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Second order slope rotatable designs with equi-spaced levels using symmetrical uneqaul block arrangements with two unequal block sizes

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Abstract. In this paper, construction of second order slope rotatable designs with equi-spaced levels using symmetrical unequal block arrangements with two unequal block sizes is suggested.

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1 Introduction

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Dey (1970) introduced response surface designs with equi-spaced doses. The study of rotatable designs mainly emphasized on the estimation of absolute response. Estimation of differences in response at two different points in the factor space will often be of great importance. If differences at two points close together, estimation of local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc., (c.f. Park, 1987).

Hader and Park (1978) introduced slope rotatable central composite designs (SRCCD). Park (1987) introduced a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham (1991) constructed second order slope rotatable designs (SOSRD) using BIBD. Victorbabu and Narasimham (2000-01) suggested a new method of construction of SOSRD. Victorbabu (2002a) suggested SOSRD with equi-spaced levels using central composite designs and BIBD. Victorbabu (2002b) constructed SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Victorbabu (2007) suggested a review on SOSRD.

2 Conditions for SOSRD

A second order response surface design $D = ((x_{iu}))$ for fitting,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum\sum_{i<j} b_{ij} x_{iu} x_{ju} + e_u \tag{2.1}$$

where x_{iu} denotes the level of the ith factor ($i = 1, 2, \ldots, v$) in the uth run ($u = 1, 2, \ldots, N$) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2. A second order response surface design D is said to be a SOSRD, if the design points satisfy the following conditions (cf. Hader and Park (1978), Victorbabu and Narasimham (1991)).

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \tag{2.2}$$

$$\text{(i)} \quad \sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$$

$$\text{(ii)} \quad \sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \tag{2.3}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4, \text{ for } i \neq j \tag{2.4}$$

$$(c + v - 1)\lambda_4 > v\lambda_2^2 \tag{2.5}$$

$$\lambda_4[v(5 - c) - (c - 3)^2] + \lambda_2^2[v(c - 5) + 4] = 0 \tag{2.6}$$

where c, λ_2 and λ_4 are constants and the summation is over the design points.

It is clear from the slope rotatability condition (2.6) that the solution for design levels (like a, b, *etc.*) depend on 'c' and central points 'n_0'. In contrast to rotatable designs, we have to choose the non-zero design levels a, b, *etc.*, in SOSRD depending on the number of central points 'n_0'. This interdependence of the parameters n_0, c and design levels is utilized to evolve a new method of construction of SOSRD.

In SOSRD usually the number of central points 'n_0' is prefixed and design levels are chosen to satisfy these conditions of SOSRD. The parameter 'c' is then evaluated. Alternatively we may prefix 'c' and choose n_0 and the design levels suitably such that the design points satisfy the conditions of SOSRD. If such designs exist with integral values for 'n_0', then this approach leads to a new method of construction of SOSRD.

3 SOSRD with equi-spaced levels using SUBA with two unequal block sizes

SUBA with two unequal block sizes (cf. Raghavarao, 1962): The arrangement of v-treatments in b blocks where b_1 blocks of size k_1 and b_2 blocks of size k_2, $(b_1 + b_2 = b)$, is said to be a symmetrical unequal block arrangements with two unequal block sizes if

(i) every treatment occurs $\frac{b_i k_i}{v}$ blocks of size k_i $(i = 1, 2)$, and

(ii) every pair of first associate treatments occurs together in u blocks of size k_1 and in $(\lambda - u)$ blocks of size k_2 while every pair of second associate treatments occurs together in λ blocks of size k_2.

From (i) each treatment occurs in $(\frac{b_1 k_1}{v}) + (\frac{b_2 k_2}{v}) = r$ blocks in all.
$(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ are known as the parameters of the SUBA with two unequal block sizes.

Let there be a SUBA with two unequal block sizes with parameters $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ and $k = \sup(k_1, k_2)$. Let us write the design in the form of a $b \times v$ matrix, the elements of which are zero and 'a'. If in any block a particular treatment occurs the element in that block corresponding to that treatment will be 'a', otherwise, zero. We denote these design points generated from the transpose of the incidence matrix of a SUBA with two unequal block sizes by $[a - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]$. Let $2^{t(k)}$ denotes the number of design points of a fractional factorial design of 2^k in ± 1 levels, such that no interaction with less than five factors is confounded. $[a-(v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)}$ is the $b2^{t(k)}$ design points generated from the SUBA with two unequal block sizes by 'multiplication' (cf. Raghavarao (1971), pp. 298–300), $(\pm b^*, 0, 0, \ldots, 0)2^1$ denote the design points generated from $(\pm b^*, 0, 0, \ldots, 0)$ point set, $\cup$ denotes the union of the design points generated from different sets of points and (n_0) be the number of central points in the design. The corresponding equi-spaced levels design using SUBA with two unequal block sizes is obtained by changing the axial points from $(\pm b^*, 0, 0, \ldots, 0)$ *etc.*, to $(\pm 2a, 0, 0, \ldots, 0)$. The axial points may be replicated n_a times and central points to be replicated n_0 times. Construction of SOSRD with equi-spaced levels (c-prefixed) is given in the following theorem.

Theorem 3.1 *The design points,*

$$\big[a - (v, b, r, k_1, k_2, b_1, b_2, \lambda)\big]2^{t(k)} \cup n_a(\pm 2a, 0, 0, \ldots, 0)2^1 \cup n_0$$

will give a v-dimensional SOSRD with equi-spaced levels in $N = b2^{t(k)} + 2n_a v + n_0$ design points if,

$$n_a = (c\lambda - r)2^{t(k)-5} \tag{3.1}$$

$$n_0 = \frac{(c\lambda + 3r)^2[v(5-c) - 4]2^{t(k)}}{16\lambda[v(5-c) - (c-3)^2]} - b2^{t(k)} - 2n_a v \tag{3.2}$$

and n_0 turns out to be an integer.

Proof. For the design points generated from SUBA with two unequal block sizes the conditions in equations (2.2) to (2.4) are satisfied. The conditions in equations (2.3) and (2.4) are true as follows:

$$\sum x_{iu}^2 = r2^{t(k)}a^2 + 8n_a a^2 = N\lambda_2 \tag{3.3}$$

$$\sum x_{iu}^4 = r2^{t(k)}a^4 + 32n_a a^4 = cN\lambda_4 \tag{3.4}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}a^4 = N\lambda_4 \tag{3.5}$$

From equations (3.4) and (3.5) we have $(r2^{t(k)} + 32n_a)a^4 = c \times \lambda 2^{t(k)}a^4$. This leads to $n_a = (c\lambda - r)2^{t(k)-5}$. From the slope rotatability condition (2.6) we have

$$\frac{\lambda_4}{\lambda_2} = \frac{v(c-5)+4}{v(c-5)+(c-3)^2}. \tag{3.6}$$

Solving equation (3.6) using equations (3.3) to (3.5), we get n_0 given in equation (3.2). The value of 'a' can be obtained from equation (3.3) by taking the scaling condition *i.e.* $\lambda_2 = 1$.

Corollary: If $k_1 = k_2 = k$, then theorem 3.1 reduces to the method of construction of Victorbabu (2002a) SOSRD with equi-spaced levels using BIBD.

Example: The design points,

$$[a - (v = 6, b = 15, r = 8, k_1 = 2, k_2 = 4, b_1 = 6, b_2 = 9, \lambda = 4)]2^4 \cup n_a(\pm 2a, 0, \ldots, 0)2^1 \cup (n_0)$$

will give a 6-factor SOSRD with equi-spaced levels using SUBA with two unequal block sizes in $N = 384$ design points with $c = 6$. Here equation (3.1) leads to $n_a = 8$ and $n_0 = 48$.

4 Conclusions

The SOSRD available in literature are not generally available with equi-spaced levels. Though designs with equi-spaced levels are not necessary, they are likely to be preferred in view of the case in handling the doses. Further in SOSRD, the calculation of the actual doses often requires approximations and hence actual dose levels can not be applied in practice. Hence, designs with equi-spaced levels are sometimes desired by the experimenter. In this paper, SOSRD with equi-spaced levels using SUBA with two unequal block sizes is obtained.

References

[1] Box, G. E. P. and Hunter, J. S. (1957). Multifactor experimental designs for exploring response surfaces, Annals of Mathematical Statistics, 28, 195–241.

[2] Das, M. N. and Narasimham, V. L. (1962). Construction of rotatable designs through balanced incomplete block designs, Annals of Mathematical Statistics, 33, 1421–1439.

[3] Dey, A. (1970). On response surface designs with equi-spaced doses, Calcutta Statistical Association Bulletin, 19, 135–144.

[4] Hader, R. J. and Park, S. H. (1978). Slope rotatable central composite designs, Technometrics, 20, 413–417.

[5] Park, S.H. (1987). A class of multifactor designs for estimating the slope of response surfaces, Technometrics, 29, 449–453.

[6] Raghavarao, D. (1962). Symmetrical unequal block arrangements with two unequal block sizes, Annals of Mathematical Statistics, 33, 620–633.

[7] Raghavarao, D. (1971). Constructions and combinatorial problems in design of experiments, John Wiley, New York.

[8] Victorbabu, B. Re. (2002a). Second order slope rotatable designs with equi-spaced levels, Proceedings of Andhra Pradesh Akademi of Sciences, 4, 211–214.

[9] Victorbabu, B. Re. (2002b). Construction of second order slope rotatable designs using symmetrical unequal block arrangements with two unequal block sizes, Journal of the Korean Statistical Society, 31, 153–161.

[10] Victorbabu, B. Re. (2007). On second order slope rotatable designs — A review, Journal of the Korean Statistical Society, 36, 373–386.

[11] Victorbabu, B. Re. and Narasimham, V.L. (1991). Construction of second order slope rotatable designs through balanced incomplete block designs, Communications in Statistics -Theory and Methods, 20, 2467–2478.

[12] Victorbabu, B. Re. and Narasimham, V.L. (2000-01). A new method of construction of second order slope rotatable designs, Journal of the Indian Society for Probability and Statistics, 5, 75–79.

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