ISSN 0974-3235

ProbStat Forum, Volume 03, July 2010, Pages 78-90

Limit distributions of the extremes of a random number of random variables in a stationary Gaussian sequence

R.Vasudeva

Alireza Yousefi Moridani

Department of Statistics, University of Mysore

Mysore, India - 570006

e-mail: *vasudeva.rasbagh@gmail.com*, *moridani@gmail.com*

Paper received on 21 october 2009; revised, 07 April 2010; accepted, 10 June 2010.

Abstract. This paper contains some results on the limit distribution of s^{th} maxima of a stationary Gaussian sequence under equi-correlated set up, when the sample size is itself a random variable.

Mathematics Subject Classification. 60G70, 62G30.

Keywords. Order statistics, Gaussian sequences, Limit distributions.

1 Introduction

Let $\{X_{n,k}, k = 1, 2, \ldots, n\}$, $n \geq 1$, be a sequence of triangular arrays of random variables (r.v.) with $EX_{n,k} = 0$, $EX_{n,k}^2 = 1$, $1 \leq k \leq n$, $n \geq 1$ and $EX_{n,k}X_{n,j} = r_{(n)}$, $k, j = 1, 2, \ldots, n$, $k \neq j$, $n \geq 1$ $(0 \leq r_{(n)} < 1)$.

Suppose that $(X_{n,1}, X_{n,2}, \ldots, X_{n,n})$ is n-variate Gaussian. Then $\{X_{n,k}, k = 1, 2, \ldots, n\}$, $n \geq 1$, is a triangular array of equi-correlated stationary Gaussian (E.C.S.G.) sequence. For such a sequence define M_n =max$(X_{n,1}, X_{n,2}, \ldots, X_{n,n})$, $n \geq 1$. Berman (1962) obtained the limit distribution of (M_n), properly normalized, by giving a representation for $X_{n,k}$, $1 \leq k \leq n$, $n \geq 1$, in terms of an i.i.d. sequence of standard normal r.v.'s. Pickands (1962), Mittal and Ylvisaker (1975), McCormick (1980) and Leadbetter et al. (1983) have established limit theorems for (M_n), assuming various rates of convergence of correlation coefficient. Galambos (1978) has studied the limiting behaviour of (M_n), over random stopping time N_n, under the condition that $\frac{N_n}{n} \xrightarrow{p} \tau$, where τ is a positive valued r.v. ($\xrightarrow{p}$, stands for convergence in probability).

In this paper, we assume that (N_n) is a sequence of integer valued r.v.'s with $P(N_n = k) = p_{n,k}$, $k = m, m+1, \ldots$, $n \geq 1$. When $N_n = k$, we suppose that $(X_{N_n,1}, X_{N_n,2}, \ldots, X_{N_n,N_n})$ reduces to $(X_{k,1}, X_{k,2}, \ldots, X_{k,k})$, a k-dimensional Gaussian random vector, with 0 means, unit variances and common covariance $r_{(k)}$. We define, $M_{s,n}$ as the s^{th} highest among $(X_{n,1}, X_{n,2}, \ldots, X_{n,n})$, $n \geq 1$, $1 \leq s \leq m$, and call it as the s^{th} maxima. Note that $M_{s,n}$ is the s^{th} upper extreme of $X_{n,1}, X_{n,2}, \ldots, X_{n,n}, n \geq 1$. In turn, for $1 \leq s \leq m$, M_{s,N_n} can be considered as the s^{th} highest among $(X_{N_n,1}, X_{N_n,2}, \ldots, X_{N_n,N_n})$. From the definition of N_n, note that M_{s,N_n} is a well defined r.v. for $1 \leq s \leq m$.

Throughout the paper, we assume that $\{X_{n,k}, 1 \leq k \leq n\}$, $n \geq 1$, and (N_n) are mutually independent and that $(\frac{N_n}{n})$ converges in distribution to a proper r.v.. Under this setup, in Section 2, we obtain the limit distribution of (M_{s,N_n}), properly normalized. This is achieved through Berman's representation described below.

Let $(Y_n, n \geq 0)$ be a sequence of i.i.d. standard normal r.v.'s. Then Berman (1962) observed that $X_{n,k} \stackrel{d}{=} r_{(n)}^{\frac{1}{2}} Y_0 + (1 - r_{(n)})^{\frac{1}{2}} Y_k$, $1 \leq k \leq n$, $n \geq 1$, which can be easily verified (here, $\stackrel{d}{=}$ means, distributionally same). Define $M^*_{s,n}$ as the s^{th} highest among $(Y_1, Y_2, \ldots, Y_n)$, $n \geq 1$, $s \geq 1$, so that M^*_{s,N_n} is the s^{th} maxima of $(Y_1, Y_2, \ldots, Y_{N_n})$, $n \geq 1$, $1 \leq s \leq m$. One can easily see that $M_{s,n} \stackrel{d}{=} r_{(n)}^{\frac{1}{2}} Y_0 + (1 - r_{(n)})^{\frac{1}{2}} M^*_{s,n}$, $n \geq 1$, $s \geq 1$. Using the above representation for E.C.S.G. sequences, the limit distribution for $(M_{1,n})$, properly normalized, (see, Theorem A below) has been established, see eg. Galambos (1978) or Leadbetter et al. (1983).

Theorem A: (Theorem 3.8.1, Galambos (1978))

Given an E.C.S.G $\{X_{n,k}, 1 \leq k \leq n\}$, $n \geq 1$, with $M_{1,n} = \max_{1\leq k\leq n} X_{n,k}$, one can find constants

$$b_n = (2\ln n)^{-\frac{1}{2}} \quad \& \quad a_n = \frac{1}{b_n} - \frac{b_n}{2}(\ln\ln n + \ln 4\pi), \quad such\ \ that$$

$$\begin{array}{ll} (i) & \lim P(\frac{M_{1,n} - a_n}{b_n} \leq x) = H(x) \quad if \quad r_{(n)} \ln n \to 0 \\ (ii) & \lim P(\frac{M_{1,n} - a_n}{b_n} \leq x) = \int_{-\infty}^{\infty} \Phi(\frac{x-y}{\sqrt{2\theta}}) dH(y) \quad if \quad r_{(n)} \ln n \to \theta, \\ (iii) & \lim P(\frac{M_{1,n} - (1-r_{(n)})^{\frac{1}{2}} a_n}{r_{(n)}^{\frac{1}{2}}} \leq x) = \Phi(x) \quad if \quad r_{(n)} \ln n \to \infty, \end{array}$$

where $H(x) = e^{-e^{-x}}$, $-\infty < x < \infty$, is the Gumbel d.f., $0 < \theta < \infty$, and $\Phi(x)$,$-\infty < x < \infty$, is the standard normal d.f. .

In Section 3, we show through some examples, that the conditions of Theorem 2.1 are non vacuous. In the last section, we deduce the limit distribution of (M_{s,N_n}), when N_n is a geometric r.v.. It is of interest to know that a good amount of work has been done in the study of partial sums (S_{N_n}), where (N_n) is a sequence of geometric r.v.'s, starting from the pioneering results of Gnedenko (1983) and of Klebanov et al. (1985). In fact, Gnedenko (1983) also mentions about the limit distribution of (M_{N_n}), when N_n is a geometric r.v.. In the study of $GI/G/1$ queues, Szelki (1986) observed that the number of customers in the waiting line is a r.v. having geometric distribution. Here, the maximal service time correspond to the partial maxima of a geometric number of r.v.'s and it plays an important role in the study of such queueing systems. As such, the last section in devoted for the study of extremes, when N_n is geometric.

2 Main Results

Recall that $\{X_{n,k}, 1 \leq k \leq n\}$ is a Gaussian vector with zero means, unit variances and common covariance $r_{(n)}$, $n \geq 1$ and that $\{Y_n\}$, $n \geq 0$, is a sequence of i.i.d.

standard normal r.v.'s. Define $\xi_{n,k} = r_{(n)}^{\frac{1}{2}} Y_0 + (1 - r_{(n)})^{\frac{1}{2}} Y_k$, $k = 1, 2, \dots, n$, $n \geq 1$. Then we have the following lemma.

Lemma 2.1 If the sequence (N_n) of r.v's is independent of $\{X_{n,1}, X_{n,2}, \dots, X_{n,n}\}$, $n \geq 1$, and $\{Y_n\}$, $n \geq 0$, then $\{X_{N_n,1}, X_{N_n,2}, \dots, X_{N_n,N_n}\} \stackrel{d}{=} \{\xi_{N_n,1}, \xi_{N_n,2}, \dots, \xi_{N_n,N_n}\}$.

Proof: We show that the two characteristic functions (ch.f.) are equal and hence prove the lemma.

The ch.f. of $(\xi_{N_n,1}, \xi_{N_n,2}, \dots, \xi_{N_n,N_n})$ is

$$\begin{aligned}
Ee^{i\sum_{j=1}^{N_n} t_j \xi_{N_n,j}} &= \sum_{k=m}^{\infty} Ee^{i\sum_{j=1}^{k} t_j \xi_{k,j}} P(N_n = k) \\
&= \sum_{k=m}^{\infty} Ee^{i\sum_{j=1}^{k} t_j \left(r_{(k)}^{\frac{1}{2}} Y_0 + (1-r_{(k)})^{\frac{1}{2}} Y_j\right)} P(N_n = k) \\
&= \sum_{k=m}^{\infty} e^{-\frac{r_{(k)}}{2}(\sum_{j=1}^{k} t_j)^2} e^{-\frac{(1-r_{(k)})}{2}(\sum_{j=1}^{k} t_j^2)} P(N_n = k) \\
&= \sum_{k=m}^{\infty} e^{-\frac{1}{2}(\sum_{j=1}^{k} t_j^2 + r_{(k)} \sum_{j,l=1, j\neq l}^{k} t_j t_l)} P(N_n = k) \qquad (2.1)
\end{aligned}$$

Similarly, the ch.f. of $(X_{N_n,1}, X_{N_n,2}, \dots, X_{N_n,N_n})$ is

$$Ee^{i\sum_{j=1}^{N_n} t_j X_{N_n,j}} = \sum_{k=m}^{\infty} Ee^{i\sum_{j=1}^{k} t_j X_{k,j}} P(N_n = k)$$

Recalling that $(X_{k,1}, X_{k,2}, \dots, X_{k,k})$ is k-variate Gaussian vector with zero means unit variances and common covariance $r_{(k)}$, one gets,

$$Ee^{i\sum_{j=1}^{N_n} t_j X_{N_n,j}} = \sum_{k=m}^{\infty} e^{-\frac{1}{2}(\sum_{j=1}^{k} t_j^2 + r_{(k)} \sum_{j,l=1, j\neq l}^{k} t_j t_l)} P(N_n = k) \qquad (2.2)$$

(2.1) and (2.2) complete the proof.

In the next lemma, we obtain the limit distribution of (M^*_{s,N_n}), properly normalized.

Lemma 2.2 Let $b_n = (2 \ln n)^{-\frac{1}{2}}$ & $a_n = \frac{1}{b_n} - \frac{b_n}{2}(\ln \ln n + \ln 4\pi)$.

If $\lim P(N_n \leq xn) = A(x)$, $x \in R$, where A is a d.f. with $A(0+) = 0$, then $\lim P(M^*_{s,N_n} \leq a_n + b_n x) = G^{(s)}(x)$, $x \in R$, where

$$G^{(s)}(x) = \sum_{j=0}^{s-1} \int_0^\infty e^{-ze^{-x}} \frac{(ze^{-x})^j}{j!} dA(z).$$

Proof: Note that M^*_{s,N_n} is the s^{th} maxima of $(Y_1, Y_2, \ldots, Y_{N_n})$, where (Y_n) is a sequence of i.i.d. standard normal r.v.'s. From the fact that $\frac{M^*_{1,n} - a_n}{b_n}$ converges to a Gumbel law, by the univariate version of Theorem 2.1 of Barakat (1997) one can show that $\lim P(M^*_{s,N_n} \leq a_n + b_n x) = G^{(s)}(x)$, $x \in R$. The details are omitted.

Lemma 2.3 Let (S_n) be a sequence of r.v.'s and (C_n) and (D_n), D_n positive, be sequences of real constants such that $\lim P(S_n \leq C_n + D_n x) = F(x)$, at all continuity points of F(.). Let (C^*_n) and (D^*_n) be any two sequences of r.v.'s such that $\frac{C^*_n - C_n}{D_n} \xrightarrow{p} \lambda$ and $\frac{D_n}{D^*_n} \xrightarrow{p} 1$, where λ is some real constant. Then $\lim P(S_n < C^*_n + D^*_n x) = F(x + \lambda)$, at all continuity points of F(.).

Proof: Note that

$$\frac{S_n - C^*_n}{D^*_n} \stackrel{d}{=} \frac{D_n}{D^*_n}\Big(\frac{S_n - C_n}{D_n} - \frac{C^*_n - C_n}{D_n}\Big) \tag{2.3}$$

$\frac{S_n - C_n}{D_n} \xrightarrow{d} X$, a r.v. with d.f. F(.), and $\frac{C^*_n - C_n}{D_n} \xrightarrow{p} \lambda$ implies (by Slutsky's theorem) $\frac{S_n - C^*_n}{D_n} \xrightarrow{d} X - \lambda$. Further, $\frac{D_n}{D^*_n} \xrightarrow{p} 1$ implies that $\frac{S_n - C^*_n}{D^*_n} \xrightarrow{d} X - \lambda$ or equivalently that $\lim P(S_n < C^*_n + D^*_n x) = F(x + \lambda)$, at all continuity points of F(.).

Lemma 2.4 Let (S_n, Q_n) be a sequence of random vectors such that

$$\lim P(S_n \leq s, Q_n \leq q) = F(s)E(q), \ \ -\infty < s, q < \infty,$$

where F(.) and E(.) are continuous d.f.s. Then for any $x \in R$,

$$\lim P(S_n + Q_n \leq x) = \int_{-\infty}^{\infty} E(x - y) dF(y).$$

Proof: For proof, see, Lemma 2.9.1, Galambos (1978).

We now move on to the main result of this paper. Recall that M_{s,N_n} is the s^{th} maxima of $(X_{N_n,1}, X_{N_n,2}, \ldots, X_{N_n,N_n})$, $1 \leq s \leq m$.

Theorem 2.1 Let $\lim P(N_n \leq xn) = A(x)$, $x \in R$, where $A(.)$ is a d.f. with $A(0+) = 0$. Then for $b_n = (2\ln n)^{-\frac{1}{2}}$ and $a_n = \frac{1}{b_n} - \frac{b_n}{2}(\ln\ln n + \ln 4\pi)$

$$(i) \qquad P(M_{s,N_n} \leq a_n + b_n x) \to G^{(s)}(x), x \in R, \ if \quad r_{(N_n)} \ln n \xrightarrow{p} 0$$

$$(ii) \qquad P(M_{s,N_n} \leq a_n + b_n x) \to \int_{-\infty}^{\infty} \Phi(\frac{x-y}{\sqrt{2\theta}}) dG^{(s)}(y), x \in R, \ if$$

$r_{(N_n)} \ln n \xrightarrow{p} \theta$, *where* $0 < \theta < \infty$ *is a constant.*

$$(iii) \qquad P(M_{s,N_n} \leq (1 - r_{(N_n)})^{\frac{1}{2}} a_n + r^{\frac{1}{2}}_{(N_n)} x) \to \Phi(x), \quad x \in R, \ if$$

$r_{(N_n)} \ln n \xrightarrow{p} \infty$.

Where $G^{(s)}(x) = \sum_{j=0}^{s-1} \int_0^{\infty} e^{-ze^{-x}} \frac{(ze^{-x})^j}{j!} dA(z)$ and $\Phi(.)$ is the standard normal d.f.

Proof: By Lemma 2.1, note that

$$M_{s,N_n} \stackrel{d}{=} r^{\frac{1}{2}}_{(N_n)} Y_0 + (1 - r_{(N_n)})^{\frac{1}{2}} M^*_{s,N_n}$$

Define

$$\frac{M_{s,N_n} - a_n}{b_n} = U_{N_n} + V_{N_n} \tag{2.4}$$

where $U_{N_n} = (2r_{(N_n)} \ln n)^{\frac{1}{2}} Y_0$ and $V_{N_n} = (1 - r_{(N_n)})^{\frac{1}{2}} \frac{M^*_{s,N_n} - (1 - r_{(N_n)})^{-\frac{1}{2}} a_n}{b_n}$. Let $\pi_n = 2r_{(N_n)} \ln n$ and $W_n = Y_0$, $n \geq 1$. Suppose that $r_{(N_n)} \ln n \xrightarrow{p} 0$. Then $\pi_n \xrightarrow{p} 0$, $W_n \xrightarrow{p} Y_0$, imply that

$$U_{N_n} = (2r_{(N_n)} \ln n)^{\frac{1}{2}} Y_0 \xrightarrow{p} 0 \tag{2.5}$$

Let $a^*_n = (1 - r_{(N_n)})^{-\frac{1}{2}} a_n$. Since $r_{(N_n)} \ln n \xrightarrow{p} 0$ as $n \to \infty$, we have $\frac{a^*_n - a_n}{b_n} \xrightarrow{p} 0$. Using the facts that $\frac{M^*_{s,N_n} - a_n}{b_n} \xrightarrow{d} Y^* \sim G^{(s)}(.)$, $\frac{a^*_n - a_n}{b_n} \xrightarrow{p} 0$ and $(1 - r_{(N_n)})^{\frac{1}{2}} \xrightarrow{p} 1$, one gets from Lemma 2.3, $V_{N_n} \xrightarrow{d} Y^*$. Along with (2.4) and (2.5), we have

$$\lim P(M_{s,N_n} < a_n + b_n x) = G^{(s)}(x), \quad -\infty < x < \infty. \tag{2.6}$$

Consider the case, $r_{(N_n)} \ln n \xrightarrow{p} \theta$, $0 < \theta < \infty$. Define $W_n = Y_0$, $n \geq 1$. Note that $r_{(N_n)} \ln n \xrightarrow{p} \theta$, $W_n \xrightarrow{p} Y_0$ imply that $U_{N_n} \xrightarrow{p} \sqrt{2\theta} Y_0$. With a^*_n as defined above, we show that $\frac{a^*_n - a_n}{b_n} \xrightarrow{p} \theta$. For any given $\epsilon > 0$, we show that

$$\lim P\Big(|\frac{a^*_n - a_n}{b_n} - \theta| < \epsilon\Big) = 1.$$

Note that $r_{(N_n)} \ln n \xrightarrow{p} \theta$, as $n \to \infty$, implies that

$$\begin{aligned}
& \lim P\Big(|r_{(N_n)} \ln n - \theta| < \epsilon\Big) = 1 \\
\Leftrightarrow\ & \lim P\Big((1-\frac{\theta+\epsilon}{\ln n})^{\frac{1}{2}} < (1-r_{(N_n)})^{\frac{1}{2}} < (1-\frac{\theta-\epsilon}{\ln n})^{\frac{1}{2}}\Big) = 1 \\
\Leftrightarrow\ & \lim P\Big(\frac{a_n}{b_n}\big((1-\frac{\theta-\epsilon}{\ln n})^{-\frac{1}{2}}-1\big) < \frac{a_n^*-a_n}{b_n} < \frac{a_n}{b_n}\big((1-\frac{\theta+\epsilon}{\ln n})^{-\frac{1}{2}}-1\big)\Big) = 1 \quad (2.7)
\end{aligned}$$

Expanding $\big(1-\frac{\theta-\epsilon}{\ln n}\big)^{-\frac{1}{2}}$ up to second term, one can observe that for some $c > 0$ and for n large,

$$\begin{aligned}
\frac{a_n}{b_n}\big((1-\frac{\theta-\epsilon}{\ln n})^{-\frac{1}{2}}-1\big) &\simeq 2\ln n\Big((1+\frac{1}{2}\frac{\theta-\epsilon}{\ln n}+\frac{c}{(\ln n)^2})-1\Big) \\
&\simeq \theta-\epsilon+\frac{2c}{\ln n} > \theta-2\epsilon.
\end{aligned}$$

Similarly, for n large, one can show that

$$\frac{a_n}{b_n}\big((1-\frac{\theta+\epsilon}{\ln n})^{-\frac{1}{2}}-1\big) \quad < \quad \theta+2\epsilon.$$

In turn, (2.7) implies that

$$\lim P\Big(\theta-2\epsilon < \frac{a_n^*-a_n}{b_n} < \theta+2\epsilon\Big) = 1$$

or that $\frac{a_n^*-a_n}{b_n} \xrightarrow{p} \theta$, as $n \to \infty$. Using the information that $\frac{M^*_{s,N_n}-a_n(1-r_{(N_n)})^{-\frac{1}{2}}}{b_n} \xrightarrow{d} Y^*$ and $(1-r_{(N_n)})^{\frac{1}{2}} \xrightarrow{p} 1$, by Lemma 2.3, we note that $P(V_{N_n} \le x) = G^{(s)}(\theta+x)$, $x \in R$. Applying Lemma 2.4, one can now show that for any $x \in R$,

$$\lim P(\frac{M_{s,N_n}-a_n}{b_n} \le x) = \int_{-\infty}^{\infty} \Phi(\frac{x-y}{\sqrt{2\theta}})dG^{(s)}(y)$$

Now consider the case, $r_{(N_n)} \ln n \xrightarrow{p} \infty$, as $n \to \infty$. We show that

$$\eta_n = \frac{M_{s,N_n} - (1-r_{(N_n)})^{\frac{1}{2}} a_n}{r_{(N_n)}^{\frac{1}{2}}}$$

Converges to a normal r.v. as $n \to \infty$. Note that,

$$\begin{aligned}
\eta_n &= Y_0 + \frac{(1-r_{(N_n)})^{\frac{1}{2}}}{r_{(N_n)}^{\frac{1}{2}}}(M^*_{s,N_n}-a_n) \\
&= Y_0 + T_{N_n}.
\end{aligned}$$

We complete the proof by showing that $T_{N_n} \stackrel{p}{\to} 0$, as $n \to \infty$. We have

$$T_{N_n} \;=\; \frac{(1 - r_{(N_n)})^{\frac{1}{2}} b_n}{r_{(N_n)}^{\frac{1}{2}}} \frac{(M^*_{s,N_n} - a_n)}{b_n}$$

But,

$$\frac{(1 - r_{(N_n)})^{\frac{1}{2}} b_n}{r_{(N_n)}^{\frac{1}{2}}} = \frac{(1 - r_{(N_n)})^{\frac{1}{2}}}{\sqrt{2 r_{(N_n)} \ln n}} \stackrel{p}{\longrightarrow} 0, \quad as\ n \to \infty,$$

since $r_{(N_n)} \ln n \stackrel{p}{\to} \infty$. Also $\frac{M^*_{s,N_n} - a_n}{b_n} \stackrel{d}{\to} Y^* \sim G^{(s)}(.)$. Slutsky's theorem implies that, $T_{N_n} \stackrel{p}{\to} 0$ as $n \to \infty$, and the proof is complete.

Remark 2.1 If N'_n is a Poisson r.v. with mean n, then identifying N'_n as a sum of n i.i.d. Poisson r.v.'s with unit mean, by strong law of large numbers. We note that $\frac{N'_n}{n} \to 1$ almost surely. Taking N_n in Theorem 2.1 as $N_n = N'_n + m$ (shifted Poisson distribution), we see that $\frac{N_n}{n} \to 1$ almost surely. In this case, Lemma 2.2 yields

$$\begin{aligned} \lim P(M^*_{s,N_n} \le a_n + b_n x) &= \lim P(M^*_{s,n} \le a_n + b_n x) \\ &= \sum_{j=0}^{s-1} e^{-e^{-x}} \frac{(e^{-x})^j}{j!} = H_s(x),\ say, -\infty < x < \infty. \end{aligned}$$

which is the limit distribution of the s^{th} maxima, $M_{s,n}$ (non-random). Consequently, Theorem 2.1 gives

$$\begin{aligned} &\lim P(M_{s,n} \le a_n + b_n x) = H_s(x), \quad x \in R,\ if \quad r_n \ln n \to 0 \\ &\lim P(M_{s,n} \le a_n + b_n x) = \int_{-\infty}^{\infty} \Phi(\frac{x-y}{\sqrt{2\theta}}) dH_s(y), \quad x \in R,\ if \quad r_n \ln n \to \theta \\ &\lim P(M_{s,n} \le (1 - r_{(n)})^{\frac{1}{2}} a_n + r_{(n)}^{\frac{1}{2}} x) = \Phi(x), \quad x \in R,\ if \quad r_n \ln n \to \infty. \end{aligned}$$

Similarly, if N'_n is Binomial $(n^2, \frac{1}{n})$, one can show that N'_n is the sum of n i.i.d. Binomial $(n, \frac{1}{n})$ r.v.'s. By strong law, one gets, $\frac{N'_n}{n} \to 1$ almost surely. Defining $N_n = N'_n + m$, one can precisely get the results deduced above (under the setup of Poisson distribution). When N'_n is a geometric r.v. with $P(N'_n = k) = \frac{1}{n}(1 - \frac{1}{n})^k$, $k = 0, 1, \ldots$, the form of limit distribution are given in the last section.

3 Existence of sequences $\left(r_{(N_n)}\right)$ for validity of the main result

In this section, we present examples of sequences $(r_{(n)})$, which satisfy

$$r_{(N_n)} \ln n \xrightarrow{p} 0;\ r_{(N_n)} \ln n \xrightarrow{p} \theta,\ 0 < \theta < \infty \text{ and } r_{(N_n)} \ln n \xrightarrow{p} \infty.$$

One may recall that in Theorem 2.1 above, the limit distribution of (M_{s,N_n}), normalized, have been obtained under these conditions.

Example 3.1 *Let $r_{(n)} = \frac{1}{n^\alpha}$, $n \geq 2$, $\alpha > 0$. We show that $r_{(N_n)} \ln n \xrightarrow{p} 0$, as $n \to \infty$. For any given $\epsilon > 0$, we have*

$$\begin{aligned} P(r_{(N_n)} \ln n > \epsilon) &= P(r_{(N_n)} > \frac{\epsilon}{\ln n}) = P\Big(N_n < (\frac{\ln n}{\epsilon})^{\frac{1}{\alpha}}\Big) \\ &= P\Big(\frac{N_n}{n} < \frac{1}{n}(\frac{\ln n}{\epsilon})^{\frac{1}{\alpha}}\Big) \end{aligned}$$

Given any $\delta > 0$, but small, one can find a $n_0 > 0$ such that $\frac{(\ln n)^{\frac{1}{\alpha}}}{\epsilon^{\frac{1}{\alpha}} n} < \delta$ for all $n \geq n_0$. Consequently, for all $n \geq n_0$

$$P(r_{(N_n)} \ln n > \epsilon) \leq P(\frac{N_n}{n} \leq \delta)$$

In turn,

$$\limsup P(r_{(N_n)} \ln n > \epsilon) \leq A(\delta),$$

where $A(.)$ is the limit distribution of $(\frac{N_n}{n})$. Since δ is arbitrary, as $\delta \to 0$. One gets

$$\limsup P(r_{(N_n)} \ln n > \epsilon) \leq A(0+), \tag{3.1}$$

From Theorem 2.1, note that $A(0+) = 0$. Consequently, (3.1) *implies that*

$$r_{(N_n)} \ln n \xrightarrow{p} 0$$

Example 3.2 *Let $r_{(n)} = \frac{\theta}{\ln n}$, $n \geq 2$, $0 < \theta < \infty$. We show that for any $\epsilon > 0$,*

$$\lim P(|r_{(N_n)} \ln n - \theta| < \epsilon) = 1,$$

which is equivalent to $r_{(N_n)} \ln n \xrightarrow{p} \theta$. *The event*

$$\begin{aligned}\big(|r_{(N_n)} \ln n - \theta| < \epsilon\big) \quad &\Leftrightarrow \quad \big(\theta - \epsilon < \frac{\theta \ln n}{\ln N_n} < \theta + \epsilon\big) \\ &\Leftrightarrow \quad \big(\frac{\theta}{\theta + \epsilon} \ln n < \ln N_n < \frac{\theta}{\theta - \epsilon} \ln n\big)\end{aligned}$$

Consequently, one can find a $\delta > 0$, *such that*

$$\big(|r_{(N_n)} \ln n - \theta| < \epsilon\big) \supseteq \big(n^{(1-\delta)} \leq N_n \leq n^{(1+\delta)}\big)$$

In turn,

$$p\big(|r_{(N_n)} \ln n - \theta| < \epsilon\big) \geq P\big(n^{-\delta} \leq \frac{N_n}{n} \leq n^{\delta}\big) \tag{3.2}$$

For any given $\epsilon_1 > 0$, *but small and* $M > 0$, *but larger, one can find a* $n_1 > 0$, *such that* $n^{-\delta} < \epsilon_1$ *and* $n^{\delta} > M$ *for all* $n \geq n_1$. *Hence, for* $n \geq n_1$,

$$P\big(n^{-\delta} \leq \frac{N_n}{n} \leq n^{\delta}\big) \geq P\big(\epsilon_1 \leq \frac{N_n}{n} \leq M\big),$$

which implies that

$$\lim P\big(n^{-\delta} \leq \frac{N_n}{n} \leq n^{\delta}\big) \geq A(M) - A(\epsilon_1),$$

Taking $\epsilon_1 \to 0$, $M \to \infty$ *and using the fact that* $A(0+) = 0$, *one gets*

$$\lim P\big(n^{-\delta} \leq \frac{N_n}{n} \leq n^{\delta}\big) = 1,$$

which along with (3.2) *yields, the required result.*

Example 3.3 *Take* $r_{(n)} = \rho$, $n \geq 2$, $0 < \rho < 1$. *Then note that* $r_{(N_n)}$ *is degenerate at* ρ. *Consequently,* $r_{(N_n)} \ln n \xrightarrow{p} \infty$ *as* $n \to \infty$.

4 Limit distribution of (M_{s,N_n}), when N_n is a geometric r.v.

In the study of partial sums and partial maxima of random number of r.v.'s, considerable work has been done, in particular, when (N_n) is a sequence of geometric r.v.'s, as mentioned in the introductory section. In this section, we obtain the limit distribution of (M_{s,N_n}) when N_n has the p.m.f. $P(N_n = k) = p_n q_n^{k-m}$, $k = m, m+1, \ldots$; $p_n = \frac{1}{n}$, $n \geq 2$. Let N_n' be a r.v. with p.m.f. $P(N_n' = k) = p_n q_n^k$, $k = 0, 1, 2, \ldots$; $p_n = \frac{1}{n}$, $n \geq 2$. Note that $N_n = N_n' + m$ and that $\left(\frac{N_n'}{n}\right)$ converges to a unit exponential r.v.. Consequently, $\left(\frac{N_n}{n}\right)$ also convergence to a unit exponential r.v. and as such, in Theorem 2.1, $A(z) = 1 - e^{-z}$, $z > 0$. From Lemma 2.2, we hence get

$$\lim P\big(M^*_{s,N_n} \leq a_n + b_n x\big) = e^x \sum_{j=1}^{s} \frac{1}{(1+e^x)^j} = G^{(s)}(x), \quad -\infty < x < \infty.$$

It is interesting to note that the random maxima, properly normalized, i.e. (M^*_{1,N_n}), converges to $G^{(1)}(x) = \frac{e^x}{1+e^x}$, $-\infty < x < \infty$, which is the logistic distribution. Also, Theorem 2.1 yields

$$\begin{aligned} \lim P(M_{s,N_n} \leq a_n + b_n x) &= G^{(s)}(x), \quad if\ r_{(N_n)} \ln n \xrightarrow{p} 0, \\ &= \sum_{j=1}^{s} \int_{-\infty}^{\infty} \Phi\Big(\frac{x-y}{\sqrt{2\theta}}\Big) d\Big(\frac{e^y}{(1+e^y)^j}\Big) \quad if \\ r_{(N_n)} \ln n \xrightarrow{p} \theta,\ 0 < \theta < \infty \quad and & \\ &= \Phi(x) \quad if\ r_{(N_n)} \ln n \xrightarrow{p} \infty. \end{aligned}$$

Remark 4.1 *The above result continues to hold whenever* $np_n \to 1$ *as* $n \to \infty$.

Acknowledgement

The authors are highly grateful to the anonymous referees for constructive suggestions and in particular, for bringing to their notice the work of Barakat.The work of first author has been supported by UGC-DSA program, University of Mysore, Mysore, India, and that of the second author is supported by the grants of Malayer University, Malayer, Iran.

References

Barakat H.M. & El-Shandidy M.A. (1990). On the limit distribution of the extremes of a random number of independent random variables , *J. Statist. Plann. Inference*, *26*, 353–361.

Barakat H.M. (1997). Asymptotic properties of bivariate random extremes, *J. Statist. Plann. Inference*, *61*, 203–217.

Berman S.M. (1962). Equally correlated random variables, *Sankhya*, *A, 24*, 155–156.

Berman S.M. (1964). Limit theorems for the maximum term in stationary sequences, *Ann. Math. Statit.* , *35*, 502–516.

Galambos J. (1978). The asymptotic theory of extreme order statistics, *Wiley, New York.*

Gnedenko B. V. (1983). On limit theorems for a random number of random variables, *In Probability Theory and Mathematical Statistics*, *Lecture Notes in Mathematics*, *Springer Berlin*, 167–176.

Kelbanov L.B. & Maniya G.M. & Melamed I.A. (1985). A Problem of zolotarev and analogs of infinitely divisible and stable distributions in a scheme for summing a random number of random variables, *Theory Prob. Appl.*, *29*, 791–794.

Leadbetter M.R. & Lindgren G. & Rootzen H. (1983). Extremes and related properties of random sequences and processes, *Springer series in statistics*, *Springer-Verlag*, New York.

McCormick W.P. (1980). Weak convergence for the maxima of stationary Gaussian processes using random normalization, *Ann. of Probab.*, *8*, 483–497.

Mittal Y. & Ylvisaker D. (1975). Limit distributions for the maxima of stationary Gaussian processes, *Stochastic Process. Appl.*, *3*, 1-18.

Pickands J. (1962). Maxima of stationary Gaussian processes, *Z. WahrschVerwGebiete*, *7*, 190-223.

Szelki (1986). On concavity of the waiting time distribution in GI/G/1 queues , *J. Appl. Probab.*, *23*, 553-561.

ProbStat Forum is an e-journal. For details please visit; *www.probstat.org.in.*