ISSN 0974-3235
ProbStat Forum, Volume 4, January 2011, Pages 25–38

Lower generalized order statistics from exponentiated gamma distribution and its characterization

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Paper received on 01 April 2010; revised, 13 October 2010; accepted, 11 November 2010.

Abstract.
In this paper we establish explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics from exponentiated gamma distribution. The results include as particular cases the above relations for moments of order statistics and lower records. Further, using a recurrence relation for single moments we obtain characterization of exponentiated gamma distribution.

Keywords. Lower generalized order statistics, order statistics, lower records, single moments, product moments, recurrence relations, exponentiated gamma distribution and characterization
AMS Subject Classification. 62G30, 62E10

1 Introduction

The concept of generalized order statistics (gos) was introduced by Kamps [6] as below: Let $F()$ be an absolutely continuous distribution function (df) with probability density function (pdf) $f()$. Further, let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$, for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called gos if their joint pdf is given by

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\Big)[1-F(x_n)]^{k-1} f(x_n)$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^{n-1}$. The model of gos contains as special cases, order statistics, sequential order statistics, Stigler's order statistics and record values. But when $F()$ is an inverse distribution function, we need a concept of lower generalized order statistics (l *gos*), which was introduced by Pawlas and Szynal [11] as follows:

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + n - r + M_r > 0,\ M_r = \sum_{j=r}^{n-1} m_j, \quad \forall\, 1 \leq r \leq n.$$

By the l *gos* from an absolutely continuous distribution function $F()$ with density function $f()$ we mean random variables $X'\ (1, n, \tilde{m}, k), \ldots, X'(n, n, \tilde{m}, k)$ having joint density function of the form

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} [F(x_i)]^{m_i} f(x_i)\Big)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0)$.
For simplicity we shall assume $m_1 = m_2 = \cdots = m_{n-1} = m$.
The *pdf* of r-th l *gos* is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{1.2}$$

The joint *pdf* of r-th and s-th l *gos* is

$$\begin{aligned} f_{X'(r,n,m,k),X'(s,n,m,k)}(x, y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad \alpha \leq y < x \leq \beta, \end{aligned} \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1} x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0, 1).$$

We shall also take $X'(0, n, m, k) = 0$. If $m = 0$, $k = 1$, then $X'(r, n, m, k)$ reduces to the $(n-r+1)$-th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots,$ X_n and when $m = -1$, then $X'(r, n, m, k)$ reduces to the r-th k-lower record value [Pawlas and Szynal [11]]. The work of Burkschat *et al.* [3] may also refer for lower generalized order statistics.

Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution are derived by Pawlas and Szynal [11]. Khan *et al.* [8] have established recurrence relations for moments of lower generalized order statistics from exponentiated Weibull distribution. Ahsanullah [1] and Mbah and Ahsanullah [9] characterized the uniform and power function distributions based on distributional properties of lower generalized order statistics respectively. Kamps [7] investigated the importance of recurrence relations of order statistics in characterization.

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of *l gos* from exponentiated gamma distribution. Result for order statistics and k-th lower record values are deduced as special cases and a characterization of exponentiated gamma distribution has been obtained on using a recurrence relation for single moments.

A random variable X is said to have exponentiated gamma distribution (Gupta *et al.* [4]) if its *pdf* is of the form

$$f(x) = \theta x e^{-x}[1 - e^{-x}(x+1)]^{\theta-1}, \quad x > 0,\ \theta > 0 \tag{1.4}$$

and the corresponding *df* is

$$F(x) = [1 - e^{-x}(x+1)]^{\theta}, \quad x > 0,\ \theta > 0. \tag{1.5}$$

For $\theta = 1$, the above distribution corresponds to the gamma distribution $G(2,1)$.

For application of the distribution one may refer to Nadarajah [10] and Shawky and Bakoban [12].

2 Relations for single moments

Note that for exponentiated gamma distribution

$$F(x) = \frac{e^{x}(1 - e^{-x}(x+1))}{\theta x} f(x). \tag{2.1}$$

The *pdf* (1.2) can be written for the exponentiated gamma distribution with *pdf* (1.4) and *df* (1.5) in the following form

$$f_{X'(r,n,m,k)}(x) = \begin{cases} \frac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} x e^{-x}(1 - e^{-x}(x+1))^{\theta\gamma_r - 1} \\ \qquad [1 - (1 - e^{-x}(x+1))^{\theta(m+1)}]^{r-1}, & m \neq -1 \\ \frac{\theta^r k^r}{(r-1)!} x e^{-x}(1 - e^{-x}(x+1))^{\theta k - 1} \\ \qquad [-\ln(1 - e^{-x}(x+1))]^{r-1}, & m = -1 \end{cases} \tag{2.2}$$

By using binomial and logarithmic expansions we can rewrite (2.2) as

$$f_{X'(r,n,m,k)}(x) = \begin{cases} \dfrac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1} \sum_{b=0}^{\infty} \sum_{c=0}^{b} (-1)^{a+b} \binom{r-1}{a} \\ \binom{\theta(\gamma_r+(m+1)a)-1}{b} \binom{b}{c} x^{c+1} e^{-(b+1)x}, & m \neq -1 \\ \dfrac{\theta^r k^r}{(r-1)!} \sum_{a=0}^{\infty} \sum_{t=0}^{\infty} \sum_{b=0}^{r-1+t+a} (-1)^a \alpha_t(r-1) \binom{\theta k-1}{a} \\ \binom{r-1+t+a}{b} x^{b+1} e^{-(r+t+a)x}, & m = -1 \end{cases} \tag{2.3}$$

where $\alpha_t(r-1)$ is the coefficient of $e^{-(r-1+t)x}(x+1)^{r-1+t}$ in the expansion of $(\sum_{u=1}^{\infty} \frac{e^{-ux}(x+1)^u}{u})^{r-1}$ [see Balakrishnan and Cohan [2], Shawky and Bakoban [13]].

We shall first establish the explicit formula for $E[X'^j(r,n,m,k)]$. Using (2.3), we obtain when $m \neq -1$

$$\begin{aligned} E[X'^j(r,n,m,k)] &= \frac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1} \sum_{b=0}^{\infty} \sum_{c=0}^{b} (-1)^{a+b} \binom{r-1}{a} \\ &\times \binom{\theta(\gamma_r+(m+1)a)-1}{b} \binom{b}{c} \int_0^{\infty} x^{j+c+1} e^{-(b+1)x} dx \\ &= \frac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1} \sum_{b=0}^{\infty} \sum_{c=0}^{b} (-1)^{a+b} \binom{r-1}{a} \\ &\times \binom{\theta(\gamma_r+(m+1)a)-1}{b} \binom{b}{c} \frac{\Gamma(j+c+2)}{(b+1)^{j+c+2}} \end{aligned} \tag{2.4}$$

and when $m = -1$ that

$$\begin{aligned} E[X'^j(r,n,-1,k)] &= \frac{\theta^r k^r}{(r-1)!} \sum_{a=0}^{\infty} \sum_{t=0}^{\infty} \sum_{b=0}^{r-1+t+a} (-1)^a \alpha_t(r-1) \binom{\theta k-1}{a} \\ &\times \binom{r-1+t+a}{b} \int_0^{\infty} x^{j+b+1} e^{-(r+t+a)x} dx \\ &= \frac{\theta^r k^r}{(r-1)!} \sum_{a=0}^{\infty} \sum_{t=0}^{\infty} \sum_{b=0}^{r-1+t+a} (-1)^a \alpha_t(r-1) \binom{\theta k-1}{a} \\ &\times \binom{r-1+t+a}{b} \frac{\Gamma(j+b+2)}{(r+t+a)^{j+b+2}}. \end{aligned} \tag{2.5}$$

If θ is a positive integer, the relations (2.4) and (2.5) then give

$$E[X'^j(r,n,m,k)] = \frac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1} \sum_{b=0}^{\theta(\gamma_r+(m+1)a)-1} \sum_{c=0}^{b} (-1)^{a+b} \binom{r-1}{a} \times \binom{\theta(\gamma_r+(m+1)a)-1}{b} \binom{b}{c} \frac{\Gamma(j+c+2)}{(b+1)^{j+c+2}} \tag{2.6}$$

and

$$E[X'^j(r,n,-1,k)] = \frac{\theta^r k^r}{(r-1)!} \sum_{a=0}^{\theta k-1} \sum_{t=0}^{\infty} \sum_{b=0}^{r-1+t+a} (-1)^a \alpha_t(r-1) \binom{\theta k-1}{a} \times \binom{r-1+t+a}{b} \frac{\Gamma(j+b+2)}{(r+t+a)^{j+b+2}}. \tag{2.7}$$

Special cases

1. Putting $m=0$, $k=1$ in (2.6), the explicit formula for single moments of order statistics of the exponentiated gamma distribution can be obtained as

$$E(X^j_{n-r+1:n}) = \theta C_{r:n} \sum_{a=0}^{r-1} \sum_{b=0}^{\theta(n-r+1+a)-1} \sum_{c=0}^{b} (-1)^{a+b} \binom{r-1}{a} \times \binom{\theta(n-r+1+a)-1}{b} \binom{b}{c} \frac{\Gamma(j+c+2)}{(b+1)^{j+c+2}},$$

   where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$

   For $n-r+1=r$, the result was obtained by Shawky and Bakoban [14].

2. Putting $k=1$ in (2.5) and (2.7), we deduce the explicit expressions for the moments of lower record values for the exponentiated gamma distribution, established by Shawky and Bakoban [13].

A recurrence relation for moments of *l gos* from *df* (1.5) can be obtained in the following theorem.

Theorem 2.1 *For the distribution given in* (1.5) *and for* $2 \le r \le n$, $n \ge 2$ *and* $k = 1, 2, \ldots$

$$E[X'^j(r,n,m,k)] - E[X'^j(r-1,n,m,k)] = \frac{j}{\theta\gamma_r}\{E[X'^{j-1}(r,n,m,k)] + E[X'^{j-2}(r,n,m,k)] - E[\phi(X'(r,n,m,k))]\}, \tag{2.8}$$

where $\phi(x) = x^{j-2}e^x$.

Proof. From (1.2), we have

$$E[X'^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \qquad (2.9)$$

Integrating by parts taking $[F(x)]^{\gamma_r-1}f(x)$ as the part to be integrated, we get

$$E[X'^j(r,n,m,k)] = E[X'^j(r-1,n,m,k)] - \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx$$

the constant of integration vanishes since the integral considered in (2.9) is a definite integral. On using (2.1), we obtain

$$\begin{aligned}&E[X'^j(r,n,m,k)]\\ &=E[X'^j(r-1,n,m,k)] - \frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\Big\{\int_0^\infty x^{j-2}e^x[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\ &\quad - \int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\ &\quad - \int_0^\infty x^{j-2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\Big\}\end{aligned}$$

and hence the result.

Remark 2.1 *Putting $m=0$, $k=1$, in (2.8), we obtain a recurrence relation for single moments of order statistics of the exponentiated gamma distribution in the form*

$$E(X^j_{n-r+1:n}) - E(X^j_{n-r+2:n}) = \frac{j}{\theta(n-r+1)}\{E(X^{j-1}_{n-r+1:n}) + E(X^{j-2}_{n-r+1:n}) - E(\phi(X_{n-r+1:n}))\}.$$

Remark 2.2 *Setting $m=-1$ and $k \geq 1$, in Theorem 2.1, we get a recurrence relation for single moments of lower k-th record values from exponentiated gamma distribution in the form*

$$\begin{aligned}&E[X'^j(r,n,-1,k)] - E[X'^j(r-1,n,-1,k)]\\ &= \frac{j}{\theta k}\{E[X'^{j-1}(r,n,-1,k)] + E[X'^{j-2}(r,n,-1,k)] - E[\phi(X'(r,n,-1,k))]\}.\end{aligned}$$

3 Relations for product moments

On using (1.3), (1.4), (1.5) and binomial, logarithmic expansions the explicit expressions for the product moments of *l gos* $X'^i(r,n,m,k)$ and $X'^j(s,n,m,k)$,

$1 \le r < s \le n$, can be obtained when $m \ne -1$ as

$$E[X'^{i}(r,n,m,k)\,X'^{j}(s,n,m,k)] \\ = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{a=0}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b} \\ \times \int_0^\infty y^j [F(y)]^{\gamma_{s-b}-1} f(y) I(y) dy, \quad (3.1)$$

where

$$\begin{aligned} I(y) &= \int_y^\infty x^i [F(x)]^{(s-r+a-b)(m+1)-1} f(x) dx \\ &= \theta \sum_{c=0}^{\infty} \sum_{d=0}^{c} (-1)^c \binom{\theta(s-r+a-b)(m+1)-1}{c} \binom{c}{d} \int_y^\infty x^{i+d+1} e^{-(1+c)x} dx \\ &= \theta \sum_{c=0}^{\infty} \sum_{d=0}^{c} (-1)^c \binom{\theta(s-r+a-b)(m+1)-1}{c} \binom{c}{d} \\ &\times \sum_{t=0}^{i+d+1} \frac{e^{-(1+c)y}((1+c)y)^t}{t!} \frac{\Gamma(i+d+2)}{(1+c)^{i+d+2}}. \end{aligned}$$

On substituting the above expression of $I(y)$ in (3.1), we find that

$$E[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)] \\ = \frac{\theta^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{a=0}^{r-1} \sum_{b=0}^{s-r-1} \sum_{c=0}^{\infty} \sum_{d=0}^{c} \sum_{t=0}^{i+d+1} \sum_{u=0}^{\infty} \sum_{v=0}^{u} (-1)^{a+b+c+u} \\ \times \binom{r-1}{a} \binom{s-r-1}{b} \binom{\theta(s-r+a-b)(m+1)-1}{c} \binom{c}{d} \\ \times \binom{\theta\gamma_{s-b}-1}{u} \binom{u}{v} \frac{\Gamma(i+d+2)}{t!(1+c)^{i+d+2-t}} \frac{\Gamma(j+t+v+2)}{(c+2+u)^{j+t+v+2}} \quad (3.2)$$

and when $m = -1$ that

$$E[X'^{i}(r,n-1,k)X'^{j}(s,n,-1,k)] \\ = \frac{k^s}{(r-1)!(s-r-1)!} \int_0^\infty y^j [F(y)]^{k-1} f(y) I(y) dy, \quad (3.3)$$

where

$$\begin{aligned}I(y) &= \int_y^\infty x^i[-\ln F(x)]^{r-1}[-\ln F(y)+\ln F(x)]^{s-r-1}\frac{f(x)}{F(x)}dx\\ &= \sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{t=0}^{\infty}\sum_{c=0}^{s-2-a+t+b}\sum_{d=0}^{i+c+1}(-1)^{s-r-1}\theta^{s-1-a}\alpha_t(s-2-a)\\ &\times\binom{s-r-1}{a}\binom{s-2-a+t+b}{c}\\ &\times\frac{\Gamma(i+2+c)}{d!(s-1-a+t+b)^{i+2+c-d}}[\ln F(y)]^a e^{-(s-1-a+t+b)y}y^d.\end{aligned}$$

On substituting the above expression of $I(y)$ in (3.3) and simplifying the resulting equation, we obtain

$$\begin{aligned}&E[X'^i(r,n,-1,k)X'^j(s,n,-1,k)]\\ &=\frac{\theta^s k^s}{(r-1)!(s-r-1)!}\sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{t=0}^{\infty}\sum_{c=0}^{s-2-a+t+b}\sum_{d=0}^{i+c+1}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\sum_{w=0}^{a+u+v}(-1)^{s-r-1+a+v}\\ &\times\alpha_t(s-2-a)\alpha_u(a)\binom{s-r-1}{a}\binom{s-2-a+t+b}{c}\binom{\theta k-1}{v}\binom{a+u+v}{w}\\ &\times\frac{\Gamma(i+2+c)\Gamma(j+d+w+2)}{d!(s-1-a+t+b)^{i+2+c-d}(s+b+t+v+u)^{j+d+w+2}}.\end{aligned}\tag{3.4}$$

If θ is a positive integer, then the relations (3.2) and (3.4) take the form

$$\begin{aligned}&E[X'^i(r,n,m,k)X'^j(s,n,m,k)]\\ &=\frac{\theta^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{\theta(s-r+a-b)(m+1)-1}\sum_{d=0}^{c}\sum_{t=0}^{i+d+1}\sum_{u=0}^{\theta\gamma_{s-b}-1}\\ &\times\sum_{v=0}^{u}(-1)^{a+b+c+u}\binom{r-1}{a}\binom{s-r-1}{b}\binom{\theta(s-r+a-b)(m+1)-1}{c}\\ &\times\binom{c}{d}\binom{\theta\gamma_{s-b}-1}{u}\binom{u}{v}\\ &\times\frac{\Gamma(i+d+2)\Gamma(j+t+v+2)}{t!(1+c)^{i+d+2-t}(c+2+u)^{j+t+v+2}}.\end{aligned}\tag{3.5}$$

$$\begin{aligned}
&E[X'^{i}(r,n,-1,k)X'^{j}(s,n-1,k)]\\
&= \frac{\theta^s k^s}{(r-1)!(s-r-1)!}\sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{t=0}^{\infty}\sum_{c=0}^{s-2-a+t+b}\sum_{d=0}^{i+c+1}\sum_{v=0}^{\theta k-1}\sum_{u=0}^{\infty}\sum_{w=0}^{a+u+v}\\
&\times(-1)^{s-r-1+a+v}\alpha_t(s-2-a)\alpha_u(a)\binom{s-r-1}{a}\binom{s-2-a+t+b}{c}\\
&\times\binom{\theta k-1}{v}\binom{a+u+v}{w}\\
&\times\frac{\Gamma(i+2+c)\Gamma(j+d+w+2)}{d!(s-1-a+t+b)^{i+2+c-d}(s+b+t+v+u)^{j+d+w+2}}. \qquad (3.6)
\end{aligned}$$

Special cases

1. Putting $m=0$, $k=1$ in (3.5) the explicit formula for the product moments of order statistics of the exponentiated gamma distribution to be obtained as

$$\begin{aligned}
&E(X^i_{n-r+1:n}X^j_{n-s+1:n}) = \theta^2 C_{r,s:n}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{\theta(s-r+a-b)-1}\sum_{d=0}^{c}\sum_{t=0}^{i+d+1}\\
&\times\sum_{u=0}^{\theta(n-s+1+b)-1}\sum_{v=0}^{u}(-1)^{a+b+c+u}\binom{r-1}{a}\\
&\times\binom{s-r-1}{b}\binom{\theta(s-r+a-b)-1}{c}\binom{c}{d}\binom{\theta(n-s+1+b)-1}{u}\binom{u}{v}\\
&\times\frac{\Gamma(i+d+2)\Gamma(j+t+v+2)}{t!(1+c)^{i+d+2-t}(c+2+u)^{j+t+v+2}},
\end{aligned}$$

   where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

   For $n-s+1=r$, $n-r+1=s$, the result was obtained by Shawky and Bakoban [14].

2. Putting $k=1$ in (3.4) and (3.6), we deduce the explicit expressions for the product moments of lower record values for the exponentiated gamma distribution, established by Shawky and Bakoban [13].

Making use of (2.1), we can derive recurrence relations for product moments of *l gos* from (1.5).

Theorem 3.1 *For the distribution given in* (1.5) *and for* $1 \le r < s \le n$, $n \ge 2$ *and* $k = 1, 2, \ldots$

$$\begin{aligned}
&E[X'^i(r,n,m,k)X'^j(s,n,m,k)] - E[X'^i(r,n,m,k)X'^j(s-1,n,m,k)] \\
&\quad = \frac{j}{\theta\gamma_r}\{E[X'^i(r,n,m,k)X'^{j-1}(s,n,m,k)] + E[X'^i(r,n,m,k)X'^{j-2}(s,n,m,k)] \\
&\quad - E[\phi(X'(r,n,m,k)X'(s,n,m,k)]\}, \qquad (3.7)
\end{aligned}$$

where $\phi(x,y) = x^i y^{j-2} e^y$.

Proof. From (1.3), we have

$$\begin{aligned}
&E[X'^i(r,n,m,k)X'^j(s,n,m,k)] \\
&\qquad = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i[F(x)]^m f(x) g_m^{r-1}(F(x)) I(x)dx, \quad (3.8)
\end{aligned}$$

where

$$I(x) = \int_0^x y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y)dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.8), we get

$$\begin{aligned}
&E[X'^i(r,n,m,k)X'^j(s,n,m,k)] \\
&\quad = E[X'^i(r,n,m,k)X'^j(s-1,n,m,k)] \\
&\quad - \frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^x x^i y^{j-1}[F(x)]^m f(x) \\
&\quad \times g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s} dydx
\end{aligned}$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (2.1), we obtain

$$\begin{aligned}
&E[X'^i(r,n,m,k)X'^j(s,n,m,k)] \\
&= E[X'^i(r,n,m,k)X'^j(s,n,m,k)] - \frac{jC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\Big\{\int_0^\infty\int_0^x x^i y^{j-2} e^y \\
&\quad \times [F(x)]^m f(x) g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)dydx \\
&\quad - \int_0^\infty\int_0^x x^i y^{j-1}[F(x)]^m f(x) g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} \\
&\quad \times [F(y)]^{\gamma_s-1} f(y)dydx - \int_0^\infty\int_0^x x^i y^{j-2}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\
&\quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)dydx\Big\}
\end{aligned}$$

and hence the result.

Remark 3.1 *Putting $m = 0$, $k = 1$, in (3.7), we obtain recurrence relations for product moments of order statistics of the exponentiated gamma distribution in the form*

$$\begin{aligned}
&E(X^i_{n-r+1:n}X^j_{n-s+1:n}) - E(X^i_{n-r+1:n}X^j_{n-s+2:n})\\
&\quad= \frac{j}{\theta(n-r+1)}\Big\{E(X^i_{n-r+1:n}X^{j-1}_{n-s+1:n}) + E(X^i_{n-r+1:n}X^{j-2}_{n-s+1:n})\\
&\quad\quad - E(\phi(X_{n-r+1:n}X_{n-s+1:n}))\Big\}.
\end{aligned}$$

Remark 3.2 *Setting $m = -1$ and $k \geq 1$, in Theorem 3.1, we obtain the recurrence relations for product moments of lower k-th record values from exponentiated gamma distribution in the form*

$$\begin{aligned}
&E[X'^i(r,n,-1,k)X'^j(s,n,-1,k)] - E[X'^i(r,n,-1,k)X'^j(s-1,n,-1,k)]\\
&\quad= \frac{j}{\theta k}\{E[X'^i(r,n,-1,k)X'^{j-1}(s,n,-1,k)]\\
&\quad+ E[X'^i(r,n,-1,k)X'^{j-2}(s,n,-1,k)]\\
&\quad- E[\phi(X'(r,n,-1,k)X'(s,n,-1,k))]\}.
\end{aligned}$$

4 Characterization

Theorem 4.1 *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$\begin{aligned}
E[X'^j(r,n,m,k)] &= E[X'^j(r-1,n,m,k)] - \frac{j}{\theta\gamma_r}E[\phi(X'(r,n,m,k))]\\
&\quad+ \frac{j}{\theta\gamma_r}E[(X'^{j-1}(r,n,m,k))] + \frac{j}{\theta\gamma_r}E[(X'^{j-2}(r,n,m,k))] \quad (4.1)
\end{aligned}$$

if and only if

$$F(x) = [1 - e^{-x}(1+x)]^\theta.$$

Proof. The necessary part follows immediately from equation (2.8). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.2), we have

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad=\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx \\
&\quad-\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-2}e^x[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad+\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad+\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \qquad (4.2)
\end{aligned}$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx \\
&\quad+\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad-\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-2}e^x[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad+\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&\quad+\frac{jC_{r-1}}{\theta\gamma_r(r-1)!}\int_0^\infty x^{j-2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx
\end{aligned}$$

which reduces to

$$\begin{aligned}
&\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))dx \\
&\qquad\left\{F(x)-\frac{1}{\theta xe^{-x}}f(x)+\frac{1}{\theta}f(x)+\frac{1}{\theta x}f(x)\right\}dx=0. \qquad (4.3)
\end{aligned}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [5]) to equation (4.3), we get

$$\frac{f(x)}{F(x)}=\frac{\theta xe^{-x}}{[1-e^{-x}(1+x)]}$$

which prove that

$$F(x)=[1-e^{-x}(1+x)]^{\theta}, x\geq 0, \theta>0.$$

Acknowledgement

The authors would like to thank the referee for carefully reading the paper and for helpful suggestions which greatly improved the paper.

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