MULTI-SALVAGEABLE PLAN - AN ALTERNATIVE APPROACH

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Abstract One of the important methods for formulating single sampling plan with a fixed sample size is maximizing probability of correct decisions based on Binomial distribution and later the result is extended to Poisson distribution. This method is further used to identify a lot into one of several categories. By minimizing the sum of risks on sampling inspection inferences, a lot may be classified as Good, Salvageable or Bad and it is an extended to multi-salvageable groups. This paper proves that division of a multi salvageable plan by minimizing probability of risks is equivalent to the assignment of a Single Sampling Plan into one of several groups by maximizing probability of correct decisions.

Keywords. Maximisation of correct decision, Minimization of Risk, Consumer's Risk, Producer's Risk, Salvageable Lot.

1 Introduction

Golub [1] has suggested a sampling plan by maximizing probability of correct decisions using Binomial model for a fixed sample size and extended the result to assign a lot in to one of the several categories based on defined qualities. Soundararajan [2] has obtained the acceptance numbers in the case of Poisson distribution by maximizing the probability of correct decisions as

$$c = -\frac{1}{2} + \frac{n}{\frac{(Lnp_2 - Lnp_1)}{(p_2 - p_1)}} \text{ for } p_1 < p_2.$$

The acceptance and rejection numbers for single sampling plan by placing a lot into one of the 3 categories are given by

$$c_1 = -\frac{1}{2} + \frac{n}{\frac{(Lnp_2 - Lnp_1)}{(p_2 - p_1)}} \text{ and } c_2 = -\frac{1}{2} + \frac{n}{\frac{(Lnp_3 - Lnp_2)}{(p_3 - p_2)}} \text{ for } p_1 < p_2 < p_3$$

where c_1 and c_2 are adjusted to the nearest integers for a fixed sample size.

If it is extended to k categories by maximizing the sum of probability of

correct decisions, the corresponding divisions were determined by

$$c_1 = -\frac{1}{2} + \frac{n}{\frac{(Lnp_{i+1} - Lnp_i)}{(p_{i+1} - p_i)}}, \quad i = 1, 2, \ldots (k-1),$$

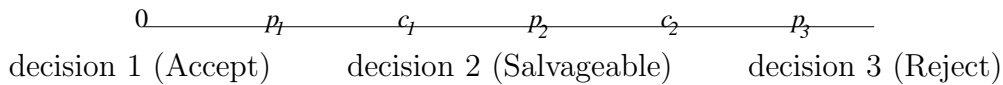$$\text{for } p_1 < p_2 < \cdots p_i < p_{i+1} \cdots < p_k.$$

Dey [3] has introduced acceptance sampling plans for salvageable lots by minimizing the total of consumer's risk and producer's risk, when sample size is prefixed. Two numbers c_1 and c_2 are determined such that if $c_1 <$ number of defects $\leq c_2$, then the lot is considered salvageable. Dey's salvageable plan is extended to multi-salvageable plan and compared with Golub's multi-decision single sampling plan and it is found that both are identical.

2 Maximising correct decision/ Minimising Error

Case 1. $n = 3$
Golub's Three decision plan
    Suppose a single sampling plan has 3 decisions with quality of items specified as p_1, p_2 and p_3 $(p_1 < p_2 < p_3)$. Golub's acceptance numbers c_1 and c_2 with fixed sample size are obtained by maximizing the sum of probability of correct decisions.

$$\underset{\substack{\text{decision 1 (Accept)} \qquad \text{decision 2 (Salvageable)} \qquad \text{decision 3 (Reject)}}}{\overline{\quad 0 \qquad\qquad p_1 \qquad\qquad c_1 \qquad\qquad p_2 \qquad\qquad c_2 \qquad\qquad p_3 \qquad}}$$

    $P = P_1(p_1) + P_2(p_2) + P_3(p_3)$ where $P_i(p_i)$ is the probability of decision i, when quality is p_i $(i = 1, 2, 3)$.
    Let the number of defectives/defects follow Poisson distribution, then maximise

$$P = \sum_{r=0}^{c_1} \frac{e^{-np_1}(np_1)^r}{r!} + \sum_{r=c_1+1}^{c_2} \frac{e^{-np_2}(np_2)^r}{r!} + \sum_{r=c_2+1}^{\infty} \frac{e^{-np_3}(np_3)^r}{r!} \cdots \quad (2.1)$$

Dey's salvageable plan- A, S, R plan
    Let $n =$ the sample size, $c_1, c_2 =$ acceptance numbers and $d =$ number of defectives in the lot, then the operating procedure is:

If $d \leq c_1$, the lot is accepted as good.

$c_1 < d \leq c_2$, the lot is salvageable with limited defectives (salvageable)

$d > c_2$, the lot is rejected as bad.

Thus the lot is classified as good (p_1) salvageable (p_2) and bad (p_3).
Then the 1st kind of risks involved in the plans are:

1. Categorizing a good lot as salvageable

2. Rejecting a good lot as bad

3. Categorizing a salvageable lot as bad

Number of expressions of 1st kind of risks $= 2 + 1 = 3$. These risks are producer's risks and corresponding probabilities are $\alpha_{1.2}, \alpha_{1.3}$ and $\alpha_{2.3}$.
The 2nd kind of risks are:

1. Accepting a salvageable lot as good

2. Accepting a bad lot as good

3. Categorizing a bad lot as salvageable

Number of expressions of 2nd kind of risks $= 2 + 1 = 3$. These are consumer's risks and corresponding probabilities are $\beta_{2.1}, \beta_{3.1}$ and $\beta_{3.2}$.

Total number of risks in a three decision ASR plan is $3 \times 2 = 6$. The best set of acceptance numbers c_1, c_2 are those for which the sum of risks is minimum. It is obtained by minimizing $(\alpha_{1.2} + \alpha_{1.3} + \alpha_{2.3} + \beta_{2.1} + \beta_{3.1} + \beta_{3.2})$.

If the number of defective or defects follows Poisson distribution then the risks become

$$\alpha_{1.2} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_1}(np_1)^r}{r!}, \alpha_{1.3} = \sum_{r=c_2+1}^{\infty} \frac{e^{-np_1}(np_1)^r}{r!}, \alpha_{2.3} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_2}(np_3)^r}{r!}$$

$$\alpha_{2.1} = \sum_{r=0}^{c_1} \frac{e^{-np_2}(np_2)^r}{r!}, \alpha_{3.1} = \sum_{r=0}^{c_1} \frac{e^{-np_3}(np_3)^r}{r!}, \alpha_{3.2} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_3}(np_3)^r}{r!}$$

Sum of risks to be minimized is

$$\sum_{r=c_1+1}^{c_2} \varphi(np_1) + \{\sum_{r=0}^{c_1} \varphi(np_2) + \sum_{c_2+1}^{\infty} \varphi(np_2)\} + \sum_{r=0}^{c_2} \varphi(np_3)$$

where

$$\varphi(np_3) = \frac{e^{-np_i}(np_i)^r}{r!}$$

or Equivalently maximising its complement

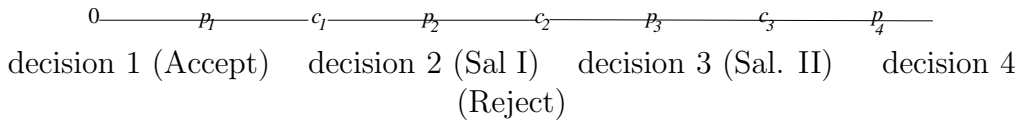$$\sum_{r=0}^{c_1} \varphi(np_1) + \sum_{c_1+1}^{c_2} \varphi(np_2) + \sum_{c_2+1}^{\infty} \varphi(np_3) \tag{2.2}$$

which is the same expression obtained in Golub's plan from (2.1).

Case 2. $n = 4$

Golub's four decision plan

Suppose a SSP has 4 decisions with quality specified as p_1, p_2, p_3 and p_4 ($p_1 < p_2 < p_3 < p_4$).

According to Golub c_1, c_2, c_3 are determined by maximizing the probability of correct decisions $P = P_1(p_1) + P_2(p_2) + P_3(p_3) + P_4(p_4)$, where $P_i(p_i)$ is the probability of placing a lot of quality p_i in the ith ($i = 1, 2, 3, 4$) category

$$0 \rule{1cm}{0.4pt} p_1 \rule{1cm}{0.4pt} c_1 \rule{1cm}{0.4pt} p_2 \rule{1cm}{0.4pt} c_2 \rule{1cm}{0.4pt} p_3 \rule{1cm}{0.4pt} c_3 \rule{1cm}{0.4pt} p_4$$

decision 1 (Accept)   decision 2 (Sal I)   decision 3 (Sal. II)   decision 4
(Reject)

Let the number of defectives/defects follow Poisson distribution, then maximize the probability of correct decisions

$$P = \sum_{r=0}^{c_1} \varphi(np_1) + \sum_{r=c_1+1}^{c_2} \varphi(np_2) + \sum_{c_2+1}^{c_3} \varphi(np_3) + \sum_{c_3+1}^{\infty} \varphi(np_4) \qquad (2.3)$$

where

$$\varphi(np_i) = \frac{e^{-np_i}(np_i)^r}{r!}, \quad i = 1, 2, 3, 4$$

Dey's salvageable plan-Extension-A, S_1, S_2, R plan

Let n = sample size, Acceptance numbers = c_1, c_2, c_3. d = Number of defectives/defects in the lot, then the operating procedure is as follows:

If $d \le c_1$, the lot is accepted as good

$c_1 < d \le c_2$, the lot is salvageable with less defects. (salvageable I)

$c2 < d \le c_3$, the lot is salvageable with more defects. (salvageable II)

$d > c3$, the lot is rejected as bad.

Thus the lot is classified as good (p_1), salvageable I (p_2), salvageable II (p_3), and bad (p_4).

Let the quality levels of the categories are $p_1 < p_2 < p_3 < p_4$. Then the 1st kind of risks involved in the plans are:

1. Categorizing a good lot as salvageable I

2. Categorizing a good lot as salvageable II

3. Rejecting a good lot as bad

4. Categorizing a salvageable I as salvageable II

5. Rejecting a salvageable I as bad

6. Rejecting a salvageable II as bad

Number of expressions of 1st kind of risks $= 3 + 2 + 1 = 6$. These risks are producer's risks and corresponding probabilities are $\alpha_{1.2}, \alpha_{1.3}, \alpha_{1.4}, \alpha_{2.3}, \alpha_{2.4}$, and $\alpha_{3.4}$.

The 2nd kind of risks are

1. Accepting a salvageable I as good

2. Accepting a salvageable II as good

3. Categorizing a salvageable II as salvageable I

4. Accepting a bad as good

5. Categorizing a bad as salvageable I

6. Categorizing a bad as salvageable II

Number of expressions of 2nd kind of risks $= 3 + 2 + 1 = 6$. These are consumer's risks and corresponding probabilities are $\beta_{2.1}, \beta_{3.1}, \beta_{3.2}, \beta_{4.1}, \beta_{4.2}$, and $\beta_{4.3}$. Total number of risks in a Four decision ASR plan is $4 \times 3 = 12$. The best set of acceptance numbers c_1, c_2, c_3 are those for which the sum of risks is to be minimum. It is obtained by minimizing $(\alpha_{1.2} + \alpha_{1.3} + \alpha_{1.4} + \alpha_{2.3} + \alpha_{2.4} + \alpha_{3.4} + \beta_{2.1} + \beta_{3.1} + \beta_{3.2} + \beta_{b.1} + \beta_{4.2} + \beta_{4.3})$

$$\alpha_{1.2} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_1}(np_1)^r}{r!}, \alpha_{1.3} = \sum_{r=c_2+1}^{c_3} \frac{e^{-np_1}(np_1)^r}{r!}, \alpha_{1.4} = \sum_{r=c_3+1}^{\infty} \frac{e^{-np_1}(np_1)^r}{r!}$$

$$\alpha_{2.3} = \sum_{r=c_2+1}^{c_3} \frac{e^{-np_2}(np_2)^r}{r!}, \alpha_{2.4} = \sum_{r=c_3+1}^{\infty} \frac{e^{-np_2}(np_2)^r}{r!}, \beta_{3.4} = \sum_{r=c_3+1}^{\infty} \frac{e^{-np_3}(np_3)^r}{r!}$$

$$\beta_{2.1} = \sum_{r=0}^{c_1} \frac{e^{-np_2}(np_2)^r}{r!}, \beta_{3.1} = \sum_{r=0}^{c_1} \frac{e^{-np_3}(np_3)^r}{r!}, \beta_{3.2} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_3}(np_3)^r}{r!}$$

$$\beta_{4.1} = \sum_{r=0}^{c_1} \frac{e^{-np_4}(np_4)^r}{r!}, \beta_{4.2} = \sum_{r=c_1+1}^{c_2} \frac{e^{-np_4}(np_4)^r}{r!}, \beta_{4.3} = \sum_{r=c_2+1}^{c_3} \frac{e^{-np_4}(np_4)^r}{r!}$$

Total Risk to be minimized is

$$\sum_{r=c_1+1}^{\infty} \frac{e^{-np_1}(np_1)^r}{r!} + \{\sum_{r=0}^{c_1} \frac{e^{-np_2}(np_2)^r}{r!} + \sum_{r=c_2+1}^{\infty} \frac{e^{-np_2}(np_2)^r}{r!}\}$$

$$+ \{\sum_{r=0}^{c_2} \frac{e^{-np_3}(np_3)^r}{r!} + \sum_{r=c_3+1}^{\infty} \frac{e^{-np_3}(np_3)^r}{r!}\} + \sum_{r=0}^{c_3} \frac{e^{-np_4}(np_4)^r}{r!}.$$
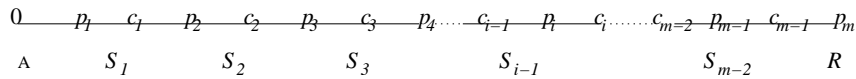
Or maximizing

$$\sum_{r=0}^{c_1} \frac{e^{-np_1}(np_1)^r}{r!} + \sum_{r=c_1+1}^{c_2} \frac{e^{-np_2}(np_2)^r}{r!} + \sum_{r=c_2+1}^{c_3} \frac{e^{-np_3}(np_3)^r}{r!} + \sum_{r=c_3+1}^{\infty} \frac{e^{-np_4}(np_4)^r}{r!}$$

$$\text{i.e., } \sum_{r=0}^{c_1} \varphi(np_1) + \sum_{r=c_1+1}^{c_2} \varphi(np_2) + \sum_{r=c_2+1}^{c_3} \varphi(np_3) + \sum_{r=c_3+1}^{\infty} \varphi(np_4) \qquad (2.4)$$

where $\varphi(np_i) = \frac{e^{-np_i}(np_i)^r}{r!}$, $i = 1, 2, 3, 4$, which is same as (2.3) by Golub's method.

Case 3. $n = m$
Golub's m decision plan

Let the number of defects follow Poisson distribution then by Golub's method, maximise the probability of correct decisions

$$\text{i.e., } \sum_{r=0}^{c_1} \varphi(np_1) + \sum_{r=c_1+1}^{c_2} \varphi(np_2) + \sum_{r=c_2+1}^{c_3} \varphi(np_3) + \cdots + \sum_{r=c_{m-1}+1}^{\infty} \varphi(np_m) \quad (2.5)$$

by which $c_1, c_2, \ldots, c_{m-1}$ can be determined in a SSP with m decisions.

Dey's salvageable plan-Extension $(A, S_1, S_2, \ldots, S_{m-2}, R)$

Dey's salvageable plan can be extended to $(m-2)$ salvageable qualities as follows:

Let d be the number of defects counted from a sample of size n.

If $d \leq c_1$, the lot is good.

$c_1 < d \leq c_2$, the lot is salvageable with 1st kind of defects. (sal 1),

$c_2 < d \leq c_3$, the lot is salvageable with 2nd kind of defects (sal 2),

......

$c_i < d \leq c_{i+1}$, the lot is salvageable with ith kind of defects (sal i),

......

$c_{m-2} < d \leq c_{m-1}$, the lot is salvageable with (m-2)th kind of defects. (sal m-2),

$d > c_{m-1}$, the lot is rejected as bad.

Following are the risks associated with Dey's extended sampling plan.
By definition, the producer's risks are

1. Categorizing an ith sal. lot with quality p_i as $(i + 1)$th, $(i + 2)$th, $\ldots (i + m - 1)$the sal.lot or rejectable lot having quality $p_{i+1}, p_{i+2}, \ldots, p_m$.
Number of expressions of producer's risks are $(m-1)+(m-2)+\cdots 2+1 = m(m-1)/2$.

$$\alpha \cdot i \cdot j = \sum_{r=c_{(j-1)+1}}^{c_j} \frac{e^{-np_i} \cdot (npi)^r}{r!} \ i < j, \ i = 1, 2, \ldots, (m-2),$$

$$j = 2, 3, \ldots (m-1)$$

$$\alpha \cdot i \cdot m = \sum_{r=c_{(m-1)+1}}^{\infty} \frac{e^{-np_i} \cdot (npi)^r}{r!} \ i = 1, 2, \ldots, (m-1).$$

The Consumer's risks are

2. Categorizing an ith sal. lot with quality p_i as $(i - 1)$th, $(i - 2)$th, $\ldots$ 1st lot with quality $p_{i-1}, p_{i-2}, \ldots, p_1$ and number of expressions of Consumer's risks are $1 + 2 + 3 + \cdots + (m - 1) = m(m - 1)/2$.

$$\beta_{j \cdot i} = \sum_{r=c_{j-1}+1}^{c_j} \frac{e^{-npj}(np_j)^r}{r!}, \ i < j, \ i = 2, 3, \ldots, (m-1), \ j = 3, 4, \ldots, m.$$

$$\beta_{j \cdot 1} = \sum_{r=0}^{c_1} \frac{e^{-np_j}(np_j)^r}{r!}, \ i = 2, 3, \ldots, m.$$

Sum of risks is

$$(\alpha_{1.2} + \alpha_{1.3} + \ldots \alpha_{1.m}) + (\beta_{2.1} + \alpha_{2.3} + \alpha_{2.4} + \ldots \alpha_{2.m})$$
$$+ (\beta_{3.1} + \beta_{3.2} + \alpha_{3.4} + \alpha_{3.5} + \ldots \alpha_{3.m}) + \cdots + (\beta_{i.1} + \beta_{i.2}$$
$$+ \cdots + \beta_{i(i-1)} + \alpha_{i(i+1)} + \ldots \alpha_{i.m}) + (\beta_{m.1} + \beta_{m.2} + \cdots + \beta_{m.m-1}).$$

Total number of expressions of risks are $(m - 1) + (m - 1) + \cdots + (m - 1) = m(m - 1)$

$$\{\sum_{r=c_1+1}^{\infty} \frac{e^{-np_1}(np_1)^r}{r!}\} + \{\sum_{r=0}^{c_1} \frac{e^{-np_2}(np_2)^r}{r!} + \sum_{r=c_2+1}^{\infty} \frac{e^{-np_2}(np_2)^r}{r!}\}$$

$$+ \{\sum_{r=0}^{c_2} \frac{e^{-np_3}(np_3)^r}{r!} + \sum_{r=c_3+1}^{\infty} \frac{e^{-np_3}(np_3)^r}{r!}\} + \cdots + \{\sum_{r=0}^{c_{i-1}} \frac{e^{-np_i}(np_i)^r}{r!}$$

$$+ \sum_{r=c_i+1}^{\infty} \frac{e^{-np_i}(np_i)^r}{r!}\} + \cdots + \{\sum_{r=0}^{c_{m-1}} \frac{e^{-np_m}(np_m)^r}{r!}\}.$$

Sum of risks to be minimized is

$$(1 - \sum_{r=0}^{c_1} \varphi(np_1)) + (1 - \sum_{r=c_1+1}^{c_2} \varphi(np_2)) + (1 - \sum_{r=c_2+1}^{c_3} \varphi(np_3)) + \cdots$$
$$+ (1 - \sum_{r=c(m-1)+1}^{\infty} \varphi(np_m)).$$

Equivalently maximizing the compliment,

$$\sum_{r=0}^{c_1} \varphi(np_1) + \sum_{r=c_1+1}^{c_2} \varphi(np_2) + \cdots + + \sum_{r=c_{m-1}+1}^{\infty} \varphi(np_m) \qquad (2.6)$$

which is (2.5) by the Golub's method for finding $c_1, c_2, \ldots, c_{m-1}$. Thus to distinguish a lot falling into one of m decisive categories by maximizing probability of correct decisions is equivalent to the partitioning of $(m-2)$ salvageable lots along with good and rejectable lots.

Conclusion

The Classification of a lot into one of the k grades of a product is usually done by finding the error in the classification of k quality levels and minimizing it. But when the number of classes were large it is difficult to minimize the sum of errors by finding all types of errors. In such situation, for fixed sample size, Golub's maximizing probability of correct decisions will give the same sets of numbers, using the formula suggested by Soundararajan [].In this paper a procedure is given to construct a multi salvageable plan which is an alternate approach to the construction of sampling plans is provided. The following two examples explain the procedure adopted in this paper.

Example 1.
    The proposed Quality grades of a product is

$$A : p_1 = 2\%, \ p_2 = 3\%, \ B : p_2 = 3\%, \ p_3 = 5\%, \ C : p_3 = 5\%, \ p_4 = 8\% D :> 8\%$$

The objective of the company is to minimize the producer's risk and consumer's risk. If 200 units were inspected from a lot of 2000 units and by using the Multi salvageable lot sentencing by risk minimization, which is equivalent to Golubs optimization procedure for a fixed sample size the corresponding acceptance numbers are obtained from Table 1 and the classification is done as below.

$$\text{If } d \leq 5, \ \text{Grade A,}$$
$$6 \leq d \leq 8, \ \text{Grade B,}$$
$$9 \leq d \leq 13, \ \text{Grade C,}$$
$$d \geq 14, \ \text{Grade D}$$

Table 1: The acceptance numbers for specified fraction defectives on 4 classes

$p\%$	$p_1 = 1, p_2 = 2$ $p_3 = 3, p_4 = 4$			$p_1 = 2, p_2 = 3$ $p_3 = 5, p_4 = 8$			$p_1 = 3, p_2 = 6$ $p_3 = 10, p_4 = 15$		
$n(\downarrow)$ \ Ac.no $(\rightarrow)$	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3
100	2	3	4	3	4	7	5	8	13
200	2	5	7	5	8	13	9	16	25
500	8	13	18	13	20	32	22	40	62
1000	15	25	35	25	40	64	44	79	124
2000	29	50	70	50	79	128	87	157	247
5000	73	124	174	124	196	320	217	392	617

where d is the number of defectives in the sample.

Example 2.

A type of rubber tube is classified as A, B, and C according to its strength in the final inspection. Proportion of defects per unit admissible in each category is specified as 6%, 10% and 15% respectively. An inspection wing spot the defectives in the stretched area as to classify the tube. From a lot of size 1000, random samples of 100 units are inspected. The acceptance numbers c_1 and c_2 were determined by minimising the sum of errors in the inspection. The values of acceptance numbers are same as that of maximizing the expression of probability of correct decisions.

From the Table 1, the acceptance numbers for the classification of the lot are 8 and 13, i.e., if each sample of 100 units contain only 8 or less defectives, the lot is categorised as A, (Good quality), in the case of 9 to 13 defects, it is classified as B (salvageble) and greater than 13, the lot is categorised as C (bad).

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References

[1] Golub, A (1953). Designing Single Sampling Inspection Plans when the sample size is fixed, Journal of the American Statistical Association, 48, 278–288.

[2] Soundararajan, V. (1967). Construction of Single Sampling Plan for fixed sample size, Proceedings of the IVth All India Conference on Quality Control, published by Indian Statistical Institute, Calcutta.

[3] Dey, B. R. (1970). Acceptance Sampling of salvageable lots. ISQC Bulletin XIII, 1–6.