ISSN 0974-3235

ProbStat Forum, Volume 02, July 2009, Pages 52-67

Reliability estimation in log-logistic distribution from censored samples

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Paper received on 19 September 2008; revised, 31 March 2009; accepted, 01 May 2009.

Abstract. Reliability function of the well-known log-logistic model is considered and is viewed as parametric function. For certain known combinations of its shape parameter, its scale parameter is estimated by different methods of estimation from right censored samples. The methods of estimation are shown to be asymptotically equivalent to ML method of estimation. However the small sample performances are studied through simulation. The situation of unknown shape parameter as on lines of Tiku at. al. [13] and Akkaya and Tiku [1]. The estimates are used in reliability function on far with ML estimation (invariance property) to get reliability estimates. The reliability estimates are also shown to be asymptotically equally efficient. The small sample comparisons of reliability are also presented across various methods of estimations.
Keywords. Log-logistic distribution, maximum likelihood estimation, modified maximum likelihood estimation, reliability estimation, right censored sample.

1 Introduction

The probability density function and cumulative distribution function of log-logistic distribution with shape parameter β and scale parameter σ are

$$f(x) = \frac{\frac{\beta}{\sigma}(\frac{x}{\sigma})^{\beta-1}}{[1+(\frac{x}{\sigma})^{\beta}]^2};\ x \geq 0,\ \beta > 1,\ \sigma > 0 \tag{1.1}$$

$$F(x) = \frac{(\frac{x}{\sigma})^{\beta}}{[1+(\frac{x}{\sigma})^{\beta}]};\ x \geq 0,\ \beta > 1,\ \sigma > 0 \tag{1.2}$$

In this paper for a given β the scale parameter in (1.1) is estimated by classical method of estimation namely maximum likelihood estimation (MLE), and it is an iterative solution of ML equation. Approximations and modifications to the maximum likelihood (ML) method of estimation in certain distributions to overcome iterative solutions of ML equations for the parameters were suggested by many authors [for example Tiku [9]; Mehrotra and Nanda [6]; Persson and Rootzen [7]; and Cohen and Whitten [2]]. Tiku et. al. [12] obtained modified maximum likelihood (MML) estimates by making linear approximation(s) to certain function(s) in ML equations of the parameters of normal, log-normal, logistic, exponential and Rayleigh distributions. Tiku [10] extended MML method to estimate the parameters of bivariate normal population. There are some situations in which ML method even from complete samples does not yield an analytical estimator. One such example is the wellknown logistic distribution. Log-logistic distribution is a model generated by a transformation of logistic variate. Kantam and Srinivasa Rao [4] studied ML estimation of the scale parameter from complete samples in log-logistic distribution when its shape parameter is known. Kantam and Srinivasa Rao [5] studied reliability estimation of log-logistic distribution from complete samples. Here we study the estimation of scale and shape parameters and reliability function from Type-II right censored samples. Estimation of scale parameter when shape is known is given in Section 2. The case of unknown shape parameter is discussed in Section 3. Reliability estimation is presented in Section 4.

2 MML Estimation from Censored Samples

Let $X_1 < X_2 < \cdots < X_n$ be an ordered sample of size n from a log-logistic distribution with scale parameter (unknown) and shape parameter (known). Suppose that the largest s observations are censored. Let $X_1 < X_2 < \cdots < X_{n-s}$ is a type II right censored sample from log-logistic distribution (1.1). The log likelihood function to estimate σ from the given censored sample (after simplification) is

$$\log L = \text{constant} + \sum_{i=1}^{n-s} \log f(t_i) + s \log[1 - F(t_{n-s})] \tag{2.1}$$

where $t_i = \frac{x_i}{\sigma}$; $i = 1, 2, \ldots, n-s$ and $f(\cdot)$ and $F(\cdot)$ are given in (1.1) and (1.2). The 'constant' is a term free from unknown parameters. The log likelihood equation for estimating σ is given by

$$\frac{\partial \log L}{\partial \sigma} = 0 \Rightarrow \frac{\beta}{\sigma}\Big[2\sum_{i=1}^{n-s} F(t_i) + sF(t_{n-s}) - (n-s)\Big] \tag{2.2}$$

It can be seen that Equation (2.2) cannot be solved analytically for σ. The MLE of σ has to be obtained as an iterative solution of (2.2) say $\hat{\sigma}_1$. To overcome the inconvenience of iterative solution of (2.2), we approximate $F(t_i)$ by a linear function (for $i = 1, 2, \ldots, n-s$).

$$F(t_i) \cong \gamma_i + \delta_i t_i \tag{2.3}$$

where γ_i and δ_i are to be suitably found. Here we suggest two different methods of finding γ_i and δ_i. After using this approximation in Equation (2.2) an approximate likelihood equation for σ is given by

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &\cong\Rightarrow 2\sum_{i=1}^{n-s}(\gamma_i + \delta_i t_i) + s(\gamma_{n-s} + \delta_{n-s} t_{n-s}) - (n-s) = 0 \\ \Rightarrow \hat{\sigma} &= \frac{s\delta_{n-s} x_{n-s} + 2\sum_{i=1}^{n-s}\delta_i x_i}{n - s - s\gamma_{n-s} - 2\sum_{i=1}^{n-s}\gamma_i}.\end{aligned} \tag{2.4}$$

The above estimator is named MMLE of σ, which is a linear estimator in x_i's. Hence its variance can be computed using the variances and covariances of standard order statistics provided we have the values of γ_i and δ_i. We propose two methods to get γ_i and δ_i. The basic work for our two methods is similar to those in Tiku [9], Tiku and Suresh [11] respectively. From Equation (2.3) the mean, variance of $\hat{\sigma}_1$ are given by

$$E(\hat{\sigma}_1) = \sigma \sum_{i=1}^{n} l_i \alpha_i \text{ and } V(\hat{\sigma}_1) = \sigma^2 \sum_{i=1}^{n}\sum_{j=1}^{n} l_i l_j b_{ij}.$$

where α_i, b_{ij} are the means, variances and covariances of standard order statistics, available in Srinivasa Rao [8].

Method-I: Let $p_i = 1 - q_i = i/(n+1)$; $i = 1, 2, \ldots, n$. Let Z_i, Z_i^* be the solutions of the following equations

$$F(Z_i) = p_i - \sqrt{\frac{p_i q_i}{n}} \text{ and } F(Z_i^*) = p_i + \sqrt{\frac{p_i q_i}{n}}, \quad \text{where } q_i = 1 - p_i.$$

The intercept γ_i and slope δ_i of the linear approximation in the Equation (2.3) are respectively given by (these can be arrived at by using coordinate geometry and also see, Tiku et.al. [9]),

$$\delta_i = \frac{F(Z_i^*) - F(Z_i)}{Z_i^* - Z_i} \tag{2.5}$$

$$\gamma_i = F(Z_i^*) - \delta_i Z_i^*. \tag{2.6}$$

For the log-logistic cumulative distribution function $F(\cdot)$ the expression for Z_i, Z_i^* are given by (see the APPENDIX at the end)

$$Z_i = \left[\frac{p_i - \sqrt{\frac{p_i q_i}{n}}}{1 - \left(p_i - \sqrt{\frac{p_i q_i}{n}}\right)}\right]^{\frac{1}{\beta}} \tag{2.7}$$

$$Z_i^* = \left[\frac{p_i + \sqrt{\frac{p_i q_i}{n}}}{1 - \left(p_i + \sqrt{\frac{p_i q_i}{n}}\right)}\right]^{\frac{1}{\beta}}. \tag{2.8}$$

Substituting the values of γ_i and δ_i of Equations (2.6) and (2.5) in Equation (2.4) we can get the MMLE $\hat{\sigma}_2$ of σ for log-logistic distribution from type-II right censored samples.

Method-II: In this method we consider the Taylor's series expansion of $F(\cdot)$ around the expected value of ith order statistic in standard log-logistic distribution say α_i. Then we get

$$F(t_i) \cong \gamma_i + \delta_i t_i \quad \text{where } \delta_i = F'(\alpha_i) \tag{2.9}$$

$$\gamma_i = F(\alpha_i) - \alpha_i \delta_i \tag{2.10}$$

Using the means of order statistics in log-logistic distribution we have calculated γ_i and δ_i of this method. The resulting MMLE of this method would be again another linear estimator of the type in equation (2.4) says $\hat{\sigma}_3$, whose coefficients depends on (2.9) and (2.10).

In these Methods the basic principle is that the distribution function $F(\cdot)$ is approximated by a linear function in some neighborhood of the population. It can be seen that the construction of the neighborhood over which the distribution function is linearized depends on the size of the sample also. The larger the size, the closer the approximation. That is the exactness of the approximation becomes finer and finer for large values of n. Hence, the approximate log likelihood equation and the exact log likelihood equation differ by little quantities for large n. Therefore, the solutions of exact and approximate log likelihood equations tend to each other as $n \to \infty$. Hence the exact and Modified MLEs are asymptotically identical (Tiku et al. [12]). However, the same cannot be said in small samples. We, compared these estimates in small samples through Monte Carlo simulation, as exact MLE is an iterative solution.

The bias and MSE of the estimates by the two methods of modification and that of the exact MLE obtained through simulation for $n = 10, 15, 20$ and $\beta = 3(1)6$ with all possible combinations of right censored samples are computed by us. But we are appending the results for $\beta = 3$ only in Table 1.

Our computations reveal that the preference goes to Tiku's method for smaller values of β. As β increases the MSEs of all the estimates are close to each other without conforming a priority of one over the other. For the sake of comparison in most of the places the estimator on lines of Tiku [9] occupied the least position. To sum up we conclude that if ML method is adopted along with the modifications to overcome the iterative solutions, the one based on lines of Tiku [9] is preferable to that based on Tiku and Suresh [11].

3 Estimation of shape parameter β

The methods presented in Section 2 are based on a known value of β. However in a given practical situation generally both the parameters are unknown. At the same time any arbitrary specification of β may not be consistent with the live situation. Also to estimate β by ML method we have to solve the following log likelihood equation.

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n-s} \log t_i + (n-s)/\beta - 2\sum_{i=1}^{n-s} \log t_i F(t_i) - s \log t_{n-s} F(t_{n-s}) \quad (3.1)$$

The above equation also does not admit analytical solution. The modifications suggested to estimate σ in Section 2 are not useful here because β is a shape parameter of the model. Hence on lines of Tiku et al. [13] and Akkaya and Tiku [1] we adopt the following procedure in any given situation. Based on a given sample compute the exact MLE, the two MMLEs over a set of strategic values of β in its admissible parametric space, that is $(1, \infty)$. Then substitute the estimate of σ with the corresponding strategic value of β in the expression of $\log L$ given in (2.1). The MLE of β is taken as that strategic value at which $\log L$ is maximum with a corresponding estimate of σ. When once an MLE of β in the above manner is identified, with that value of β, the exact MLE by solving the iterative equation (2.2) and the two MMLEs in the form of equation (2.3) can be calculated. This is how we handle the situation of unknown β.

We present the above method by an illustration of a live data given in Gupta et al. [3] about days of survival for lung cancer patients. Gupta et al. [3] fitted this data for log-logistic model. We consider the data as random samples of size 10, 15, 20 from a log-logistic distribution [for this sample the strategic β are chosen as 1(0.1) 20] as presented below.
Random sample of size 10 are 10, 117, 7, 278, 33, 99, 73, 15, 162, 72.
Random sample of size 15 are 132, 25, 30, 587, 95, 63, 384, 278, 80, 110, 45, 27, 111, 30, 72.
Random sample of size 20 are 389, 18, 22, 10, 112, 63, 100, 13, 151, 467, 162, 117, 122, 33, 42, 99, 283, 80, 314, 112.

For the above sample the strategic values of β that are considered are 1(0.1) 20. The exact MLE and two MMLEs of σ are calculated for each β in the above range, which are intern are substituted in equation (2.1). The value of β at which, equation (2.1) is maximum is isolated in each case. For such an identified β, the exact MLE and two MMLEs are again calculated. Our calculations are summarized in the Table 2. The mean and variance of the estimates of β are given in Table 3.

The Table 2 reveals that the MLE of β is reached to 5.9 for extreme censoring case when sample size is 10, it is 11.4 for sample size 15 and is going beyond 20 also for the combination $n = 15$, $s = 13$. But when $n = 20$, it is as low as 7.8. Hence a methodically choice for β depending on n cannot be made.

4 Reliability Estimation

We know that the reliability function of a scaled log-logistic distribution is given by

$$R(x) = 1 - F(x) = \left[1 + (\frac{x}{\sigma})^{\beta}\right]^{-1}; \quad x \geq 0,\, \sigma > 0, \beta > 1 \qquad (4.1)$$

We require the knowledge of σ to get $R(x)$. In order to get the estimate of reliability function, we explore the invariance property of ML method and extend the same to MML method also. Therefore beside exact MLE, we get MMLEs for reliability function say $\hat{R}_i(x)$; $i = 1, 2, 3$ as $\hat{R}_1(x)$ is by using exact MLE, $\hat{R}_i(x)$ are by using MMLEs of methods I, II respectively. As the expression of $\hat{R}_i(x)$ indicates we cannot get analytical expression for the variance of $\hat{R}_i(x)$. However its asymptotic variance is given by

$$\text{Asvar } (\hat{R}_i(x)) = (\frac{\partial R}{\partial \sigma})^2 \text{Asvar } (\hat{\sigma}_i); \quad i = 1, 2, 3 \qquad (4.2)$$

where $\frac{\partial R}{\partial \sigma} = \dfrac{\frac{\beta}{\sigma}(\frac{x}{\sigma})^{\beta}}{(1 + (\frac{x}{\sigma})^{\beta})^2}$.

In (4.2) Asvar $(\hat{\sigma}_i)$ means the asymptotic variance of the estimator of σ that we used in estimating the reliability function. As mentioned in Section 2 we see that Asvar $(\hat{\sigma}_i)$ for $i = 1, 2, 3$ are identical. Hence $\hat{R}_i(x)$; $i = 1, 2, 3$ are asymptotically equally efficient. But in small samples a comparison may throw a different picture. In view of the non-tractability of the analytical variances, we resorted to simulation to estimate $R(x)$ for chosen values of x corresponding to $R(x) = 0.25, 0.50, 0.75, 0.90, 0.95$ from all possible right censored samples for $n = 10, 15, 20$, $\beta = 3(1)6$. As method II is based on expectation of ordered statistics, we have chosen the values of β as 3(1)6. But results for $\beta = 3$ are only appended in Table 4. The following conclusions are drawn from Table 4.

Remarks: The results in the Table 4 are across three sizes of the sample four values of the shape parameter and five values of the true reliability (two values less than or equal 0.5 and three values more than 0.5) for all possible cases of right censored samples. In all the situations of exact and modified MLEs the results show a change in the trend at true reliability = 0.5, such as the bias changing its sign, the MSE, variance changing from increasing to decreasing. This is possible because the population is skewed and the true reliability becomes = 0.5 leads to the median life of the population. As a mater of comparison among three methods of estimation, it is the modified ML method on lines of Tiku gives the minimum value for absolute bias, MSE and variance across all combinations that are considered to estimate reliability of log-logistic distribution at values of the variate less than or equal to median. On the other hand if at any value of the variate more than the population median, the methods on the lines Tiku and Suresh is equally preferable to the one on the lines of Tiku.

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Table 1: Empirical Sample Characteristics of MLE and MMLEs of scale parameter in log-logistic distribution from right censored samples ($\beta = 3$)

n	S	BIAS			MSE		
		mle	tmmle	smmle	mle	tmmle	smmle
10	1	0.01783	0.00393	0.02786	0.03643	0.03591	0.03817
	2	0.01493	0.00467	0.01983	0.03463	0.03413	0.03542
	3	0.01481	0.00771	0.01502	0.04002	0.03944	0.04002
	4	0.01090	0.00838	0.00829	0.03829	0.03811	0.03807
	5	0.00709	0.01073	0.00231	0.04153	0.04187	0.04108
	6	-0.00025	0.01312	-0.00571	0.04342	0.04485	0.04306
	7	-0.01426	0.01455	-0.01981	0.05483	0.05829	0.05448
	8	-0.03049	0.03269	-0.03531	0.06914	0.07852	0.06883
15	1	0.01271	0.00295	0.02117	0.02515	0.02485	0.02613
	2	0.00817	-0.00068	0.01381	0.02231	0.02208	0.02291
	3	0.01124	0.00356	0.01438	0.02271	0.02238	0.02301
	4	0.01093	0.00475	0.01223	0.02555	0.02516	0.02564
	5	0.01021	0.00530	0.00968	0.02421	0.02402	0.02428
	6	0.00434	0.00122	0.00244	0.02417	0.02404	0.02413
	7	0.00737	0.00662	0.00419	0.02738	0.02736	0.02720
	8	0.00318	0.00549	-0.00052	0.02707	0.02727	0.02693
	9	-0.00013	0.00611	-0.00445	0.02882	0.02937	0.02869
	10	-0.00361	0.00847	-0.00803	0.03370	0.03460	0.03346
	11	-0.00783	0.01258	-0.01227	0.03838	0.04012	0.03816
	12	-0.02329	0.01103	-0.02751	0.04674	0.04954	0.04655
	13	-0.04386	0.02241	-0.04801	0.06189	0.06913	0.06178
20	1	0.01022	0.00273	0.01718	0.01825	0.01816	0.01891
	2	0.00424	-0.00330	0.00917	0.01595	0.01585	0.01629
	3	0.01102	0.00418	0.01451	0.01590	0.01563	0.01612
	4	0.00538	-0.00117	0.00723	0.01778	0.01750	0.01782
	5	0.01249	0.00706	0.01380	0.01969	0.01934	0.01973
	6	0.00398	-0.00083	0.00410	0.01823	0.01801	0.01824
	7	0.00492	0.00109	0.00421	0.01707	0.01697	0.01710
	8	0.00800	0.00489	0.00622	0.01933	0.01918	0.01922
	9	0.00438	0.00265	0.00220	0.01779	0.01772	0.01770
	10	0.00576	0.00526	0.00283	0.01836	0.01842	0.01828
	11	-0.00370	-0.00211	-0.00682	0.02058	0.02064	0.02047
	12	0.00740	0.01139	0.00417	0.02387	0.02412	0.02366
	13	-0.00780	-0.00116	-0.01130	0.02612	0.02643	0.02604
	14	-0.00435	0.00600	-0.00795	0.02481	0.02534	0.02468
	15	-0.00840	0.00731	-0.01197	0.02987	0.03083	0.02976
	16	-0.01112	0.01236	-0.01483	0.03658	0.03843	0.03650
	17	-0.03347	0.00294	-0.03738	0.04558	0.04783	0.04541
	18	-0.04931	0.01826	-0.05326	0.06813	0.07576	0.06799

mle: Maximum likelihood estimator

tmmle: Modified maximum likelihood estimator on lines of Tiku's.

smmle: Modified maximum likelihood estimator on lines of Tiku and Suresh.

Table 2: Estimated values of scale and shape parameters of log-logistic distribution (data of Gupta et al., [3]).

n	s	β	MLE	TMMLE	SMMLE
10	0	1.4	53.36354	49.37024	56.46609
10	1	1.4	54.06142	48.24000	55.84407
10	2	1.3	54.28680	48.43611	55.85666
10	3	1.2	56.61561	50.10754	57.82151
10	4	1.2	56.52950	51.06857	57.58857
10	5	1.1	66.22606	59.67516	67.83328
10	6	1.4	43.18548	41.35501	43.37497
10	7	3.0	19.92231	20.63826	19.85780
10	8	5.9	12.72626	13.48069	12.66586
15	0	1.8	80.47910	77.07551	81.41110
15	1	1.8	80.45891	77.06939	81.45518
15	2	1.8	80.24166	77.46997	81.33816
15	3	2.2	75.66116	73.58537	75.55345
15	4	2.2	75.17073	73.47038	75.08703
15	5	2.1	77.62327	75.98859	77.50404
15	6	2.1	76.80908	75.50510	76.68422
15	7	2.2	74.22012	73.43257	74.06579
15	8	2.2	74.18775	73.74640	74.03537
15	9	2.2	72.99467	72.91399	72.82649
15	10	3.3	54.38167	55.02882	54.15225
15	11	15.0	32.22282	32.55900	32.18305
15	12	11.4	33.87993	34.51609	33.83265
15	13	10.9	33.99145	34.89761	34.15832
20	0	1.6	88.73929	85.60456	90.31197
20	1	1.5	88.74874	84.39083	90.40606
20	2	1.5	89.04893	84.99540	90.38271
20	3	1.5	89.72922	86.09157	90.96204
20	4	1.6	86.25878	83.60962	87.40954
20	5	1.6	87.56496	84.90522	88.31153
20	6	1.6	86.21535	84.10517	87.15402
20	7	1.5	89.07071	86.46074	89.78841
20	8	1.4	92.95013	89.72464	93.47511
20	9	1.3	100.00880	95.86792	100.30935
20	10	1.3	101.30270	97.51255	101.58687
20	11	1.2	112.48710	107.72093	112.93002
20	12	1.2	108.65810	104.39046	109.01239
20	13	1.3	97.61884	95.05716	97.83625
20	14	1.6	70.50987	70.11561	70.47607
20	15	1.8	60.05375	60.47543	59.99138
20	16	2.9	35.47559	36.42691	35.41422
20	17	3.5	29.55020	30.71625	29.47459
20	18	7.8	17.26433	18.02423	17.20178

mle: Maximum likelihood estimator.

tmmle: Modified maximum likelihood estimator on lines of Tikus.

smmle: Modified maximum likelihood estimator on lines of Tiku and Suresh.

Table 3: Mean and variance values of shape parameter (β) of log-logistic distribution (data of Gupta et al., [3])

n	S	β	mean	variance
10	0	1.4	2.20557	0.04745
10	1	1.4	1.95794	0.07757
10	2	1.3	1.92478	0.10188
10	3	1.2	1.92195	0.12273
10	4	1.2	1.77461	0.15628
10	5	1.1	1.78024	0.14836
10	6	1.4	1.27247	0.24704
10	7	3	0.52853	1.12091
10	8	5.9	0.22825	4.39929
15	0	1.8	1.94496	0.04611
15	1	1.8	1.75989	0.06635
15	2	1.8	1.63728	0.09120
15	3	2.2	1.26113	0.18186
15	4	2.2	1.19428	0.23521
15	5	2.1	1.18810	0.26606
15	6	2.1	1.12898	0.31380
15	7	2.2	1.02294	0.38384
15	8	2.2	0.96821	0.40580
15	9	2.2	0.91178	0.41224
15	10	3.3	0.56773	0.92206
15	11	15	0.11510	18.90808
15	12	11.4	0.13630	11.06856
15	13	10.9	0.14722	10.96296
20	0	1.6	2.37164	0.02541
20	1	1.5	2.31211	0.02996
20	2	1.5	2.17063	0.03871
20	3	1.5	2.06227	0.04904
20	4	1.6	1.84889	0.06955
20	5	1.6	1.77492	0.08518
20	6	1.6	1.70778	0.10215
20	7	1.5	1.75481	0.10493
20	8	1.4	1.81183	0.10364
20	9	1.3	1.87958	0.09806
20	10	1.3	1.80859	0.10421
20	11	1.2	1.88172	0.09173
20	12	1.2	1.80201	0.09274
20	13	1.3	1.58630	0.10856
20	14	1.6	1.22172	0.16324
20	15	1.8	1.02022	0.20597
20	16	2.9	0.58663	0.54097
20	17	3.5	0.37899	0.82151
20	18	7.8	0.17006	4.51712

Table 4: Empirical Sample Characteristics of Reliability estimates using MLE and MMLEs of scale parameter in log-logistic distribution from right censored samples ($\beta = 3$).

n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
10	1	0.25	0.01339	0.00587	0.01880	0.01085	0.01055	0.01142
10	1	0.50	0.00032	-0.00958	0.00698	0.01666	0.01692	0.01684
10	1	0.75	-0.01274	-0.02070	-0.00765	0.01059	0.01128	0.01031
10	1	0.90	-0.01064	-0.01483	-0.00802	0.00309	0.00338	0.00293
10	1	0.95	-0.00658	-0.00888	-0.00514	0.00097	0.00108	0.00092
10	2	0.25	0.01186	0.00630	0.01451	0.01033	0.01008	0.01060
10	2	0.50	-0.00110	-0.00836	0.00219	0.01625	0.01641	0.01634
10	2	0.75	-0.01363	-0.01944	-0.01112	0.01057	0.01105	0.01044
10	2	0.90	-0.01110	-0.01414	-0.00981	0.00314	0.00334	0.00306
10	2	0.95	-0.00684	-0.00851	-0.00613	0.00100	0.00107	0.00097
10	3	0.25	0.01223	0.00842	0.01235	0.01178	0.01155	0.01179
10	3	0.50	-0.00291	-0.00785	-0.00277	0.01852	0.01861	0.01853
10	3	0.75	-0.01694	-0.02089	-0.01683	0.01234	0.01268	0.01233
10	3	0.90	-0.01352	-0.01560	-0.01345	0.00375	0.00391	0.00374
10	3	0.95	-0.00831	-0.00946	-0.00827	0.00120	0.00126	0.00120
10	4	0.25	0.01020	0.00884	0.00879	0.01117	0.01109	0.01108
10	4	0.50	-0.00494	-0.00669	-0.00674	0.01818	0.01819	0.01820
10	4	0.75	-0.01848	-0.01987	-0.01991	0.01262	0.01273	0.01274
10	4	0.90	-0.01449	-0.01522	-0.01525	0.00401	0.00406	0.00407
10	4	0.95	-0.00892	-0.00932	-0.00934	0.00132	0.00134	0.00134
10	5	0.25	0.00844	0.01039	0.00590	0.01195	0.01209	0.01177
10	5	0.50	-0.00865	-0.00615	-0.01192	0.01964	0.01962	0.01971
10	5	0.75	-0.02265	-0.02066	-0.02531	0.01401	0.01384	0.01431
10	5	0.90	-0.01721	-0.01616	-0.01867	0.00461	0.00453	0.00477
10	5	0.95	-0.01057	-0.00999	-0.01139	0.00156	0.00153	0.00162
10	6	0.25	0.00486	0.01200	0.00197	0.01239	0.01295	0.01222
10	6	0.50	-0.01440	-0.00530	-0.01818	0.02090	0.02081	0.02104
10	6	0.75	-0.02837	-0.02106	-0.03149	0.01561	0.01487	0.01600
10	6	0.90	-0.02089	-0.01694	-0.02260	0.00547	0.00508	0.00567
10	6	0.95	-0.01282	-0.01060	-0.01379	0.00193	0.00178	0.00201
10	7	0.25	-0.00078	0.01438	-0.00367	0.01490	0.01628	0.01471
10	7	0.50	-0.02693	-0.00770	-0.03072	0.02694	0.02663	0.02713
10	7	0.75	-0.04429	-0.02851	-0.04751	0.02293	0.02082	0.02346
10	7	0.90	-0.03279	-0.02389	-0.03465	0.00945	0.00820	0.00976
10	7	0.95	-0.02059	-0.01544	-0.02169	0.00377	0.00323	0.00391

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n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
10	8	0.25	-0.00709	0.02562	-0.00955	0.01770	0.02127	0.01752
10	8	0.50	-0.04096	0.00005	-0.04420	0.03390	0.03323	0.03410
10	8	0.75	-0.06290	-0.02873	-0.06573	0.03269	0.02716	0.03326
10	8	0.90	-0.04800	-0.02789	-0.04974	0.01593	0.01214	0.01632
10	8	0.95	-0.03119	-0.01910	-0.03226	0.00710	0.00521	0.00730
15	1	0.25	0.00964	0.00435	0.01431	0.00769	0.00752	0.00803
15	1	0.50	0.00029	-0.00673	0.00614	0.01197	0.01216	0.01213
15	1	0.75	-0.00892	-0.01452	-0.00451	0.00721	0.00762	0.00709
15	1	0.90	-0.00724	-0.01014	-0.00505	0.00192	0.00209	0.00186
15	1	0.95	-0.00441	-0.00599	-0.00323	0.00058	0.00063	0.00055
15	2	0.25	0.00712	0.00229	0.01023	0.00680	0.00667	0.00702
15	2	0.50	-0.00193	-0.00834	0.00198	0.01114	0.01130	0.01124
15	2	0.75	-0.01010	-0.01518	-0.00715	0.00694	0.00727	0.00684
15	2	0.90	-0.00772	-0.01033	-0.00624	0.00188	0.00201	0.00183
15	2	0.95	-0.00465	-0.00606	-0.00385	0.00057	0.00061	0.00055
15	3	0.25	0.00887	0.00467	0.01060	0.00695	0.00678	0.00705
15	3	0.50	0.00019	-0.00531	0.00239	0.01132	0.01137	0.01134
15	3	0.75	-0.00863	-0.01293	-0.00695	0.00700	0.00726	0.00694
15	3	0.90	-0.00703	-0.00923	-0.00619	0.00188	0.00199	0.00186
15	3	0.95	-0.00428	-0.00548	-0.00383	0.00056	0.00060	0.00056
15	4	0.25	0.00886	0.00548	0.00957	0.00781	0.00765	0.00785
15	4	0.50	-0.00113	-0.00551	-0.00021	0.01245	0.01248	0.01244
15	4	0.75	-0.01043	-0.01385	-0.00971	0.00762	0.00782	0.00758
15	4	0.90	-0.00815	-0.00990	-0.00778	0.00205	0.00214	0.00204
15	4	0.95	-0.00492	-0.00588	-0.00473	0.00062	0.00065	0.00061
15	5	0.25	0.00849	0.00581	0.00820	0.00733	0.00724	0.00735
15	5	0.50	-0.00099	-0.00453	-0.00141	0.01204	0.01210	0.01208
15	5	0.75	-0.01020	-0.01297	-0.01055	0.00768	0.00784	0.00771
15	5	0.90	-0.00811	-0.00953	-0.00830	0.00214	0.00221	0.00215
15	5	0.95	-0.00494	-0.00571	-0.00504	0.00065	0.00068	0.00066
15	5	0.95	-0.00494	-0.00571	-0.00504	0.00065	0.00068	0.00066
15	6	0.25	0.00547	0.00376	0.00443	0.00726	0.00720	0.00723
15	6	0.50	-0.00519	-0.00742	-0.00656	0.01245	0.01248	0.01249
15	6	0.75	-0.01386	-0.01562	-0.01496	0.00826	0.00837	0.00835
15	6	0.90	-0.01017	-0.01107	-0.01074	0.00238	0.00243	0.00242
15	6	0.95	-0.00610	-0.00659	-0.00642	0.00074	0.00075	0.00075
15	7	0.25	0.00711	0.00670	0.00538	0.00827	0.00826	0.00819
15	7	0.50	-0.00429	-0.00483	-0.00654	0.01336	0.01338	0.01340
15	7	0.75	-0.01367	-0.01410	-0.01546	0.00847	0.00850	0.00860
15	7	0.90	-0.01011	-0.01034	-0.01104	0.00236	0.00238	0.00242

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n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
15	7	0.95	-0.00606	-0.00618	-0.00657	0.00072	0.00073	0.00074
15	8	0.25	0.00489	0.00616	0.00289	0.00809	0.00817	0.00802
15	8	0.50	-0.00712	-0.00550	-0.00976	0.01344	0.01345	0.01352
15	8	0.75	-0.01605	-0.01479	-0.01817	0.00885	0.00878	0.00903
15	8	0.90	-0.01148	-0.01084	-0.01259	0.00256	0.00253	0.00264
15	8	0.95	-0.00685	-0.00650	-0.00746	0.00079	0.00078	0.00082
15	9	0.25	0.00361	0.00702	0.00129	0.00851	0.00873	0.00842
15	9	0.50	-0.00989	-0.00555	-0.01296	0.01484	0.01486	0.01495
15	9	0.75	-0.01957	-0.01619	-0.02207	0.01033	0.01012	0.01058
15	9	0.90	-0.01390	-0.01216	-0.01524	0.00313	0.00302	0.00324
15	9	0.95	-0.00831	-0.00736	-0.00906	0.00099	0.00095	0.00103
15	10	0.25	0.00223	0.00875	-0.00015	0.00982	0.01021	0.00970
15	10	0.50	-0.01395	-0.00554	-0.01706	0.01714	0.01702	0.01722
15	10	0.75	-0.02468	-0.01796	-0.02720	0.01230	0.01171	0.01256
15	10	0.90	-0.01733	-0.01377	-0.01868	0.00389	0.00361	0.00401
15	10	0.95	-0.01037	-0.00840	-0.01113	0.00126	0.00116	0.00131
15	11	0.25	0.00042	0.01137	-0.00194	0.01094	0.01167	0.01082
15	11	0.50	-0.01829	-0.00420	-0.02140	0.01921	0.01901	0.01933
15	11	0.75	-0.03000	-0.01869	-0.03257	0.01450	0.01339	0.01481
15	11	0.90	-0.02108	-0.01501	-0.02249	0.00493	0.00437	0.00509
15	11	0.95	-0.01272	-0.00932	-0.01352	0.00168	0.00146	0.00174
15	12	0.25	-0.00617	0.01202	-0.00839	0.01268	0.01397	0.01255
15	12	0.50	-0.03098	-0.00753	-0.03390	0.02424	0.02365	0.02437
15	12	0.75	-0.04515	-0.02586	-0.04760	0.02093	0.01843	0.02130
15	12	0.90	-0.03213	-0.02134	-0.03353	0.00850	0.00707	0.00870
15	12	0.95	-0.01988	-0.01367	-0.02069	0.00330	0.00268	0.00339
15	13	0.25	-0.01478	0.01968	-0.01690	0.01585	0.01895	0.01573
15	13	0.50	-0.04877	-0.00476	-0.05160	0.03195	0.03065	0.03215
15	13	0.75	-0.06712	-0.03014	-0.06961	0.03094	0.02489	0.03146
15	13	0.90	-0.04875	-0.02709	-0.05028	0.01442	0.01057	0.01476
15	13	0.95	-0.03093	-0.01804	-0.03187	0.00620	0.00440	0.00637
20	1	0.25	0.00780	0.00369	0.01166	0.00566	0.00559	0.00591
20	1	0.50	0.00092	-0.00457	0.00579	0.00916	0.00929	0.00927
20	1	0.75	-0.00628	-0.01059	-0.00261	0.00543	0.00567	0.00535
20	1	0.90	-0.00521	-0.00740	-0.00340	0.00140	0.00149	0.00136
20	1	0.95	-0.00317	-0.00435	-0.00220	0.00041	0.00044	0.00040
20	2	0.25	0.00438	0.00024	0.00711	0.00488	0.00481	0.00502
20	2	0.50	-0.00255	-0.00807	0.00095	0.00831	0.00843	0.00836
20	2	0.75	-0.00840	-0.01273	-0.00575	0.00519	0.00542	0.00513
20	2	0.90	-0.00617	-0.00836	-0.00486	0.00139	0.00148	0.00136

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n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
20	2	0.95	-0.00368	-0.00486	-0.00298	0.00042	0.00045	0.00041
20	3	0.25	0.00806	0.00427	0.00999	0.00493	0.00480	0.00501
20	3	0.50	0.00241	-0.00253	0.00491	0.00810	0.00812	0.00813
20	3	0.75	-0.00438	-0.00819	-0.00250	0.00485	0.00500	0.00480
20	3	0.90	-0.00406	-0.00597	-0.00313	0.00125	0.00131	0.00123
20	3	0.95	-0.00252	-0.00355	-0.00202	0.00037	0.00039	0.00036
20	4	0.25	0.00515	0.00154	0.00617	0.00546	0.00533	0.00548
20	4	0.50	-0.00241	-0.00714	-0.00107	0.00910	0.00913	0.00908
20	4	0.75	-0.00886	-0.01252	-0.00781	0.00561	0.00577	0.00555
20	4	0.90	-0.00655	-0.00839	-0.00601	0.00149	0.00155	0.00146
20	4	0.95	-0.00391	-0.00490	-0.00361	0.00044	0.00046	0.00043
20	5	0.25	0.00916	0.00617	0.00988	0.00607	0.00593	0.00609
20	5	0.50	0.00212	-0.00175	0.00307	0.00970	0.00969	0.00970
20	5	0.75	-0.00582	-0.00879	-0.00508	0.00582	0.00594	0.00579
20	5	0.90	-0.00517	-0.00666	-0.00480	0.00153	0.00158	0.00152
20	5	0.95	-0.00319	-0.00399	-0.00299	0.00045	0.00047	0.00045
20	6	0.25	0.00452	0.00187	0.00459	0.00555	0.00545	0.00555
20	6	0.50	-0.00352	-0.00698	-0.00343	0.00948	0.00950	0.00949
20	6	0.75	-0.01009	-0.01277	-0.01003	0.00601	0.00615	0.00602
20	6	0.90	-0.00733	-0.00869	-0.00731	0.00163	0.00169	0.00163
20	6	0.95	-0.00437	-0.00510	-0.00436	0.00049	0.00051	0.00049
20	7	0.25	0.00491	0.00280	0.00452	0.00522	0.00517	0.00522
20	7	0.50	-0.00244	-0.00522	-0.00297	0.00890	0.00893	0.00892
20	7	0.75	-0.00879	-0.01096	-0.00921	0.00559	0.00570	0.00562
20	7	0.90	-0.00653	-0.00762	-0.00674	0.00151	0.00155	0.00152
20	7	0.95	-0.00390	-0.00450	-0.00402	0.00045	0.00047	0.00046
20	8	0.25	0.00674	0.00503	0.00576	0.00596	0.00590	0.00592
20	8	0.50	-0.00107	-0.00330	-0.00234	0.00979	0.00980	0.00979
20	8	0.75	-0.00834	-0.01006	-0.00931	0.00595	0.00603	0.00599
20	8	0.90	-0.00643	-0.00729	-0.00692	0.00156	0.00160	0.00158
20	8	0.95	-0.00387	-0.00433	-0.00413	0.00046	0.00047	0.00047
20	9	0.25	0.00467	0.00372	0.00347	0.00539	0.00536	0.00535
20	9	0.50	-0.00302	-0.00426	-0.00459	0.00917	0.00919	0.00919
20	9	0.75	-0.00951	-0.01049	-0.01074	0.00591	0.00596	0.00597
20	9	0.90	-0.00705	-0.00754	-0.00767	0.00165	0.00167	0.00168
20	9	0.95	-0.00423	-0.00449	-0.00456	0.00050	0.00051	0.00051
20	10	0.25	0.00552	0.00526	0.00392	0.00562	0.00563	0.00557
20	10	0.50	-0.00228	-0.00266	-0.00440	0.00955	0.00960	0.00959
20	10	0.75	-0.00917	-0.00951	-0.01084	0.00598	0.00603	0.00609
20	10	0.90	-0.00686	-0.00705	-0.00772	0.00160	0.00162	0.00164

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n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
20	10	0.95	-0.00411	-0.00421	-0.00457	0.00048	0.00048	0.00049
20	11	0.25	0.00053	0.00140	-0.00117	0.00619	0.00622	0.00614
20	11	0.50	-0.00999	-0.00885	-0.01223	0.01071	0.01070	0.01076
20	11	0.75	-0.01603	-0.01514	-0.01781	0.00702	0.00696	0.00713
20	11	0.90	-0.01065	-0.01019	-0.01156	0.00197	0.00195	0.00202
20	11	0.95	-0.00623	-0.00598	-0.00672	0.00060	0.00060	0.00062
20	12	0.25	0.00708	0.00927	0.00531	0.00717	0.00728	0.00708
20	12	0.50	-0.00280	0.00003	-0.00509	0.01216	0.01215	0.01217
20	12	0.75	-0.01182	-0.00962	-0.01361	0.00806	0.00793	0.00817
20	12	0.90	-0.00912	-0.00799	-0.01004	0.00234	0.00228	0.00240
20	12	0.95	-0.00554	-0.00492	-0.00604	0.00073	0.00071	0.00075
20	13	0.25	-0.00092	0.00269	-0.00282	0.00770	0.00785	0.00764
20	13	0.50	-0.01468	-0.00995	-0.01719	0.01369	0.01360	0.01378
20	13	0.75	-0.02225	-0.01850	-0.02428	0.00956	0.00928	0.00975
20	13	0.90	-0.01483	-0.01288	-0.01591	0.00287	0.00275	0.00295
20	13	0.95	-0.00872	-0.00765	-0.00932	0.00091	0.00086	0.00094
20	14	0.25	0.00087	0.00651	-0.00109	0.00733	0.00758	0.00726
20	14	0.50	-0.01167	-0.00432	-0.01425	0.01306	0.01297	0.01311
20	14	0.75	-0.01945	-0.01365	-0.02150	0.00907	0.00867	0.00922
20	14	0.90	-0.01328	-0.01028	-0.01434	0.00272	0.00255	0.00279
20	14	0.95	-0.00786	-0.00622	-0.00844	0.00086	0.00080	0.00089
20	15	0.25	-0.00080	0.00771	-0.00272	0.00860	0.00903	0.00853
20	15	0.50	-0.01606	-0.00500	-0.01861	0.01536	0.01519	0.01544
20	15	0.75	-0.02494	-0.01614	-0.02700	0.01130	0.01058	0.01150
20	15	0.90	-0.01710	-0.01247	-0.01820	0.00375	0.00341	0.00384
20	15	0.95	-0.01024	-0.00768	-0.01085	0.00128	0.00115	0.00132
20	16	0.25	-0.00136	0.01125	-0.00333	0.01046	0.01127	0.01039
20	16	0.50	-0.02005	-0.00379	-0.02270	0.01874	0.01849	0.01888
20	16	0.75	-0.03103	-0.01798	-0.03324	0.01434	0.01308	0.01462
20	16	0.90	-0.02156	-0.01457	-0.02277	0.00498	0.00434	0.00511
20	16	0.95	-0.01300	-0.00909	-0.01369	0.00172	0.00147	0.00177
20	17	0.25	-0.01153	0.00775	-0.01359	0.01224	0.01345	0.01213
20	17	0.50	-0.03789	-0.01280	-0.04060	0.02450	0.02366	0.02462
20	17	0.75	-0.05091	-0.03008	-0.05319	0.02167	0.01881	0.02201
20	17	0.90	-0.03521	-0.02348	-0.03650	0.00863	0.00703	0.00881
20	17	0.95	-0.02149	-0.01473	-0.02224	0.00320	0.00254	0.00328
20	18	0.25	-0.01639	0.01855	-0.01840	0.01691	0.02021	0.01679
20	18	0.50	-0.05311	-0.00859	-0.05578	0.03476	0.03343	0.03495
20	18	0.75	-0.07399	-0.03640	-0.07635	0.03578	0.02929	0.03628
20	18	0.90	-0.05572	-0.03330	-0.05718	0.01878	0.01422	0.01913

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n	s	$R(x)$	BIAS			MSE		
			mle	tmmle	smmle	mle	tmmle	smmle
20	18	0.95	-0.03651	-0.02286	-0.03742	0.00889	0.00651	0.00909

mle: Maximum likelihood estimator.
tmmle: Modified maximum likelihood estimator on lines of Tiku's.
smmle: Modified maximum likelihood estimator on lines of Tiku and Suresh.

APPENDIX

Equations (2.7) and (2.8) are the inversion of standard *c.d.f.* of log-logistic distribution given in (1.2). Equation (2.7) is obtained as follows.

$$\begin{aligned}
F(Z_i) &= p_i\sqrt{\frac{p_iq_i}{n}} \\
&\Rightarrow \frac{z_i^\beta}{1+z_i^\beta} = p_i - \sqrt{\frac{p_iq_i}{n}} \\
&\Rightarrow z_i^\beta = (p_i - \sqrt{\frac{p_iq_i}{n}}) + z_i^\beta(p_i - \sqrt{\frac{p_iq_i}{n}}) \\
&\Rightarrow z_i^\beta[1-(p_i - \sqrt{\frac{p_iq_i}{n}})] = pi - \sqrt{\frac{p_iq_i}{n}} \\
&\Rightarrow Z_i = \left[\frac{p_i - \sqrt{\frac{p_iq_i}{n}}}{1-(p_i - \sqrt{\frac{p_iq_i}{n}})}\right]^{\frac{1}{\beta}}
\end{aligned}$$

Similarly, the equation (2.8).