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On Modified Second-Order Slope-Rotatable Designs

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Abstract. This article presents a review on modified second-order slope-rotatable designs (SOSRDs). It presents different methods of construction of modified SOSRDs, using central composite designs, balanced incomplete block designs (BIBD), pairwise balanced designs (PBD), symmetrical unequal block arrangements (SUBA) with two unequal block sizes etc. Minimum number of design points for modified SOSRDs are listed among the methods of constructions available in the literature for a fixed numbers of covariates.
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1 Introduction

Response surface methodology is a statistical technique very useful in design and analysis of scientific experiments. In many experimental situations the experimenter is concern with explaining certain aspects of a functional relationship $Y = f(x_1, x_2, \ldots, x_v) + e$, where Y is the response, $x_1, x_2, \ldots, x_v$ are v factors and e is the random error. The function $f(\cdot)$ is called response surface or response function. Designs, which are used, for the study of response surface methods, are called response surface designs. Response surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continuous and controlled by the experimenter. The response is assumed to be as random variable. For example, if a chemical engineer wishes to find the temperature (x_1) and pressure (x_2) that maximizes the yield (response) of his process, the observed response Y may be written as a function of the factors temperature (x_1) and pressure (x_2) as $Y = f(x_1, x_2) + e$.

In many applications of Response Surface Methodology, good estimation of the derivatives of the response function may be as important or perhaps more important than estimation of mean response. Certainly, the computation of a stationary point in a second-order analysis or the use of gradient techniques for example, steepest ascent or ridge analysis depends heavily on the partial derivatives of the estimated response function with respect to the design variables. Since designs that attain certain properties in Y (estimated response) do not enjoy the same properties for the estimated derivatives (slopes), it is important for the user to consider experimental designs that are constructed with the derivatives in mind.

The concept of rotatability, which is very important in response surface second-order designs, was proposed by [7]. A design is said to be rotatable if the variance of the response estimate is a function only of the distance of the point from the design center. The study of rotatable designs is mainly emphasized on the estimation of differences of yields and its precision. Estimation of differences in responses at two different points in the factor space will often be of great importance. If differences in responses at two points close together is of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal *etc* ([35]).

[15] introduced slope-rotatable central composite designs (SRCCD). For the central composite designs they modified [7] rotatability to slope-rotatability simply by adjusting the axial point distance (a), so that the variance of the estimated pure quadratic coefficients is one-fourth the variance of the estimated mixed second-order coefficients. Since the first work of [6] in this slope area, many research papers have been subsequently published such as [34, 33, 32, 15, 30, 37, 27, 22, 68, 69, 29, 31, 38] and so on. [35] extended the concept of slope-rotatability to slope-rotatability overall directions. [36] studied slope-rotatable designs for estimating the slope of response surfaces in experiments with mixtures. A measure and graphical method for evaluating slope-rotatability in response surface designs was suggested by [28]. Slope-rotatability with correlated errors was studied by [9].

Different methods of constructions of SOSRDs were suggested by various authors, including [41, 42, 43, 44, 45, 50, 54, 55, 56, 57, 58, 59, 60, 62, 63, 61, 64, 65, 66, 67, 2, 5, 3, 4] so on.

Specifically, [46] introduced modified slope-rotatable central composite designs. Different methods of constructions of modified SOSRDs were suggested by [47, 48, 49, 53] and so on.

This article presents a review on modified SOSRDs. It presents different methods of construction of modified SOSRDs, using central composite designs, BIBD, PBD, SUBA with two unequal block sizes etc. Minimum number of design points for modified SOSRDs are listed among the methods of constructions available in the literature for a fixed numbers of covariates.

2 Second-order slope-rotatable designs

A second-order response surface model is $D = ((x_{iu}))$ for fitting,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j} b_{ij} x_{iu} x_{ju} + e_u \tag{2.1}$$

where x_{iu} denotes the level of the ith factor ($i = 1, 2, \ldots, v$) in the uth run ($u = 1, 2, \ldots, N$) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2.

Definition 2.1 (Second-order slope-rotatable design)
A second-order response surface design D is said to be a SOSRD, if the variance of the estimate of first order partial derivative $(\partial \hat{Y}_u / \partial x_i)$ with respect to each of independent variables (x_i) is only a function of the distance $(d^2 = \sum_i x_{iu}^2)$ of the point $(x_{1u}, x_{2u}, \ldots, x_{vu})$ from the origin (centre) of the design. Such a spherical variance function for estimation of slopes in the second-order response surface is achieved if the design points satisfy the following conditions ([15]).

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \tag{2.2}$$

$$\begin{aligned} (i) \quad & \sum_{u=1}^{N} x_{iu}^2 = \text{ constant } = N\lambda_2, \\ (ii) \quad & \sum_{u=1}^{N} x_{iu}^4 = \text{ constant } = cN\lambda_4, \text{ for all } i \end{aligned} \tag{2.3}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{ constant } = N\lambda_4, \text{ for } i \neq j \tag{2.4}$$

$$(c + v - 1)\lambda_4 > v\lambda_2^2 \tag{2.5}$$

$$\lambda_4[v(5 - c) - (c - 3)^2] + \lambda_2^2[v(c - 5) + 4] = 0 \tag{2.6}$$

where c, λ_2 and λ_4 are constants and the summation is over the design points. The variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4(c + v - 1)\sigma^2}{N[\lambda_4(c + v - 1) - v\lambda_2^2]},$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2}, \qquad V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_2}\Big[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\Big],$$
$$\mathrm{Cov}(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$
$$\mathrm{Cov}(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]},$$

and other covariances are zero.

$$V\Big(\frac{\partial\hat{Y}}{\partial x_i}\Big) = V(\hat{b}_i) + 4x_i^2 V(\hat{b}_{ii}) + \sum_{j\neq i} x_j^2 V(\hat{b}_{ij})$$
$$= \frac{1}{N}\Big[\frac{\lambda_4+\lambda_2 d^2}{\lambda_2\lambda_4}\Big]\sigma^2.$$

3 Construction of modified SOSRD

The usual method of construction of SOSRD is to take combinations with unknown constants, associate a 2^v factorial combinations or a suitable fraction of it with factors each at ±1 levels to make the level codes equidistant. All such combinations form a design. Generally SOSRDs need at least five levels (suitably coded) at 0, ±1, $\pm a$ for all factors (0,0, …, 0—chosen center of the design, unknown level a to be chosen suitably to satisfy slope-rotatability). Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively by putting some conditions indicating some relation among $\sum x_{iu}^2$, $\sum x_{iu}^4$ and $\sum x_{iu}^2 x_{ju}^2$ some equations involving the unknown levels are obtained and their solution gives the unknown levels. In SOSRD the conditions used are $V(b_{ij}) = 4V(b_{ii})$ and $c = \frac{\sum x_{iu}^4}{\sum x_{iu}^2 x_{ju}^2}$. Other conditions are also possible though, it seems, not yet exploited. We shall investigate the condition $(\sum x_{iu}^2)^2 = N\sum x_{iu}^2 x_{ju}^2$ i.e. $(N\lambda_2)^2 = N(N\lambda_4)$ i.e., $\lambda_2^2 = \lambda_4$ to get another series of symmetrical response surface designs which provide more precise estimates of response at specific points of interest than what is available from the corresponding existing designs. By applying this new condition $\lambda_2^2 = \lambda_4$ in equation (2.6), we get $c = 1$ or $c = 5$. The non-singularity condition (2.5) leads to $c = 5$. It may be noted $\lambda_2^2 = \lambda_4$ and $c = 5$ are equivalent conditions. Further,

$$V(\hat{b}_0) = \frac{(v+4)\sigma^2}{4N}, \qquad V(\hat{b}_i) = \frac{\sigma^2}{N\sqrt{\lambda_4}},$$
$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}, \qquad V(\hat{b}_{ii}) = \frac{\sigma^2}{4N\lambda_4},$$
$$\mathrm{Cov}\ (\hat{b}_0, \hat{b}_{ii}) = \frac{-\sigma^2}{4N\sqrt{\lambda_4}}.$$

It is seen that if $\lambda_2^2 = \lambda_4$, then $\mathrm{Cov}(\hat{b}_{ii}, \hat{b}_{ij}) = 0$ and other covariances are zero.

These modifications of the variances and covariances affect the variance of estimated response at specific points considerably.

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = \left[\frac{\sqrt{\lambda_4} + d^2}{N\lambda_4}\right]\sigma^2.$$

3.1 Modified SRCCD

The most widely used design for fitting a second-order model is the central composite design. Central composite designs are constructed by adding suitable factorial combinations to those obtained from $\frac{1}{2^p} \times 2^v$ fractional factorial design (here $2^{t(v)} = \frac{1}{2^p} \times 2^v$ denotes a suitable fractional replicate of 2^v, in which no interaction with less than five factors is confounded). In coded form the points of $2^{t(v)}$ factorial have coordinates $(\pm 1, \pm 1, \ldots, \pm 1)$ and $2v$ axial points have coordinates of the form $(\pm a, 0, \ldots, 0), (0, \pm a, \ldots, 0), \ldots, (0, 0, \ldots, \pm a)$ etc., and n_0 be the number of central points. The axial points may be replicated n_a times and central points to be replicated n_0 times. A central composite design will give a v dimensional modified SRCCD in $N = \frac{(2^{t(v)} + 2n_a a^2)^2}{2^{t(v)}}$ design points if,

$$a^4 = \frac{2^{t(v)+1}}{n_a}, \tag{3.1}$$

$$n_0 = \frac{(2^{t(v)} + 2n_a a^2)^2}{2^{t(v)}} - (2^{t(v)} + 2vn_a) \text{ and } n_0 \text{ turns out to be an integer.} \tag{3.2}$$

Note: If n_0 is a positive integer then modified SOSRD using central composite designs exists. If n_0 is a non-integral positive real number, we take $[n_0]$ or $[n_0] + 1$ central points, where $[n_0]$ is Gauss symbol denoting integral part of n_0 and construct nearly modified SOSRD.

3.2 Modified SOSRD using BIBD

Balanced incomplete block design. A BIBD denoted by (v, b, r, k, λ) is an arrangement of v treatments in b blocks each containing $k(< v)$ treatments, if (i) every treatment occurs at most once in a block, (ii) every treatment occurs in exactly r blocks and (iii) every pair of treatments occurs together in λ blocks.

Let (v, b, r, k, λ) be a BIBD, $2^{t(k)}$ denotes a fractional replicate of 2^k with $+1$ or -1 levels in which no interaction with less than five factors is confounded. $[1 - (v, b, r, k, \lambda)]$ denote the design points generated from the transpose of the incidence matrix of BIBD. $[1 - (v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (see [40, pp. 298–300]). Let n_0 be the number of central points in modified SOSRD and $\cup$ denotes combination of the design points generated from different sets of points.

Let $(a,0,0,\ldots,0)2^1$ denote the design points generated from $(a,0,0,\ldots,0)$ point set. Repeat this set of additional design points say n_a times when $r<5\lambda$. Consider the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)}\cup n_a(a,0,0,\ldots,0)2^1\cup n_0$ will give a v dimensional modified SOSRD in $N=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ design points if,

$$a^4=\frac{(5\lambda-r)2^{t(k)-1}}{n_a}, \tag{3.3}$$

$$n_0=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}-[b2^{t(k)}+2n_a v] \tag{3.4}$$

and n_0 turns out to be an integer.

Consider the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)}\cup n_0$ will give a three level v dimensional modified SOSRD in $N=\frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}}$ design points when $r=5\lambda$, if,

$$n_0=\frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}}-b2^{t(k)} \tag{3.5}$$

and n_0 turns out to be an integer.

Let $(a,a,\ldots,a)2^{t(v)}$ denote the design points generated from $(a,a,\ldots,a)$ point set. Repeat this set of additional design points say n_a times when $r>5\lambda$. Consider the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)}\cup n_a(a,a,a,\ldots,a)2^{t(v)}\cup n_0$ will give a v dimensional modified SOSRD in

$$N=\frac{(r2^{t(k)}+n_a 2^{t(v)}a^2)^2}{\lambda 2^{t(k)}+n_a 2^{t(v)}a^4}$$

design points if,

$$a^4=\frac{(r-5\lambda)2^{t(k)-t(v)-2}}{n_a}, \tag{3.6}$$

$$n_0=\frac{(r2^{t(k)}+n_a 2^{t(v)}a^2)^2}{\lambda 2^{t(k)+n_a 2^{t(v)}a^4}}-(b2^{t(k)}+n_a 2^{t(v)}) \tag{3.7}$$

and n_0 turns out to be an integer.

3.3 Modified SOSRD using SUBA with two unequal block sizes

SUBA with two unequal block sizes: The arrangement of v treatments in b blocks where b_1 blocks of size k_1, and b_2 blocks of size k_2 is said to be a SUBA with two unequal block sizes, if

(i) every treatment occurs $\frac{b_i k_i}{v}$ blocks of size k_i $(i=1,2)$, and

(ii) every pair of first associate treatments occurs together in u blocks of size k_1 and in $(\lambda - u)$ blocks of size k_2 while every pair of second associate treatments occurs together in λ blocks of size k_2.

From (i) each treatment occurs in $(\frac{b_1 k_1}{v}) + (\frac{b_2 k_2}{v}) = r$ blocks in the whole design. $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ are known as the parameters of the SUBA with two unequal block sizes.

Let $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$, $k = \max(k_1, k_2)$, and $b_1 + b_2 = b$ be a SUBA with two unequal block sizes. $2^{t(k)}$ denotes a resolution V fractional factorial of 2^k with $+1$ or -1 levels, such that no interaction with less than five factors is confounded. $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]$ denote the design points generated from the transpose of incidence matrix of SUBA with two unequal block sizes, $[1-(v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from SUBA with two unequal block sizes by 'multiplication'.

Let $(a, 0, 0, \ldots, 0)2^1$ denote the design points generated from $(a, 0, 0, \ldots, 0)$ point set. Repeat this set of additional design points say n_a times when $r < 5\lambda$. Consider the design points, $[1-(v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup n_a(a, 0, \ldots, 0)2^1 \cup (n_0)$ will give a v dimensional modified SOSRD in $N = \frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ design points if,

$$a^4 = \frac{(5\lambda - r)2^{t(k)-1}}{n_a}, \tag{3.8}$$

$$n_0 = \frac{(r2^{t(k)} + 2n_a a^2)^2}{\lambda 2^{t(k)}} - [b2^{t(k)} + 2n_a v] \tag{3.9}$$

and n_0 turns out to be an integer.

Consider the design points, $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup n_0$ will give a three level v dimensional modified SOSRD in $N = \frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}}$ design points when $r = 5\lambda$, if,

$$n_0 = \frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}} - b2^{t(k)} \tag{3.10}$$

and n_0 turns out to be an integer.

Let $(a, a, \ldots, a)2^{t(v)}$ denote the design points generated from $(a, a, \ldots, a)$ point set. Repeat this set of additional design points say n_a times when $r > 5\lambda$. Consider the design points, $[1-(v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup n_a(a, a, \ldots, a)2^{t(v)} \cup n_0$ will give a v dimensional modified SOSRD in

$$N = \frac{(r2^{t(k)} + n_a 2^{t(v)} a^2)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4}$$

design points if,

$$a^4 = \frac{(r - 5\lambda)2^{t(k)-t(v)-2}}{n_a}, \tag{3.11}$$

$$n_0 = \frac{(r2^{t(k)} + n_a 2^{t(v)} a^2)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4} - (b2^{t(k)} + n_a 2^{t(v)}) \tag{3.12}$$

and n_0 turns out to be an integer.

3.4 Modified SOSRD using PBD

Pairwise balanced designs. The arrangement of v treatments in b blocks will be called a PBD of index λ and type $(v, k_1, k_2, \ldots, k_m)$ if each block contains $k_1, k_2, \ldots, k_m$ treatments $(k_i \le v, k_i \ne k_j)$ and every pair of distinct treatments occurs in exactly λ blocks of the design. If b_i is the number of blocks of size k_i $(i = 1, 2, \ldots, m)$ then $b = \sum_{i=1}^{m} b_i$ and $\lambda v(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1)$.

Let $(v, b, r, k_1, k_2, \ldots k_p, \lambda)$, be an equi-replicated PBD and $k = \max(k_1, k_2, \ldots, k_m)$. $2^{t(k)}$ denotes a resolution V fractional factorial of 2^k with 1 or -1 levels, such that no interaction with less than five factors is confounded. $[1-(v, b, r, k_1, k_1, \ldots k_p, \lambda)]$ denote the design points generated from the transpose of incidence matrix of PBD, $[1-(v, b, r, k_1, k_2, \ldots k_p, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by multiplication.

Let $(a, 0, 0, \ldots, 0)2^1$ denote the design points generated from $(a, 0, 0, \ldots, 0)$ point set. Repeat this set of additional design points say n_a times when $r < 5\lambda$. Consider the design points, $[1-(v, b, r, k_1, k_2, \ldots, k_p, \lambda)]2^{t(k)} \cup n_a(a, 0, \ldots, 0)2^1 \cup (n_0)$ will give a v dimensional modified SOSRD in

$$N = \frac{(r2^{t(k)} + 2n_a a^2)^2}{\lambda 2^{t(k)}}$$

a design points if,

$$a^4 = \frac{(5\lambda - r)2^{t(k)-1}}{n_a}, \tag{3.13}$$

$$n_0 = \frac{(r2^{t(k)} + 2n_a a^2)^2}{\lambda 2^{t(k)}} - (b2^{t(k)} + 2n_a v) \tag{3.14}$$

and n_0 turns out to be an integer.

Consider the design points, $[1-(v, b, r, k_1, k_2, \ldots, k_p, \lambda)]2^{t(k)} \cup n_0$ will give a three level v dimensional modified SOSRD in $N = \frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}}$ design points when $r = 5\lambda$, if

$$n_0 = \frac{(r2^{t(k)})^2}{\lambda 2^{t(k)}} - b2^{t(k)} \tag{3.15}$$

and n_0 turns out to be an integer.

Let $(a, a, \ldots, a)2^{t(v)}$ denote the design points generated from $(a, a, \ldots, a)$ point set. Repeat this set of additional design points say n_a times when $r > 5\lambda$. Consider the design points, $[1-(v, b, r, k_1, k_2, \ldots, k_p, \lambda)]2^{t(k)} \cup n_a(a, a, \ldots, a)2^{t(v)} \cup n_0$ will give a v dimensional modified SOSRD in

$$N = \frac{(r2^{t(k)} + n_a 2^{t(v)} a^2)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4}$$

design points if,

$$a^4 = \frac{(r - 5\lambda)2^{t(k)-t(v)-2}}{n_a}, \tag{3.16}$$

$$n_0 = \frac{(r2^{t(k)} + n_a 2^{t(v)} a^2)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4} - (b2^{t(k)} + n_a 2^{t(v)}) \tag{3.17}$$

and n_0 turns out to be an integer.

4 Concluding Remarks

Different methods of constructions of modified SOSRD were examined in detail. These methods are useful in deciding a proper design for second-order response surface polynomial models for the construction of modified SOSRD with desired properties or minimum number of design points. It is often necessary to choose a response surface design in which the number of levels of factors are unequal and in such a case modified second-order asymmetric slope-rotatable designs are useful.

Another area in which one may be interested is to study modified group-divisible second-order slope-rotatable designs. Not much work is available with regard to construction of designs in this area. It may be interesting to study some new methods of constructions of modified group-divisible second-order slope-rotatable designs using central composite designs, balanced incomplete block designs, pairwise balanced designs, etc.,

There is scope for further research to evolve new methods of constructions of modified SOSRD which will lead to designs with lesser number of design points for different v compared as per the existing methods of construction vide Table 2. A comparison of different methods of constructions of modified SOSRD for $2 \leq v \leq 16$ is given in Table 1.

A list of modified SOSRD with minimum number of design points constructed using different methods is given in Table 2.

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Table 1: Comparison of different methods of construction of modified SOSRD

No. of factors	Modified SRCCD (2005a)	Modified SOSRD using BIBD (2006a)	Modified SOSRD using PBD (2005a)	Modified SOSRD using SUBA with two unequal block sizes (2008b)
(v)	N	N	N	N
2	36	–	–	–
3	32	–	–	–
4	64	64	–	–
		(4,6,3,2,1)	–	–
5	64	150	–	–
		(5,10,6,3,3)	–	–
6	72	100	128	128
		(6,15,5,2,1)	(7,7,3,3,1)	(6,7,3,2,3,3,4,1)
7	144	128	–	–
8	144	432	392	162
		(8,14,7,4,3)	(8,15,6,4,3,2,2)	(8,12,4,2,3,4,8,1)
9	200	162	392	200
		(9,12,4,3,1)	(9,15,6,4,3,2)	(9,18,5,2,3,9,9,1)
10	200	361	–	–
		(10,18,9,5,4)		
11	200	600	–	–
		(11,55,15,3,3)		
12	400	768	676	–
		(12,33,11,4,3)	(12,16,6,6,5,4,3,2)	
13	400	400	676	–
		(13,13,4,4,1)	(13,16,6,6,5,4,3,2)	
14	400	–	676	900
			(14,16,6,6,5,4,2)	(14,35,7,2,3,7,28,1)
15	400	1200	676	–
		(15,15,7,7,3)	(15,16,6,6,5,2)	
16	400	400	–	–
		(16,20,5,4,1)		

Table 2: Modified SOSRD with minimum number of design points for $2 \le v \le 16$

No. of factors	1st best Modified SOSRD	2nd best Modified SOSRD
2	Modified SRCCD, $N = 36$	–
3	Modified SRCCD, $N = 32$	–
4	Modified SRCCD, $N = 64$ Modified SOSRD-BIBD (4,6,3,2,1), $N = 64$	–
5	Modified SRCCD, $N = 64$	Modified SOSRD-BIBD (5,10,6,3,3), $N = 150$
6	Modified SRCCD, $N = 72$	Modified SOSRD-BIBD (6,15,5,2,1), $N = 100$
7	Modified SOSRD-BIBD (7,7,3,3,1), $N = 128$	Modified SRCCD, $N = 144$
8	Modified SRCCD, $N = 144$	Modified SOSRD-SUBA (8,12,4,2,3,4,8,1), $N = 162$
9	Modified SOSRD-BIBD (9,12,4,3,1), $N = 162$	Modified SRCCD, $N = 200$ Modified SOSRD-SUBA (9,18,5,2,3,9,9,1), $N = 200$
10	Modified SRCCD, $N = 200$	Modified SOSRD-BIBD (10,18,9,5,4), $N = 361$
11	Modified SRCCD, $N = 200$	Modified SOSRD-BIBD (11,55,15,3,3), $N = 600$
12	Modified SRCCD, $N = 400$	Modified SOSRD-PBD (12,16,6,6,5,4,3,2), $N = 676$
13	Modified SRCCD, $N = 400$ Modified SOSRD-BIBD (13,13,4,4,1), $N = 400$	Modified SOSRD-PBD (13,16,6,6,5,4,3,2), $N = 676$
14	Modified SRCCD, $N = 400$	Modified SOSRD-PBD (14,16,6,6,5,4,2), $N = 676$
15	Modified SRCCD, $N = 400$	Modified SOSRD-PBD (15,16,6,6,5,2), $N = 676$
16	Modified SRCCD, $N = 400$ Modified SOSRD-BIBD (16,20,5,4,1), $N = 400$	—

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