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Approximate tolerance limits for C_p capability chart based on Range

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Abstract

Kotz and Johnson [10], Deleryd [1] indicated that there was a gap between theory and practice of process capability studies. Then how to reduce the gap and the variation between theory and practice of process capability studies has been become a serious problem.

Most of results obtained so far regarding the distributional properties of estimated capability indices are based on the assumption of a simple sample of observations from the normally distributed process, which is in-control. To use estimators based on several small subsamples and then interpret the results as if they were based on a single sample may result in incorrect conclusions. For the sake of use past in-control data from subsamples to make decision regarding process capability, the distribution of the estimated capability index with subgrouping should be considered.

And then, we know that all the information gathered from the product inspection process in the manufacturing industries can be used as the rational subgroup data in real. Therefore, we consider estimator that naturally occur when using an $\bar{X}$-chart together with a R-chart in quality control. In addition, we make use of the Patnaik's [13] approximation to the central Chi-squared distribution, to construct a procedure approximate tolerance limits for the sampling distribution of the C_p to assess the performance and enhance practical value for the capability chart based on range.

Keywords. Process Capability Chart; Approximate Tolerance Limits; Range; Relative Efficiency.

1 Introduction

Process capability indices are widely used to measure whether the product quality meets to customer's requirements. They can help companies to promote marketing sales, retain customers and reduce the process variability. The setting and communication becomes much simpler and easier by using process capability indices to express process capability between manufacturers and customers. The use of these indices provides a unitless language for evaluating not only the actual performance of production processes, but the potential performance as well. The indices are intended to provide a concise summary of importance that is readily usable. Engineers, manufacturers, and suppliers can communicate with this unitless language in an effective manner to maintain high process capabilities and enable cost savings.

Since capability analysis of a process and effectiveness of control charts are directly related, therefore, Kuo, et al. [5, 6, 7]provided a confidence bound and a test of hypothesis for C_{pm} and C_{pp} based on subsamples. Not only we make use of a better estimator of σ for moderate large sample size, which was that introduction by Kirmani, et al. [9] and Derman and Ross [2], respectively, but also consider with variable sample size, to construct a procedure lower confidence bounds in some detail in connection with minimum values for C_p.

Montgomery [12] and Juran and Gryna [8] pointed out that control charts, in addition to monitoring a process, provide estimates of the process parameters that are useful in capability studies. No matter how, not only we make use of a link between capability indices and tolerance limits, and we propose to utilize the information gathered by control charts to estimate tolerance limits of a process on a continuous basis, and also construct an approximate tolerance charts for C_p based on range. The procedures have been included.

Montgomery [12] pointed out that control chart is an on-line process control technique widely used for this purpose. Control chart is an important part of the magnificent seven major tool of SPC (Statistical Process Control). Moreover, control charts are of great use in the analysis and control of manufacturing processes, so as to produce quality that is satisfactory adequate, dependable and economic. Nevertheless, the general characteristics of control charts, and their usefulness, and emphasized two general purposes:

(a) Analysis of past data for control. (b) Process control versus given standards.

Now that we know all existing process capability indices have some weaknesses. Deleryd [1, Figure 1] indicated that C_p react to changes in process dispersion but not to change of process location. C_{pk} reacts to changes both in process dispersion and location. C_{pm} and C_{pmk} react more strongly to changes both in dispersion and location than C_{pk}. And C_{pmk} is more sensitive than C_{pm} to deviations from the target value T. The index C_{pp} is useful to evaluate process capability for a single product in common situation, it cannot be applied to evaluate the multi-process capability. The index C_{pp} is a simple transformation from the index C_{pm}, and provided individual information concerning the process accuracy and process precision. Capability indices are key measures in the context of never-ending improvement in quality. The process measures are estimated based on a single random sample of observations from the normally distributed and in statistical control.

Extensive studies have been conducted to determine the effects of nonnormality on the carious capability indices since Gunter [3] bemoaned the many drawbacks of C_{pk} in particular. Several methods for handling non-normal data have been suggested. Standard measurement-system analysis criteria assume the gauge measures a single variable. However, in some kind of manufacturing, measurement systems take data for many quality characteristics, to support using these data as a multivariate response. Majeske [11] develops multivariate extensions of gauge-approval criteria precision to tolerance ratio, percent R&R, and signal-to-noise ratio.

A successful implementation of process capability studies require that proper resources are allocated, which is a managerial responsibility. The proper education and training can be provided. So we can give proper education and training every co-worker knows the method. Then the conservative personal attitudes may be changed and the prerequisites for handling the practical problems are much more promising.

To bridge some of the gap between theory and practice, loss function can be used. In general, Spring, et al. [14] and Vännman [15] think that there is a need for simple graphical tools to bridge some of the gap between practice and theory in capability studies. And the visual impact of a plot is more effective than numbers, for example, estimates or confidence limits.

The capability of a process and effectiveness of control charts are directly related. The gap between theoreticians and practitioners is, we believe and hope, mainly through software, but there still remain numerous instances of lack of understanding of the purpose and usage of Process capability indices (PCIs) and process performance.

In this paper, we use the C_p index, but also consider various sample size to construct the approximate tolerance limits for C_p, based on range. To assess the capability of a process, it is proposed to consider estimated tolerance limits in capability analysis, along with control charts for monitoring the process mean and process standard deviation, respectively, and come from a normal distribution and are independent. The capability chart used in conjunction with the traditional Shewhart variables charts will provide evidence of improvement. It may also assist in ending the unfortunate practice of including specification limits on the $\bar{X}$ chart and R-chart will incorporate the limits into the calculation of process capability.

2 Approximate tolerance limits for $\hat{C}_p$ based on range

In any production or manufacturing process, regardless how well it was designed or carefully maintained, a certain amount of inherent or natural variability will always exists. In the framework of statistical quality control, this natural variability is often called a ‘stable system of chance causes’. A process

Table 1: The R.E. of the range method to S^2

n	2	3	4	5	6	7	10
R.E.	1.000	0.992	0.975	0.955	0.930	0.910	0.850

that is operating with only chance causes of variation is said to be in statistical control. Other kinds of variability (assignable causes) may occasionally be present in the output of a process is said to be out of control.

A quality characteristic that is measured on a numerical scale is called a variable. Usually it needs to be monitored both the mean value of the quality characteristic and its variability. The most commonly used types of control charts for variable are $\bar{X}$ charts related to the process level, and the standard deviation S chart and the range R chart related to the process variability.

Let $X_{i1}, X_{i2}, \ldots, X_{in}$, $i = 1, \ldots, m$, be m preliminary independent random samples of size n from normal distribution with mean μ and standard deviation σ. The ith subsample mean is

$$\bar{X}_i = \frac{(X_{i1} + X_{i2} + \cdots + X_{in})}{n} \tag{2.1}$$

In most cases, both μ and σ are unknown. Therefore, they need to be estimated from the preliminary sample or subsamples from process which is in statistical control. These estimates are usually based on at least 20 to 25 subsamples. Suppose that m subsamples are available, each subsample contains n observations on the quality characteristic. Then, a reasonable estimator of μ, the process mean, is the grand average, say

$$\bar{\bar{X}} = \frac{(\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_m)}{m} \tag{2.2}$$

To construct the confidence bounds for some process capability indices, we need an estimate of the standard deviation σ. We may estimate σ either by the sample standard deviation or the range of m subsamples means. Montgomery [12] point out, the relative efficiency (R.E.) of the range method to the sample variance S^2 for various sample size based on a single sample is shown in Table 1.

However, for the small sample size, $n = 4, 5$, or 6, the range chart method is often employed and it is entirely satisfactory. But, for moderate value of n, say, $n \geq 10$, the range method loses its efficiency rapidly since it ignores all the information in the sample between $X_{\max} = \max\{X_1, X_2, \ldots, X_n\}$ and $X_{\min} = \min\{X_1, X_2, \ldots, X_n\}$. In this case, the S chart is preferred to the R chart.

As stated before, in this paper, a common practice in process control is to estimate the process capability indices by analyzing the past 'in control' data. Suppose that m subsamples, each of size n, are available, and then we can estimate μ by $\bar{\bar{X}}$, given in (2.2). Since $\bar{\bar{X}}$ is equal to the average of all of the mn data values, it is the 'natural' estimator of μ.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a normal population with mean μ and standard deviation σ. The range of this single sample is defined by

$$R = X_{\max} - X_{\min}, \tag{2.3}$$

where $X_{\max} = \max\{X_1, X_2, \ldots, X_n\}$ and $X_{\min} = \min\{X_1, X_2, \ldots, X_n\}$.

Suppose the total samples are grouped into m subsamples such that each subsample contains n observations. The mean of the m ranges will be denoted by $\bar{R}_{m,n}$ and the range of a single sample of size n is denoted by $R_{1,n}$.

When $E(R) = \sigma d_2$ and $\mathrm{Var}(R) = \sigma^2 d_3^2$, the mean and variance of $\bar{R}_{m,n}/\sigma$ are given by

$$E(\bar{R}_{m,n}/\sigma) = E(R_{1,n}/\sigma) = d_2, \tag{2.4}$$

and

$$\mathrm{Var}\ (\bar{R}_{m,n}/\sigma) = \mathrm{Var}(R_{1,n}/\sigma)/m = d_3^2/m, \text{ respectively.} \tag{2.5}$$

Then $\bar{R}_{m,n}/d_2$ is an unbiased estimator of σ, where d_2 and d_3 are constants (see Hartley and Pearson [4]).

According to Patnaik [13], it has been shown that $\bar{R}_{m,n}/\sigma$ is approximately distributed as $\frac{c\sqrt{\chi^2}}{\sqrt{\nu}}$. That is,

$$(\frac{\bar{R}_{m,n}}{\sigma})^2 \equiv c^2 \frac{\chi^2_\nu}{\nu}, \tag{2.6}$$

$$\text{and} \quad (\frac{\bar{R}_{m,n}}{\sigma})^2 \times \frac{\nu}{c^2} \equiv \chi^2_\nu, \tag{2.7}$$

where χ^2_ν denotes a chi-square distribution with ν degrees of freedom, and c and ν are constants which are functions of the first two moments of the range variable, given by

$$\nu = \frac{1}{(-2 + 2\sqrt{1 + 2(d_3/d_2)^2/m})}, \tag{2.8}$$

$$\text{and } c = d_2 \times \sqrt{\nu/2} \times \Gamma(\nu/2)/\Gamma((\nu+1)/2) \approx d_2(1 + 1/(4\nu)). \tag{2.9}$$

Using these relations, the values of c and ν can be easily obtained for any n and m.

Assume that the process measurement follows $N(\mu, \sigma^2)$, the normal distribution, the index and reasonable estimator of C_p are given as following, respectively,

$$C_p = (USL - LSL)/(6\sigma), \tag{2.10}$$

$$\text{and } \hat{C}_p = (USL - LSL)/(6\hat{\sigma}), \tag{2.11}$$

where $[LSL, USL]$ is the specification interval, μ is the process mean, σ is the process standard deviation (overall process variability), under stationary controlled conditions.

From (2.10) and (2.11), we obtain

$$\left(\frac{C_p}{\hat{C}_p}\right)^2 = \left(\frac{\hat{\sigma}}{\sigma}\right)^2, \tag{2.12}$$

where $\hat{\sigma} = R'/d_2$ is an unbiased estimator of σ, and R' indicates either $\bar{R}_{m,n}$ or $R_{1,n}$, based on rang. Thus,

$$\chi_R^2 = C_p^2/\hat{C}_p^2 \times d_2^2\nu/c^2 = (R'/\sigma)^2 \times \nu/c^2 \equiv \chi_v^2. \tag{2.13}$$

Apply a simple approximation procedure based on range we can obtain the tolerance limits of the estimator of C_p.

The $100(1-\alpha)\%$ approximate tolerance limits for $\hat{C}_p$, together with $\bar{X}-R$ charts

$$\begin{aligned} 1-\alpha &= P(\chi^2_{1-\alpha/2} \leq \chi^2_R \leq \chi^2_{\alpha/2}) \\ &= P(J_1 C_p \leq \hat{C}_p \leq J_2 C_p), \end{aligned} \tag{2.14}$$

where

$$J_1 = \frac{d_2}{c} \times \sqrt{\frac{\nu}{\chi^2_{\alpha/2}}} \text{ and } J_2 = \frac{d_2}{c} \times \sqrt{\frac{\nu}{\chi^2_{1-\alpha/2}}}, \tag{2.15}$$

and $\chi^2_{\alpha/2}(\nu)$ is the upper $\alpha/2$ quantile of the chi-squared distribution with ν degrees of freedom. So, the $100(1-\alpha)\%$ approximate tolerance limits for $\hat{C}_p$ based on range, is given by

$$(J_1 C_p, J_2 C_p). \tag{2.16}$$

When using $\bar{X}-R$ charts, the mean line, denoted $\bar{C}_p$, is given by (2.17) form, as follows

$$\bar{C}_p = \frac{USL - LSL}{6(\frac{\bar{R}}{d_2})}, \tag{2.17}$$

resulting in upper and lower limits of the form $U_1 = J_2\bar{C}_p$ and $L_1 = J_1\bar{C}_p$.

Therefore, the $100(1-\alpha)\%$ approximate upper and lower limits for $\hat{C}_p$ in conjunction with $\bar{X}-R$ charts based on Range are of the form.

Approximate upper tolerance limits: $U_1 = J_2\bar{C}_p$

center line: $\bar{C}_p$

Approximate lower tolerance limits:

$$L_1 = J_1\bar{C}_p. \tag{2.18}$$

Table 2: The step of the approximate tolerance limits for C_p capability Chart

Step	C_p
1.	Determine the value of ith subsample mean $\bar{X}_i$ and range R_i.
2.	a. Compute the grand average, $\bar{X}$.
	b. Calculate the mean of the m ranges, $\bar{R}$.
3.	a. Compute $\hat{\sigma} = \bar{R}_{m,n}/d_2$.
	b. Calculate the value ν.
	c. Calculate a series of the estimates $\hat{C}_p$.
4.	a. Compute center line $\bar{C}_p$.
	b. Calculate the approximate upper tolerance limits U_1.
	c. Calculate the approximate lower tolerance limits L_1.

3 The procedure

As stated before, to check it the process meets the capability requirement, we first determine the ith subsample mean is $\bar{X}_i$ and the ith subsample range is R_i. Second, we calculate the grand average, say $\bar{\bar{X}}$, and the mean of the m ranges, say $\bar{R}$. Third, calculate an unbiased estimator of σ is $\hat{\sigma} = R'/d_2$ and a chi-square distribution with ν degrees of freedom, and a series of the estimates of C_p, $\hat{C}_p$. Finally, we compute center line $\bar{C}_p$, and the approximate upper and lower tolerance limits U_1 and L_1, respectively.

Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In sum, we summarize these steps shown in Table 2.

4 Numerical example

We use the data given in Table 3 to demonstrate this procedure. This example is about a manufacturing process with $m = 20$ subsamples, each subsample consists of $n = 5$ samples, have been taken from the process when the process was in control. A total of 100 observations were collected and are displayed in Table 3. The upper and lower specification limits are USL= 1.2 and LSL = 0.8, respectively.

Since each subgroup in the process provides a measure of location, $\bar{X}_i$, and a measure of variability either R_i, an estimator of $\hat{C}_p$ can be determined for each subsample using either equation (2.11). The result is a series of estimates for C_p over the life the process.

Control charts can indicate whether or not statistical control is being maintained and provide us with other signals from the data. If the process is in a state of statistical control, then the value of C_p at the subgroup level provides information regarding process capability. If the process is out-of-control, then

Table 3: Collected 100 observations sample data

$n\backslash m$	1	2	3	4	5	6	7	8	9	10
1	0.96	1.25	1	1	1.08	1.11	1.08	1.11	1.02	1.1
2	1.1	1.1	1.2	1.15	1.15	1.02	1.24	1.02	1.02	1.2
3	1.16	1.2	1.11	1.14	1.25	1.1	1.1	1.1	1.23	1.21.
4	1.18	1.04	1.12	1.12	1.12	1.23	1.06	1.23	1.12	1
5	1.23	1.08	1.1	1.2	1.11	1.1	1.12	1.1	1.06	1.2

$n\backslash m$	11	12	13	14	15	16	17	18	19	20
1	1.2	1.1	1.12	1.15	1.2	1.3	1.1	1.23	1.1	1.23
2	1.29	1.08	1.04	1.12	1.07	1.2	1.19	1.15	1.02	1.15
3	1.12	1.02	1.23	1.05	1.02	1.12	1.03	1.07	1.14	1.04
4	1.1	1.05	1.12	1.1	1.13	1.15	1.02	1.02	1.07	1.14
5	1.14	1.2	1.1	1.26	1.18	1.12	1.09	1.1	1	1.14

no estimates of C_p are possible. After a process has been brought into a state of statistical control, a process capability study can be initiated to determine the capability of the process in regard to meeting the specifications.

It would be illogical to undertake such a study if the process is not in control, since the objective should be study the capability of the process after all problematic causes has been eliminated, if possible. Once control has been estimated, capability can be assessed in a variety of ways, and then estimates of the process capability can be calculated from the subgroup information.

Take the first steps, it is possible to attain estimates of C_p as subgroup information is gathered. That is similar to general control chart procedures the subgroup information should be monitored and considered as all assignable cause can be both detected and removed, and this cycle should be continued until no further action can be taken. Once in-control, a typical control charts contains a center line and two other horizontal lines, called the upper control limit (UCL) and the lower control limit (LCL), are also shown on the chart, the later permitting limits for the capability chart.

As expected, the estimates of C_p vary from subgroup to subgroup. To analysis these fluctuations, limits and a mean lines are required, similar to the Shewhart control charts, the upper and lower limits for $\hat{C}_p$ will represent the interval expected to contained 99.73% of the estimates if the process has not been changes or altered.

Looking first at the R chart in Figure 1, the process variability does not appear out-off control signals, which also seems to be the case with the $\bar{X}$ chart. The usual control limits and centre lines for the $\bar{X}$ and R charts have been calculated from the subgroup information and included on the control chart. Since the process appears to be in-control, $\hat{C}_p$ has been calculated using the form (2.11) for each subgroup and the values plotted in a run chart to

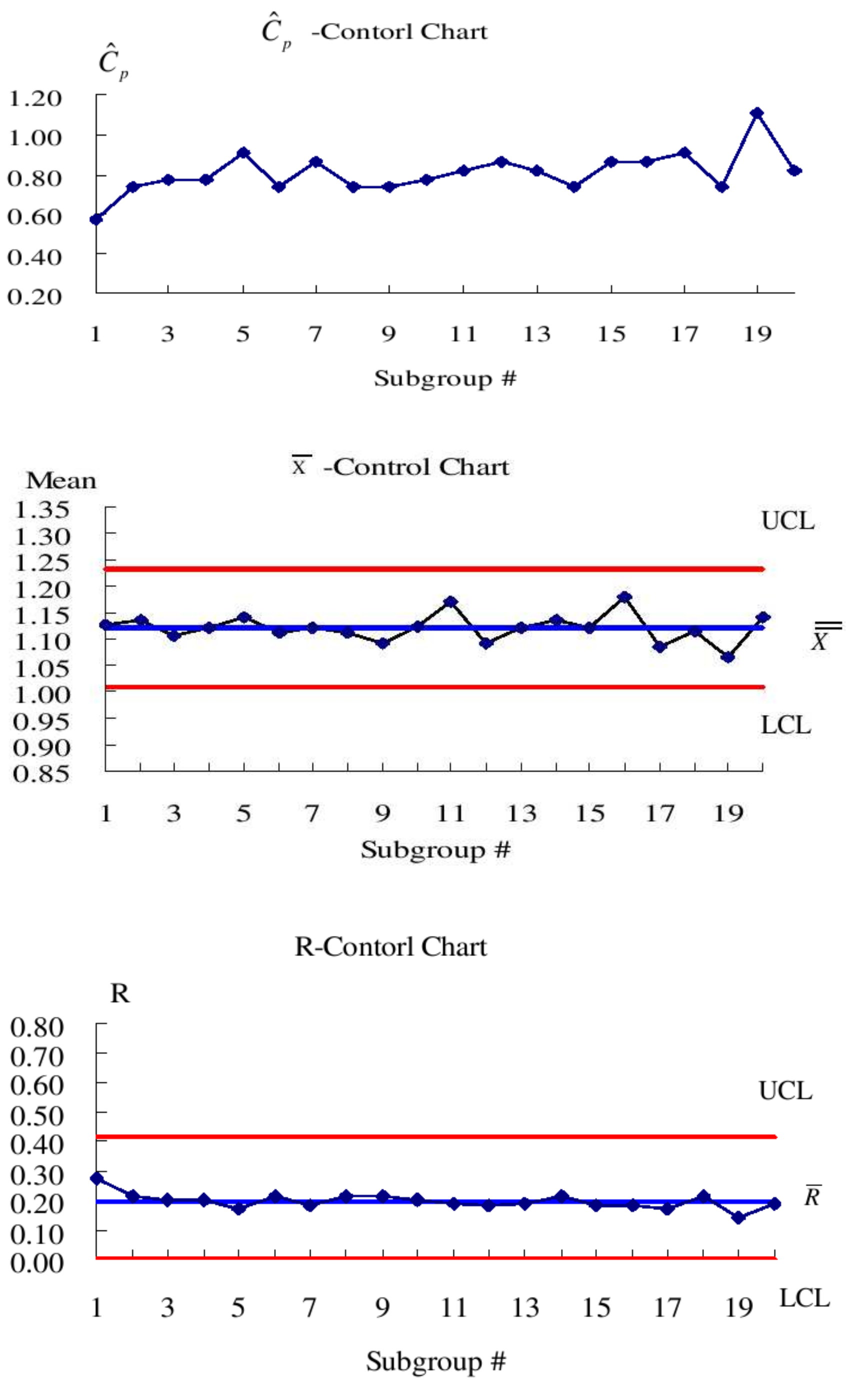

Figure 1: $\bar{X}$-R charts and a runs chart of $\hat{C}_p$.

Table 4: Values of the vary from subgroup to subgroup for the $\hat{C}_p$

$m/1$	2	3	4	5	6	7	8	9	10
0.5743	0.7384	0.7753	0.7753	0.9122	0.7384	0.8615	0.7384	0.7384	0.7753
11	12	13	14	15	16	17	18	19	20
0.8161	0.8615	0.8161	0.7384	0.8615	0.8615	0.9122	0.7384	1.1076	0.8161

show in Table 1.

To analysis these fluctuations, limits and a means line analogous to the Shewhart limits and center line are required. Similar to Shewhart control charts, the upper and lower limits for C_p will represent the interval expected to contain 99.73% of the estimates if the process has not been altered.

From the process data, we obtain that sample mean $\bar{\bar{x}} = 1.1206$, $\bar{R} = 0.1950$, $A_2 = 0.577$, $d_2 = 2.326$, $d_3 = 0.8641$, $D_3 = 0$, $D_4 = 2.114$, and $\nu = 72.7080$. The result is a series of estimates of C_p over the life of the process, shows in Table 4.

The upper and lower limits for $\hat{C}_p$ in the conjunction with $\bar{X}$ and R charts are of the form U_1 and L_1, respectively, and center line as follows is Approximate upper tolerance limits: $U_1 = J_2\bar{C}_p = 1.05205$,

$$\text{center line: } \bar{C}_p = 0.79521,$$

Approximate lower tolerance limits: $L_1 = J_1\bar{C}_p = 0.63465$.
The limits and mean line have been showed in Figure 2.

Apparently, the process capability estimates vary from subgroup to subgroup. With the exception of subgroup 19 the fluctuations in $\hat{C}_p$ appear to be due to random causes. In period 19 the process capability appears to have increased significantly and warrants investigation. Practioners would likely attempt to determine what caused the capability to rise significantly and recreate that situation in the future. If the estimated process capability had dropped below L_1 this would signals a charge in the process (for example, subgroup 1), and the process capability was not at the level required by the customer, changes in the process would be required.

Owing, in the never-ending improvement system, the process capability should be under constant influence to increase. The capability chart used in conjunction with the traditional Shewhart variables charts will provide evidence of improvement.

5 Conclusion

Control chart is an important part of the magnificent seven major tool of SPC. Thereby, control charts are of great use in the analysis and control of manufacturing processes, so as to produce quality that is satisfactory adequate,

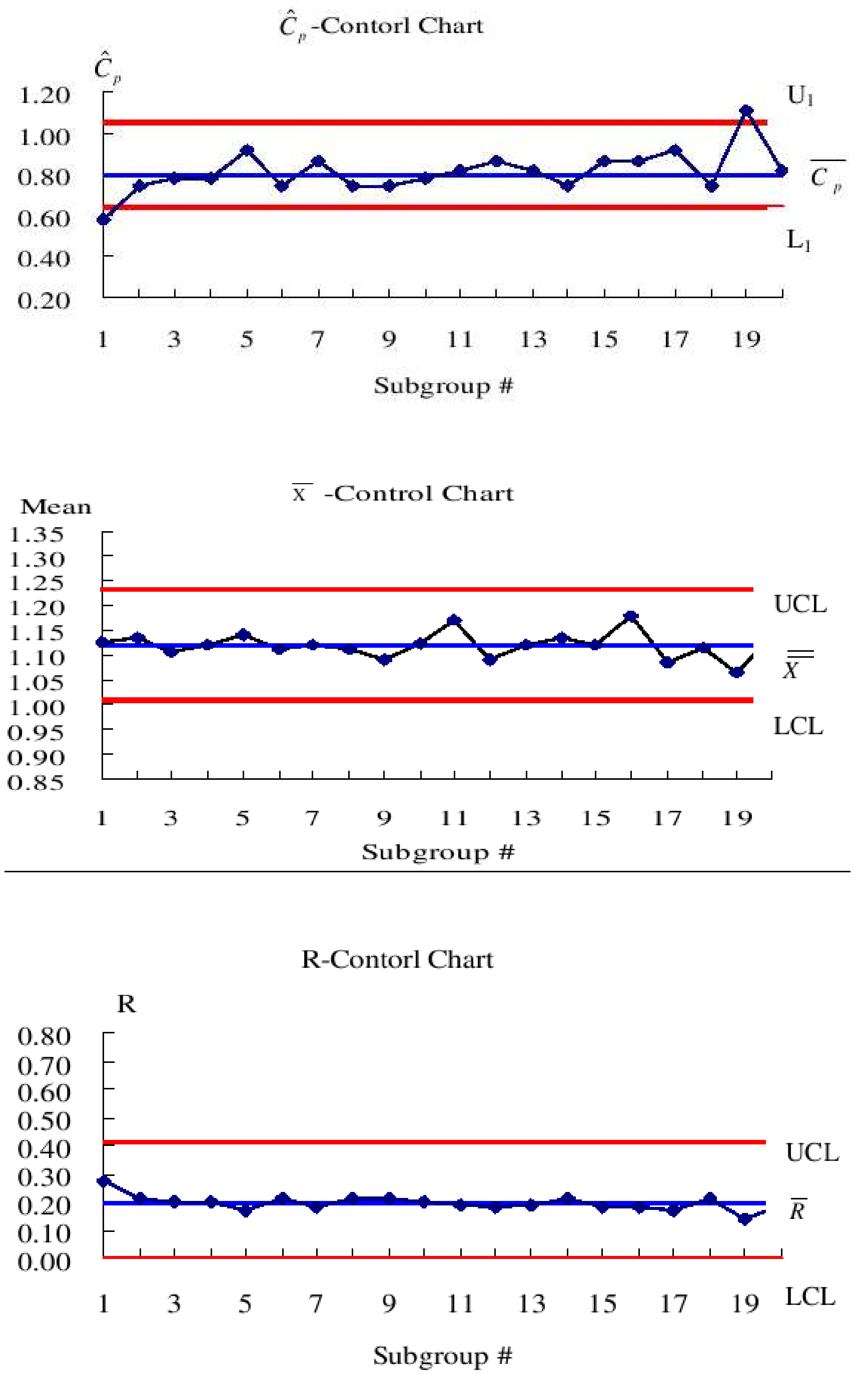

Figure 2: $\bar{X}$-R charts and $\hat{C}_p$ capability chart with upper and lower limits, and center line.

dependable and economic. And, the effect of any changes to the process will also show up on the chart, thereby providing feedbacks to the practioner regarding the effect changes to the process have on process capability.

The proposed chart is easily appended to $\bar{X}$-R charts and makes easy judgments regarding the ability of a process to meet requirements, also providing evidence of process performance. The capability chart represents a modification of the control charts that combines customer requirements and process performance in a certain degree reflecting the needs of the customer as well as providing information to the manager of the process.

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