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A Characterization of Continuous Distributions through Lower Record Statistics

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Abstract In this paper, a family of continuous probability distribution is characterized through conditional expectations, conditioned on a non-adjacent lower record statistic. Also, an equivalence between the unconditional expectation and a conditional expectation is used to characterize a family of distributions.

Keywords. Characterization, continuous distributions, conditional expectation, lower record statistics.
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1 Introduction

Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables. Let $Y_j = \max(X_1, X_2, \ldots, X_j)$ for $j \geq 1$. We say X_j an upper record if $Y_j > Y_{j-1}$, $j > 1$. An analogous definition deals with lower record values.

Let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ be the first r lower record statistics from a population whose probability density function (*pdf*) is $f(x)$ and the distribution function (*df*) is $F(x)$. Let $H(x) = -\log F(x)$. Then the *pdf* of $X_{L(r)}$ $r = 1, 2, \ldots$ is

$$f_r(x) = \frac{(H(x))^{r-1} f(x)}{\Gamma(r)}, \quad -\infty < x < \infty, \tag{1.1}$$

and the joint *pdf* of two lower records $X_{L(r)}$ and $X_{L(s)}$, $r < s$, $r, s = 1, 2, \ldots$ is

$$f_{r,s}(x, y) = \frac{1}{\Gamma(r)\Gamma(s-r)} (H(x))^{r-1} [H(y) - H(x)]^{s-r-1} h^*(x) f(y) \tag{1.2}$$

where $h^*(x) = -\frac{d\,H(x)}{dx}$, Ahsanullah [1] and Arnold *et al.* [2].

Then the conditional *pdf* of $X_{L(s)}$ given $X_{L(r)} = x$, $1 \leq r < s$, is

$$f\left(X_{L(s)} \middle| X_{L(r)} = x\right) = \frac{1}{\Gamma(s-r)} [-\ln F(y) + \ln F(x)]^{s-r-1} \frac{f(y)}{F(x)}. \tag{1.3}$$

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records, e.g.Olympic records. It is also useful in reliability theory. The record values have been extensively studied in literature. For an excellent review, one may refer to Ahsanullah [1], Arnold *et al.* [2] and Nevzorov [4] amongst others. Malinowska and Szynal [3] and Shawky and Bakoban [5] have characterized a wide class of distributions through the conditional expectation of adjacent lower record. In this paper, we characterize a wide class of distributions by considering the conditional expectations conditioned on non-adjacent lower record.

2 Characterization Theorems

Theorem 2.1 *Let X be an absolutely continuous random variable with df $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for, $1 \leq r < s$,*

$$E\left[h\left(X_{L(s)}\right) \middle| X_{L(r)} = x\right] = h(x) + (s-r)c \tag{2.1}$$

if and only if

$$F(x) = e^{-\frac{h(x)}{c}}, \ c > 0, x \in (\alpha, \beta). \tag{2.2}$$

Where $h(x)$ is a non-increasing and differentiable function of x with $h(\alpha) = \infty$, $h(\beta) = 0$, such that, $h(x)F(x) \to 0$, as $x \to \alpha$.

Proof. For necessary part, we have

$$\begin{aligned} &E\left[h\left(X_{L(x)}\right) \middle| X_{L(r)} = x\right] \\ &\qquad = \frac{1}{\Gamma(s-r)} \int_\alpha^x h(y) \left[-\ln F(y) + \ln F(x)\right]^{s-r-1} \frac{f(y)}{F(x)} dy \\ &\qquad = \frac{1}{\Gamma(s-r)} \frac{1}{c^{s-r}} \frac{1}{F(x)} \\ &\qquad \times \int_\alpha^x h(y) \left[h(y) - h(x)\right]^{s-r-1} (-)h'(y) e^{-\frac{h(y)}{c}} dy. \end{aligned} \tag{2.3}$$

After simplification, we get

$$E\left[h\left(X_{L(s)}\right) \middle| X_{L(r)} = x\right] = h(x) + (s-r)c.$$

For the sufficiency part, consider

$$g_{s|r}(x) = h(x) + (s-r)c.$$

Hence, by (2.1), we have

$$\frac{1}{\Gamma(s-r)} \int_\alpha^x h(y) \left[-\ln F(y) + \ln F(x)\right]^{s-r-1} f(y) dy = g_{s|r}(x) F(x). \tag{2.4}$$

Differentiating both sides of (2.4) w.r.t. x, we get

$$\frac{(s-r-1)}{\Gamma(s-r)}\int_{\alpha}^{x} h(y)\left[-\ln F(y)+\ln F(x)\right]^{s-r-2}\frac{f(x)}{F(x)}f(y)dy \\ = g'_{s|r}(x)F(x)+g_{s|r}(x)f(x)$$

and hence

$$g_{s|r+1}(x)=g'_{s|r}(x)\frac{F(x)}{f(x)}+g_{s|r}(x).$$

Therefore,

$$\frac{f(x)}{F(x)}=\frac{g'_{s|r}(x)}{g_{s|r+1}(x)-g_{s|r(x)}} \text{ and } F(x)=e^{-\int_x^{\beta}A(t)dt}$$

where

$$A(x)=\frac{g'_{s|r}(x)}{\left[g_{s|r+1}(x)-g_{s|r}(x)\right]}=\frac{-}{h'(x)}c$$

and hence the result in (2.2).

Theorem 2.2 *Under the conditions as given in the Theorem 2.1 and, for* $1\le r\le s<t$,

$$E\left[h\left(X_{L(t)}\right)-h\left(X_{L(s)}\right)\middle|X_{L(r)}=x\right]=(t-s)c \tag{2.5}$$

if and only if (2.2) *holds.*

Proof. Now it is easy to see that (2.2) implies (2.5) and hence the necessary part. For the sufficiency part, let $c*=(t-s)c$, then

$$\frac{1}{\Gamma(t-r)}\int_{\alpha}^{x} h(y)\left[-\ln F(y)+\ln F(x)\right]^{t-r-1}f(y)dy \\ -\frac{1}{\Gamma(s-r)}\int_{\alpha}^{x} h(y)\left[-\ln F(y)+\ln F(x)\right]^{s-r-1}f(y)dy=c*F(x). \tag{2.6}$$

Differentiating $(s-r)$ times both sides of (2.6) w.r.t. x, we get

$$\frac{1}{\Gamma(t-s)}\int_{\alpha}^{x} h(y)\left[-\ln F(y)+\ln F(x)\right]^{t-s-1}\frac{f(y)}{F(x)}dy=h(x)+c*. \tag{2.7}$$

Integrating left-hand side of (2.7) by parts and simplifying, we have

$$\frac{1}{\Gamma(t-s-1)F(x)}\int_{\alpha}^{x} h(y)\left[-\ln F(y)+\ln F(x)\right]^{t-s-2}f(y)dy \\ -\frac{1}{\Gamma(t-s)F(x)}\int_{\alpha}^{x} h'(y)\left[-\ln F(y)+\ln F(x)\right]^{t-s-1}F(y)dy=h(x)+c*. \tag{2.8}$$

Thus in view of (2.7), reduces to

$$\frac{1}{\Gamma(t-s)}\int_{\alpha}^{x} h'(y)\left[-\ln F(y)+\ln F(x)\right]^{t-s-1} F(y)dy = -cF(x). \tag{2.9}$$

Differentiating $(t-s)$ times both sides of (2.9) w.r.t. x, we obtain

$$h'(x)F(x) = -cf(x)$$

and hence the Theorem.

Remark 2.1 *At $r = s$ Theorem 2.2 reduces to Theorem 2.1.*

Corollary 2.2.1 *Under the conditions as given in Theorem 2.1 and, for $1 \le r < s < t$,*

$$E\left[h\left(X_{L(t)}\right) - h\left(X_{L(s)}\right)\right] + h(x) = E\left[h\left(X_{L(t)}\right)\middle| X_{L(s)} = x\right] \tag{2.10}$$

if and only if (2.2) holds.

Proof. Corollary follows from Theorem 2.1 and Theorem 2.2.

A number of distributions can be characterized by the proper choice of c and $h(x)$.

Table 1: Examples based on distribution function $F(x) = e^{-\frac{h(x)}{c}}$

Distribution	$F(x)$	c	$h(x)$
Power function	$\left(\frac{x}{a}\right)^p \; 0 < x < a$	$-\frac{1}{p}$	$\log(x/a)$
Logistic	$\left(1+e^{-x}\right)^{-1} \; -\infty < x < \infty$	1	$\log\left(1+e^{-x}\right)$
Burr Type II	$\left(1+e^{-x}\right)^{-\theta} \; -\infty < x < \infty$	$\frac{1}{\theta}$	$\log\left(1+e^{-x}\right)$
Burr Type III	$\left(1+x^{-c}\right)^{-k} \; 0 < x < \infty$	$\frac{1}{k}$	$\log\left(1+x^{-c}\right)$
Burr Type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k} \; 0 < x < c$	$\frac{1}{k}$	$\log\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]$
Burr Type V	$\left(1+ce^{-\tan x}\right)^{-k} \; -\frac{\pi}{2} < x < \frac{\pi}{2}$	$\frac{1}{k}$	$\log\left(1+ce^{-\tan x}\right)$
Burr Type VI	$\left(1+ce^{-k\sinh x}\right)^{-k} \; -\infty < x < \infty$	$\frac{1}{k}$	$\log\left(1+ce^{-k\sinh x}\right)$
Burr Type VII	$\left(\frac{1+\tanh x}{2}\right)^{k} \; -\infty < x < \infty$	$-\frac{1}{k}$	$\log\left(\frac{1+\tanh x}{2}\right)$
Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1} e^{x}\right)^{k} \; -\infty < x < \infty$	$-\frac{1}{k}$	$\log\left(\frac{2}{\pi}\tan^{-1} e^{x}\right)$
Burr Type X	$\left(1-e^{-x^2}\right)^{k} \; 0 < x < \infty$	$-\frac{1}{k}$	$\log\left(1-e^{-x^2}\right)$
Burr Type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k} \; 0 < x < 1$	$-\frac{1}{k}$	$\log\left(x-\frac{1}{2\pi}\sin 2\pi x\right)$
Gumbel	$\exp\left[-e^{-x}\right] \; -\infty < x < \infty$	1	e^{-x}
Inverse Weibull	$e^{-\left(\frac{\theta}{x}\right)^p} \; 0 < x < \infty$	$\frac{1}{\theta^p}$	x^{-p}

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