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Construction of Second Order Slope-Rotatable Designs

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Abstract. *In this paper, a new method of construction of three and five level second order slope-rotatable designs (SOSRD) using two suitably chosen dissimilar incomplete block designs like balanced incomplete block designs and symmetrical unequal block arrangements with two unequal block sizes is developed. The method of construction of SOSRD by Victorbabu and Narasimham [11] is shown to be a particular case of this new method.*
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1 Introduction

Box and Hunter [1] introduced rotatable designs for the exploration of response surfaces. Das and Narasimham [2] constructed rotatable designs through balanced incomplete block designs (BIBD). The study of rotatable designs mainly emphasized on the estimation of absolute response. Estimation of differences in response at two different points in the factor space will often be of great importance. If differences at two points close together, estimation of local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc., (*cf.* Park [4]).

Hader and Park [3] introduced slope-rotatable central composite designs. Park [4] introduced a class of multifactor designs for estimating the slope of response surfaces. Different methods of construction of second order slope-rotatable designs (SOSRDs) using BIBD, a pair of BIBD, and pairwise balanced designs *etc.*, were developed by Victorbabu and Narasimham [8, 9, 10, 11]. A method of construction of SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes, and its review was developed by Victorbabu [6, 7].

2 Conditions for SOSRD

Assuming the response surface is of second order, we adopt the model:

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j} b_{ij} x_{iu} x_{ju} + e_u, \tag{2.1}$$

where x_{iu} denotes the level of the i^{th} factor ($i = 1, 2, \ldots, v$) in the u^{th} run ($u = 1, 2, \ldots, N$) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2. Here b_0, b_i, b_{ii}, b_{ij} are the parameters of the model and Y_u is the response observed at the u^{th} design point. A second order response surface design D is said to be a SOSRD, if the design points satisfy the following conditions (*cf.* Hader and Park [3]).

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \tag{2.2}$$

$$(i) \sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$$

$$(ii) \sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \tag{2.3}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4, \text{ for } i \neq j \tag{2.4}$$

$$(c + v - 1)\lambda_4 > v\lambda_2^2 \tag{2.5}$$

$$\lambda_4[v(5 - c) - (c - 3)^2] + \lambda_2^2[v(c - 5) + 4] = 0 \tag{2.6}$$

where c, λ_2 and λ_4 are constants and the summation is over the design points.

It is clear from the slope-rotatability condition (2.6) that the solution for design levels (like a, b, *etc.*,) depend on 'c' and central points 'n_0'. In contrast to rotatable designs, we have to choose the non-zero design levels a, b, *etc.*, in SOSRD depending on the number of central points 'n_0'. This interdependence of the parameters n_0, c and design levels is utilized to evolve a new method of construction of SOSRD.

In SOSRD usually the number of central points 'n_0' is prefixed and design levels are chosen to satisfy these conditions of SOSRD. The parameter 'c' is then evaluated. We refer to such designs as exact SOSRD. Alternatively we may prefix 'c' and choose n_0 and the design levels suitably such that the design points satisfy the conditions of SOSRD. If such designs exist with integral values for 'n_0', then this approach leads to a new method of construction of SOSRD. We refer to such designs as exact SOSRD. For given v if n_0 as determined by above is non-integral positive real number, we take $[n_0]$ or $[n_0]+1$, { where $[n_0]$-Gauss symbol denoting integral part of n_0} central points and call such designs as nearly SOSRD. The extent of disturbance in the spherical variance function of such designs is found to be insignificant through sensitivity analysis of the spherical variance function of the estimated slope to small changes in the value of n_0.

3 Method of construction of SOSRD

3.1 Three level SOSRD using a pair of dissimilar incomplete block designs

A new method of construction of three level SOSRD using a pair of dissimilar incomplete block designs like SUBA with two unequal block sizes and BIBD without any additional set of points, is developed.

Let $D_1 = (v, b_1, r_1, k_{11}, k_{12}, b_{11}, b_{12}, \lambda_1)$ denote a SUBA with two unequal block sizes (where b_{11} blocks each of size k_{11} and b_{12} blocks each of size k_{12}, $b_{11} + b_{12} = b_1$) with $r_1 \leq > c\lambda_1$, $k_1 = \max(k_{11}, k_{12})$, and $D_2 = (v, b_2, r_2, k_2, \lambda_2)$ denote a BIBD with $r_2 \geq < c\lambda_2$ in v-treatments. Let $2^{t(k_1)}$ and $2^{t(k_2)}$ denote Resolution V fractional replicates of 2^{k_1} and 2^{k_2} factorials with levels ± 1 respectively. Let $[a-(v, b_1, r_1, k_{11}, k_{12}, \lambda_1)]2^{t(k_1)}$ denote the $b_1 2^{t(k_1)}$ design points generated from SUBA with two unequal block sizes and $[a-(v, b_2, r_2, k_2, \lambda_2)]2^{t(k_2)}$ denote the $b_2 2^{t(k_2)}$ design points generated from the BIBD by 'multiplication' (see Raghavarao [5, pp.298–300]). Let n_0 be the number of central points in SOSRD. SOSRD can be constructed with three levels factors using a pair of dissimilar incomplete block designs as follows. Repeat the set of $b_2 2^{t(k_2)}$ design points generated from BIBD-D_2 n_a times. Then with the above design points along with n_0 central points we can construct a three level SOSRD which is given in the following theorem.

Theorem 3.1. *The design points,*

$$[a-(v, b_1, r_1, k_{11}, k_{12}, b_{11}, b_{12}, \lambda_1)]2^{t(k_1)} \bigcup n_a[a-(v, b_2, r_2, k_2, \lambda_2)]2^{t(k_2)} \bigcup n_0$$

will give a three level v-dimensional SOSRD with $N = b_1 2^{t(k_1)} + n_a b_2 2^{t(k_2)} + n_0$ design points if $(r_1 - c\lambda_1)(r_2 - c\lambda_2) \leq 0$ and

$$n_a = \frac{(c\lambda_1 - r_1)2^{t(k_1)-t(k_2)}}{(r_2 - c\lambda_2)}, \tag{3.1}$$

$$n_0 = \frac{[v(c-5)+4]c^2[\lambda_1 2^{t(k_1)} + n_a \lambda_2 2^{t(k_2)}]}{[v(5-c)-(c-3)^2]} - b_1 2^{t(k_1)} - n_a b_2 2^{t(k_2)} \tag{3.2}$$

Here we may choose D_1 and D_2 suitably such that n_0 in equation (3.2) is a positive integer. For a given 'v' if n_0 as determined above is a non-integral positive real number, we may take $[n_0]$ or $[n_0]+1$, { where $[n_0]$-Gauss symbol denoting integral part of n_0} central points and construct nearly SOSRD.

Proof. For the design points generated from SUBA with two unequal block sizes-D_1 and n_a-repetitions of points from BIBD-D_2, conditions of SOSRD are true as follows:

$$\sum x_{iu}^2 = r_1 2^{t(k_1)} a^2 + n_a r_2 2^{t(k_2)} a^2 = N\lambda_2 = \text{constant} \tag{3.3}$$

$$\sum x_{iu}^4 = r_1 2^{t(k_1)} a^4 + n_a r_2 2^{t(k_2)} a^4 = cN\lambda_4 = \text{constant} \tag{3.4}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda_1 2^{t(k_1)} a^4 + n_a \lambda_2 2^{t(k_2)} a^4 = N\lambda_4 = \text{constant} \tag{3.5}$$

From equations (3.4) and (3.5) we have,

$$r_1 2^{t(k_1)} a^4 + n_a r_2 2^{t(k_2)} a^4 = c(\lambda_1 2^{t(k_1)} a^4 + n_a \lambda_2 2^{t(k_2)} a^4),$$

which leads to n_a given in equation (3.1). Equation (3.1) has a real solution only if $(r_1 - c\lambda_1)(r_2 - c\lambda_2) \leq 0$. From the slope-rotatability condition (2.6) we have

$$\frac{\lambda_4}{\lambda_2^2} = \frac{v(c-5)+4}{v(c-5)+(c-3)^2}. \tag{3.6}$$

From equations (3.3) and (3.5), we get λ_2 and λ_4 respectively as follows:

$$\lambda_2 = \frac{r_1 2^{t(k_1)} a^2 + n_a r_2 2^{t(k_2)} a^2}{N}, \tag{3.7}$$

$$\lambda_4 = \frac{\lambda_1 2^{t(k_1)} a^4 + n_a \lambda_2 2^{t(k_2)} a^4}{N}. \tag{3.8}$$

Solving equation (3.6) using equations (3.7) and (3.8), we get n_0 given in equation (3.2). The value of 'a' can be obtained from equation (3.3) by taking the scaling condition *i.e.* $\lambda_2 = 1$. □

Corollary 3.1.1. *The method of construction of three level SOSRD using a pair of BIBD suggested by Victorbabu and Narasimham [11] can be obtained from Theorem 3.1 by substituting $D_1 = (v, b_1, r_1, k_{11} = k_{12} = k_1, \lambda_1)$ and $D_2 = (v, b_2, r_2, k_2, \lambda_2)$.*

Example 3.1. *We illustrate the construction of three level SOSRD for 12-factors, with the help of a pair of dissimilar incomplete block designs. Consider the design points,*

$$[a - (v = 12, b_1 = 13, r_1 = 4, k_{11} = 3, k_{12} = 4, b_{11} = 4, b_{12} = 9, \lambda_1 = 1)]2^4$$
$$\bigcup n_a[a - (v = 12, b_2 = 44, r_2 = 11, k_2 = 3, \lambda_2 = 2)]2^3 \bigcup (n_0)$$

will give a three level 12-factor SOSRD with $N = 1200$ design points for $c = 5$ and $n_a = 2$. Here (3.2) leads to $n_0 = 288$.

3.2 Five level SOSRD using a pair of dissimilar incomplete block designs

The method of construction of five level SOSRD using a pair of dissimilar incomplete block designs like SUBA with two unequal block sizes and BIBD without any additional set of points can be obtained as follows.

Let $D_1 = (v, b_1, r_1, k_{11}, k_{12}, b_{11}, b_{12}, \lambda_1)$ denote a SUBA with two unequal block sizes $(b_{11} + b_{12} = b_1)$ with $r_1 \leq> c\lambda_1$, $k_1 = \max(k_{11}, k_{12})$, and $D_2 = (v, b_2, r_2, k_2, \lambda_2)$ denote a BIBD with $r_2 \geq< c\lambda_2$ in v-treatments with 'c' pre-fixed. The method of construction is given in the following theorem.

Theorem 3.2. *The design points,*

$$[1 - (v, b_1, r_1, k_{11}, k_{12}, b_{11}, b_{12}, \lambda_1)]2^{t(k_1)} \bigcup [a - (v, b_2, r_2, k_2, \lambda_2)]2^{t(k_2)} \bigcup (n_0)$$

will give a five level SOSRD with $N = b_1 2^{t(k_1)} + b_2 2^{t(k_2)} + n_0$ *if* $(r_1 - c\lambda_1)(r_2 - c\lambda_2) \leq 0$ *and*

$$a^4 = \frac{(r_1 - c\lambda_1)2^{t(k_1)-t(k_2)}}{(c\lambda_2 - r_2)}, \tag{3.9}$$

$$n_0 = \frac{-[v(c-5)+4][r_1 2^{t(k_1)} + r_2 2^{t(k_2)} a^2]^2}{[v(5-c)-(c-3)^2][\lambda_1 2^{t(k_1)} + \lambda_2 2^{t(k_2)} a^4]} - b_1 2^{t(k_1)} - b_2 2^{t(k_2)} \tag{3.10}$$

Proof. For the design points generated from SUBA with two unequal block sizes-D_1 and from BIBD-D_2, conditions of SOSRD are true as follows:

$$\sum x_{iu}^2 = r_1 2^{t(k_1)} + r_2 2^{t(k_2)} a^2 = N\lambda_2 = \text{constant} \tag{3.11}$$

$$\sum x_{iu}^4 = r_1 2^{t(k_1)} + r_2 2^{t(k_2)} a^4 = cN\lambda_4 = \text{constant} \tag{3.12}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda_1 2^{t(k_1)} + \lambda_2 2^{t(k_2)} a^4 = N\lambda_4 = \text{constant} \tag{3.13}$$

From equations (3.12) and (3.13) we have,

$$r_1 2^{t(k_1)} + r_2 2^{t(k_2)} a^4 = c(\lambda_1 2^{t(k_1)} + \lambda_2 2^{t(k_2)} a^4),$$

which leads to 'a^4' given in equation (3.9). $a^2 > 0$, since $(r_1 - c\lambda_1)(r_2 - c\lambda_2) \leq 0$. From the slope rotatability condition (2.6) we have

$$\frac{\lambda_4}{\lambda_2^2} = \frac{v(c-5)+4}{v(c-5)+(c-3)^2}. \tag{3.14}$$

Solving equation (3.14) using equations (3.11) and (3.13), we get n_0 given in equation (3.10). □

Corollary 3.2.1. *The method of construction of five level SOSRD using a pair of BIBD suggested by Victorbabu and Narasimham [11] can be obtained from Theorem 3.2 by substituting* $D_1 = (v, b_1, r_1, k_{11} = k_{12} = k_1, \lambda_1)$ *and* $D_2 = (v, b_2, r_2, k_2, \lambda_2)$.

Example 3.2. *We illustrate the construction of five level SOSRD for 8-factors, with the help of a pair of dissimilar incomplete block designs. Consider the design points,*

$$[1-(v=8, b_1=26, r_1=10, k_{11}=4, k_{12}=3, b_{11}=2, b_{12}=24, \lambda_1=3)]2^4 \bigcup [a-(v=8, b_2=28, r_2=7, k_2=2, \lambda_2=1)]2^2 \bigcup (n_0).$$

Let us fix $c = 5$, *then equation* (3.9) $\Rightarrow a = 1.7782$ *and equation* (3.10) $\Rightarrow n_0 = 173.9773$ *which is not an integer. In such cases we take* $[n_0]$ *or* $[n_0] + 1$ *(*$[n_0]$ *denoting integral part of* n_0*) central points and construct five level 8-dimensional nearly SOSRD with 701 or 702 design points.*

4 Conclusion

In this study, a new method of construction of three and five level second order slope-rotatable designs using two suitably chosen dissimilar incomplete block designs like balanced incomplete block designs and symmetrical unequal block arrangements with two unequal block sizes is developed. The method of construction of SOSRD by Victorbabu and Narasimham [11] is shown to be a particular case of this new method. It can be observed that a necessary condition for the existence of a positive integral solution for n_0 is $c \geq 5$. As such one can construct above type of SOSRD (if they exist) with any other value of $c \geq 5$, for example $c = 5, 6, 7, 7.5, 8$ *etc.* However, we may mention here that large number of design points are required in this method for the construction of SOSRD.

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