:n}^{(0,0,...,1,...,0)^{(r,s+\delta)}} - (n-j+1)$$
$$\times\mu_{i,j-1:n-1:n}^{(0,0,...,1,...0)^{(r,s+\delta)}} + \mu_{i,j:n}^{(r,s+\delta)}\Bigg].$$

4. Characterizations

In this section, we introduce the characterization of the additive Weibull distribution using recurrence relation for single moments, Hazard rate function and truncated moment.

Theorem 4.1 For $2 \leq i \leq m-1$, $m \leq n$ and $k \geq 0$, a necessary and sufficient condition for a random variable X to be distributed with *pdf* given in (3) is that

$$\mu_{i:m:n}^{(R_1,...,R_m)^{(k)}} = \left(\frac{\alpha\beta}{k+\beta}\right)\Bigg[(n-R_1-R_2-...-R_i-i)\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(k+\beta)}}$$
$$-(n-R_1-R_2-...-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(k+\beta)}}$$
$$+(1+R_i)\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\beta)}}\Bigg]$$
$$+\left(\frac{\theta\delta}{k+\delta}\right)\Bigg[(n-R_1-R_2-...-R_i-i)\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(k+\delta)}}$$
$$-(n-R_1-R_2-...-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(k+\delta)}}$$
$$+(1+R_i)\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\delta)}}\Bigg]. \tag{22}$$

Proof. The necessary part follows from (11). On the other hand if the recurrence relation (22) is satisfied, then on using (2), we have

$$\mu_{i:m:n}^{(R_1,...,R_m)^{(k+a)}} = A(n,m-1)\int\int ... \int_{0<x_1<...<x_{i-1}<x_{i+1}<...<x_m<\infty} I(x_{i-1}, x_{i+1})$$

$$\times f(x_1)[1-F(x_1)]^{R_1}...f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}}f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}}$$
$$...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m, \tag{23}$$

where

$$I(x_{i-1},x_{i+1})=\int_{x_{i-1}}^{x_{i+1}} x_i^{k+a}f(x_i)[1-F(x_i)]^{R_i}dx_i. \tag{24}$$

Integrating (24) by parts, we get

$$I(x_{i-1},x_{i+1})=\frac{1}{(R_i+1)}\Big\{-x_{i+1}^{k+a}[1-F(x_{i+1})]^{R_i+1}+x_{i-1}^{k+a}[1-F(x_{i-1})]^{R_i+1}$$
$$+(k+a)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+a-1}[1-F(x_i)]^{R_i+1}dx_i\Big\}. \tag{25}$$

Upon substituting (25) in (23), and simplifying the resulting expression, we get

$$\mu_{i:m:n}^{(R_1,...,R_m)^{(k+a)}}=\frac{1}{R_i+1}\Big\{-(n-R_1-R_2-...-R_i-i)$$
$$\times\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(k+a)}}+(n-R_1-R_2-...-R_{i-1}-i+1)$$
$$\times\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(k+a)}}+(k+a)A(n,m-1)$$
$$\times\int\int...\int_{0<x_1<...<x_m<\infty} x_i^{k+a-1}f(x_1)[1-F(x_1)]^{R_1}$$
$$...f(x_i)[1-F(x_i)]^{R_i+1}...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m\Big\}. \tag{26}$$

Now substituting for $\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\beta)}}$ and $\mu_{i:m:n}^{(R_1,...,R_m)^{(k+\delta)}}$ in (22) and simplifying the resulting expression, we get

$$A(n,m-1)\int\int...\int_{0<x_1<...<x_m<\infty} x_i^k\{f(x_i)-(\alpha\beta x_i^{\beta-1}+\theta\delta x_i^{\delta-1})[1-F(x_i)]\}$$
$$\times f(x_1)[1-F(x_1)]^{R_1}...f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}}f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}}$$
$$\times...f(x_m)[1-F(x_m)]^{R_m}dx_1...dx_m=0. \tag{27}$$

Now applying a generalization of the M*ü*ntz-Sz*á*sz Theorem (see for example Hwang and Lin (1984)) to (27), we obtain

$$f(x_i)=(\alpha\beta x_i^{\beta-1}+\theta\delta x_i^{\delta-1})[1-F(x_i)]$$

which proves the result.

Remark 4.1 On using (3) and (4), we can characterize this distribution through hazard function.

Following theorem contains characterization of this distribution based on truncated moment.

Theorem 4.2 Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the *df* $F(x)$ and *pdf* $f(x)$ for $0<x<\infty$, such that $f'(x)$ and $E(X|X\le x)$ exist for all x, $0<x<\infty$, then

$$E(X|X\le x)=g(x)\eta(x), \tag{28}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = -\frac{x}{(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})} + \frac{e^{(\alpha x^{\beta}+\theta x^{\delta})}}{(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})} \int_0^x e^{-(\alpha u^{\beta}+\theta u^{\delta})} du.$$

if and only if

$$f(x) = (\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1}) e^{-(\alpha x^{\beta}+\theta x^{\delta})}, \quad x \geq 0,\ \alpha,\ \beta,\ \theta,\ \delta > 0.$$

Proof. From (3), we have

$$\begin{aligned} E(X|X \leq x) &= \frac{1}{F(x)} \int_0^x u f(u) du \\ &= \frac{1}{F(x)} \int_0^x u(\alpha\beta u^{\beta-1} + \theta\delta u^{\delta-1}) e^{-(\alpha u^{\beta}+\theta u^{\delta})}\, du. \end{aligned} \tag{29}$$

Integrating by parts, taking '$(\alpha\beta u^{\beta-1} + \theta\delta u^{\delta-1}) e^{-(\alpha u^{\beta}+\theta u^{\delta})}$' as the part to be integrated and the rest of the integrand for differentiation, we get

$$E(X|X \leq x) = \frac{1}{F(x)} \bigg\{ - x e^{-(\alpha x^{\beta}+\theta x^{\delta})} \} + \int_0^x e^{-(\alpha u^{\beta}+\theta u^{\delta})} du \bigg\}. \tag{30}$$

Multiplying and dividing by $f(x)$ in (30), we have the result given in (28).
To prove sufficient part, we have from (28)

$$\frac{1}{F(x)} \int_0^x u f(u) du = \frac{g(x) f(x)}{F(x)} \tag{31}$$

or

$$\int_0^x u f(u) du = g(x) f(x).$$

Differentiating (31) on both sides with respect to x, we find that

$$x f(x) = g'(x) f(x) + g(x) f'(x)$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} \qquad \text{[Ahsanullah } et\ al. \text{ (2016)]}$$

$$= -(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1}) + \frac{(\alpha\beta(\beta-1) x^{\beta-2} + \theta\delta(\delta-1) x^{\delta-2})}{(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})}, \tag{32}$$

where

$$g'(x) = x + g(x) \left((\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1}) - \frac{(\alpha\beta(\beta-1) x^{\beta-2} + \theta\delta(\delta-1) x^{\delta-2})}{(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})} \right).$$

Integrating both the sides in (32) with respect to x, we have

$$f(x) = c(\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\theta x^{\delta})}.$$

It is known that

$$\int_0^\infty f(x)dx = 1$$

Thus,

$$\frac{1}{c} = \int_0^\infty (\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\theta x^{\delta})}dx = 1$$

which proves that

$$f(x) = (\alpha\beta x^{\beta-1} + \theta\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\theta x^{\delta})}, \quad x \geq 0, \alpha, \beta, \theta, \delta > 0.$$

5. Conclusion

In this paper, we have established several recurrence relations satisfied by single and product moments of progressively Type –II right censored order statistics from additive Weibull distribution in a simple recursive manner. These relations enable us to compute all means and variances for different censoring schemes. Further, using a recurrence relation for single moments we obtain a characterization of this distribution, which play an integral role for identification of population distribution from the properties of sample.

6. Acknowledgement

The authors would like to thank two anonymous referees and the Editor-in-Chief for their fruitful suggestions and comments which led to an improvement in the manuscript significantly.

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